LECTURE 1. STATEMENT OF CONNES' EMBEDDING CONJECTURE

0.1. **Introduction.** Connes' embedding conjecture comes from the sentence: "*We now construct an approximate imbedding of N in R. Apparently such an imbedding ought to exist for all II_1-factors because it does for the regular representation of free groups. However the construction below relies on condition 6*" (see [8], p.105). Alain Connes himself observed in p.106, Lemma 5.22, that such an approximate embedding would become an exact embedding in the ultraproduct $R^{\mathcal{U}}$. Therefore, we can say that the *original statement* of the problem is: does every (separable) II_1-factor embeds into R^{ω}? Since 1976, there had not been any news in merit for about twenty years, when Eberhard Kirchberg obtained in [22] a series of equivalent statements, the most unexpected one is probably the following: Connes' embedding conjecture holds true if and only if

$$C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) \tag{1}$$

What is really fascinating about this formulation is not only the apparent farness between the two problems, but also the fact that Connes' embedding conjecture is a problem regarding a class of objects (separable II_1-factors), while (1) is a problem regarding a property of a particular object: the universal C^*-algebra of the free group on countably many generators. Even more unexpected is the topological proof of Kirchberg's theorem appeared in [17], as a corollary of another reformulation of Connes' embedding conjecture, this time a topological reformulation. Over the same years, Voiculescu was developing his free entropy theory and found that Connes' embedding conjecture is related to the existence of microstates (see [34], [35]). All these findings inspired many other papers (see, e.g., [1], [27], [7], [16]) and increased the interest around this conjecture. About ten years ago, Florin Rădulescu tried to attack the problem from a simpler point of view, looking only at II_1-factors arising from groups and discovered the so-called hyperlinear groups [32] that allowed Gabor Elek and Endre Szabó to prove that an important conjecture in Symbolic Dynamics, called Gottschalk surjunctivity conjecture, is related to Connes' embedding conjecture. In particular they proved that Gottschalk's conjecture implies Connes' embedding conjecture for group algebras. Even more, Elek-Szabó theorem uses sofic groups and, in fact, they showed that every sofic group is hyperlinear [9]. Starting from this point, Connes' embedding problem went out his original field, Operator Algebras, to get in the field of Geometric Group Theory. In 2005, again Florin Rădulescu found that Connes' embedding conjecture is equivalent to some non-commutative analogue of Hilbert's 17th problem [33]; this inspired a work of Klep and Schweighofer, who found a purely algebraic reformulation of Connes' embedding conjecture, leading the problem of interest also in the field of Real Algebra [23]. Another algebraic reformulation of Connes' embedding conjecture has been very recently proposed by Juschenko and Popovych [21]. Of the same flavor is the observation in [6] that Connes' embedding conjecture is related to a problem of embedding Euclidean spaces into Hilbert spaces with an additional structure, called *cyclic Hilbert spaces*. Other very recent discoveries include the fact that Connes' embedding conjecture is related to an important problem in Quantum Information Theory, the so-called Tsirelson's problem (see [13], [20], and [29]) and that II_1-factors verifying Connes' embedding conjecture admit a new invariant that can help to classify them in terms of rigidity properties. This geometric invariant has been first abstractly introduced by Nate Brown in [2], while more recently it has been observed that it can be realized as a convex subset of a suitable Banach space (see [5], [3]).

0.2. **II_1-factors.** Connes' embedding conjecture concerns the so-called (separable) II_1-factors. They are a sort of a continuous analogue of matrix algebras and indeed, roughly speaking, Connes' embedding problem is equivalent to ask whether any II_1-factor can be approximated by matrix algebras in a suitable sense.

Let H be a Hilbert space and denote by $B(H)$ the set of all linear and bounded operators from H to itself. Recall that $B(H)$ can be endowed with several different topologies:

- The norm topology is the one induced by the usual norm of an operator $x \in B(H)$
$$||x|| = \sup\{||x\xi|| : \xi \in H, ||\xi|| \leq 1\}.$$
- The weak operator topology is the weakest (locally convex) topology on $B(H)$ making continuous all maps belonging to the following family:
$$\{B(H) \ni x \to |(x\xi, \eta)| : \xi, \eta \in H\},$$
where $(\cdot, \cdot)$ stands for the inner product in H.
- The strong operator topology is the weakest (locally convex) topology on $B(H)$ making continuous all maps belonging to the following family:
$$\{B(H) \ni x \to ||x\xi|| : \xi \in H\}.$$

All these topologies coincide if and only if H is finite dimensional.

Let A be a subset of $B(H)$, the commutant of A, denoted by A', is the set of operators that commute with every operator in A; namely,

$$A' = \{x \in B(H) : ax = xa, \text{ for all } a \in A\}. \tag{2}$$

The double commutant of A, denoted by A'', is the set of operators that commute with all operators that commute with every operator in A.

Exercise 0.1. Prove that for all subsets $A \subseteq B(H)$, one has $A \subseteq A''$ and $A' = A'''$.

Theorem 0.2. (von Neumann's bicommutant theorem) *Let A be a *subalgebra*[1] *of $B(H)$ containing the identity. One has*

$$\overline{A}^s = \overline{A}^w = A''$$

A matrix algebra $M_n\mathbb{C}$ can be seen as the algebra of linear (and automatically bounded) operators on a finite dimensional complex Hilbert space. It then verifies the following properties:

(1) $M_n(\mathbb{C})$ is a unital *algebra; i.e. a unital algebra equipped with an involution * that is compatible with the structure of algebra,
(2) $M_n(\mathbb{C})$ is closed with respect to *any* topology.
(3) $M_n(\mathbb{C})$ is a factor, meaning that $Z(M_n\mathbb{C}) := M_n\mathbb{C} \cap (M_n(\mathbb{C}))' = \mathbb{C}1$.
(4) $M_n(\mathbb{C})$ admits a unique normalized trace; i.e. there is a unique linear (and automatically continuous) functional $\tau : M_n(\mathbb{C}) \to \mathbb{C}$ such that
   - $\tau(x^*x) = \tau(xx^*) \geq 0$, for all $x \in M$,
   - $\tau(x^*x) = 0$ implies $x = 0$,
   - $\tau(1) = 1$,
   - Let $P(M_n(\mathbb{C})) := \{e \in M_n\mathbb{C} : e^*e = e\}$ denote the set of projections, one has
$$\tau(P(M_n(\mathbb{C}))) = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

As mentioned before, II_1-factors are the continuous analogue of matrix algebras.

Definition 0.3. *A II_1-factor is a subset M of $B(H)$ verifying the following properties*

*(1) M is a unital *algebra;*
(2) M is closed in the weak operator topology[2]*.*
(3) M is a factor, in the sense that $M \cap M' = \mathbb{C}1$.
(4) M admits a unique linear weakly continuous functional τ such that
   - $\tau(xx^*) = \tau(x^*x) \geq 0$, *for all* $x \in M$,
   - $\tau(x^*x) = 0$ *implies* $x = 0$,
   - $\tau(1) = 1$,
   - $\tau(P(M)) = [0, 1]$, *where* $P(M) = \{e \in M : e^*e = e\}$.

The very last property shows the meaning of the rough sentence that II_1-factors are the continuous analogue of matrix algebras. If M verifies only the first two properties above, then it is called von Neumann algebra. A von Neumann algebra is called separable if H is a separable Hilbert space. The simplest example of von Neumann algebra is $B(H)$ itself, for any Hilbert space.

II_1-factors may seem quite wild objects but actually they can be constructed very easily starting from groups through a procedure that was already known to Murray and von Neumann.

[1] A *subalgebra of $B(H)$ is simply a subalgebra A of $B(H)$ such that $a \in A$ implies $a^* \in A$, where a^* is the adjoint of the operator a.

[2] This is equivalent to require that M is closed in the strong topology, by von Neumann's bicommutant theorem.

Example 0.4. Let G be a countable group with property i.c.c. (i.e. every conjugacy class is infinite except the one containing the identity 1_G) and let $\ell^2(G)$ be the Hilbert space of all square-summable complex-valued functions on G. Each $g \in G$ defines an operator $\lambda_g : \ell^2(G) \to \ell^2(G)$ in the following way:

$$\lambda_g(f)(x) = f(g^{-1}x)$$

Exercise 0.5. Prove that $g \to \lambda_g$ is a group monomorphism from G to the unitary group[3] of $B(\ell^2(G))$.

The group von Neumann algebra of G, denoted by $L(G)$, is the weak operator closure of the subalgebra of $B(\ell^2(G))$ generated by all the λ_g's, equipped with the trace obtained extending by linearity and continuity the conditions: $tr(1_G) = 1$ and $tr(g) = 0, \forall g \neq 1_G$.

Exercise 0.6. Prove that $L(G)$ is a separable II_1-factor (*Hint:* property i.c.c. should be reflected in the fact that $L(G)$ has trivial center).

Exercise 0.7. Let S_{fin}^{∞} be the group of permutations of a countable set that fix all but finitely many elements. Prove that S_{fin}^{∞} has the property i.c.c.

Definition 0.8. *The hyperfinite II_1-factor, denoted by R, is the group von Neumann algebra of the group S_{fin}^{∞}.*

The hyperfinite II_1-factor is the smallest II_1-factor: it is contained in every II_1-factor and it is the unique, up to isomorphisms, II_1-factor having this property [8]. It can be described in several ways: Murray and von Neumann showed in [25] that it is the unique factor, up to isomorphisms, which contains an increasing chain of copies of matrix algebras whose union is weakly dense; Alain Connes was the first to observe that R could be described as an infinite tensor product of 2×2-matrices; i.e. $R \cong \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$. This description makes essentially trivial the useful property that $R\bar{\otimes}R$ is isomorphic to R [4].

0.3. **Ultrafilters and Ultraproducts.**

Definition 0.9. *A free ultrafilter on the natural numbers is a family $\mathcal{U}$ of subsets of $\mathbb{N}$ such that*

1. *If A is a finite subset of $\mathbb{N}$, then $A \notin \mathcal{U}$,*
2. *$A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$,*
3. *For each $A \subseteq \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$*

Exercise 0.10. Prove that if $A \in \mathcal{U}$ and $B \supseteq A$, then $B \in \mathcal{U}$.

Exercise 0.11. Prove, making use of Zorn's lemma, that there exists at least one free ultrafilter (*Hint:* condition (3) is a maximality condition).

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[3]The unitary group of a von Neumann algebra M is the multiplicative group

$$U(M) = \{u \in M : u^*u = uu^* = 1\}$$

.

[4]The formal definition of tensor product of von Neumann algebras will be given later in this course.

Exercise 0.12. Prove that free ultrafilters coincide with finitely additive probability measures $\mathcal{U} : P(\mathbb{N}) \to \{0,1\}$ such that $\mathcal{U}(A) = 0$ for all finite subsets of $\mathbb{N}$.

Ultrafilters are very useful to define a weak notion of convergence.

Definition 0.13. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers, $x \in \mathbb{R}$ and $\mathcal{U}$ a free ultrafilter on A. We say that $\lim_{n\to\mathcal{U}} x_n = x$ if for any $\varepsilon > 0$ one has*

$$\{n \in \mathbb{N} : |x_n - x| < \varepsilon\} \in \mathcal{U}$$

Exercise 0.14. Prove that every bounded sequence converges along any free ultrafilter (*Hint:* Bolzano-Weierstrass).

Exercise 0.15. Prove that the operator $\lim_{n\to\mathcal{U}}$ is linear, multiplicative and verifies the property that $\lim_{n\to\mathcal{U}} x_n$ belongs to the set of limit points of the sequence x_n. Conversely, show that if $L : \ell^\infty \to \mathbb{R}$ is a linear and multiplicative operator with the property that $L(x_n)$ belongs to the set of limit points of the sequence (x_n), then there is a free ultrafilter $\mathcal{U}$ such that $L = \lim_{n\to\mathcal{U}}$.

Let $\{(M_n, \tau_n)\}_{n\in\mathbb{N}}$ be a family of II_1-factors equipped with normalized traces τ_n and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Set

$$M = \left\{ x = (x_n) \in \prod_{n\in\mathbb{N}} M_n : \sup_n ||x_n|| < \infty \right\} \tag{3}$$

and

$$J = \left\{ x = (x_n) \in M : \lim_{n\to\mathcal{U}} \tau_n(x_n^* x_n)^{\frac{1}{2}} = 0 \right\} \tag{4}$$

The quotient M/J turns out to be a II_1-factor with component-wise operations and with trace $\tau(\{x_n\}_n) = lim_{n\to\mathcal{U}}\tau_n(x_n)$, that does not depend on the choice of the representative sequence. The proof that M/J is a II_1-factor is not easy. A sketch of the proof can be found at pp. 18-19 of [30], or in the first papers about tracial ultraproducts by Wright ([36]), McDuff ([24]) and Janssen ([18]). We will give a proof using logic of metric structures in the next chapter. The quotient M/J is called *tracial ultraproduct of the M_n's.* The word *ultrapower* is used when $M_n = M_m$ for every $m, n \in \mathbb{N}$.

Definition 0.16. *$R^{\mathcal{U}}$ is the tracial ultrapower of R with respect to a free ultrafilter $\mathcal{U}$ on the natural numbers.*

0.4. **Original statement of Connes' Embedding Conjecture.**

Conjecture 0.17. (A. Connes, [8] **pp.105-106)** *Every separable II_1-factor is embeddable into some $R^{\mathcal{U}}$?*

Assuming Continuum Hypothesis (CH), Ge and Hadwin [14] proved that all ultrapowers of a fixed separable II_1-factor with respect to a free ultrafilter on the natural numbers are isomorphic among themselves. More recently, Farah, Hart and Sherman proved also the converse: for any separable II_1-factor M, CH is *equivalent* to the statement that all tracial ultrapowers of M (with respect to a free ultrafilter on the natural numbers) are isomorphic among themselves (see [10], Th.3.1). Even more: if CH fails, there are $2^{2^{\aleph_0}}$ many non-isomorphic ultrapowers (see [11]). It follows that CH together with Connes' embedding conjecture implies the existence of a *universal* II_1-factor; *universal* in the sense that it should contain every separable type II_1 factors. Ozawa proved in [28] that such a universal II_1-factor cannot be separable. So, the first thing to check is that $R^{\mathcal{U}}$ is not separable.

Proposition 0.18. *$R^{\mathcal{U}}$ is not separable.*

Non-separability of $R^{\mathcal{U}}$ is well-known and proved by several authors (see [12] and, in greater generality, [31] Prop. 4.3). We will present a new proof, appeared in [4], because it uses a new construction that will be useful in the next lecture to give a short proof of Rădulescu's theorem.

Proof of Proposition 0.18. We have to prove that $R^{\mathcal{U}}$ is not faithfully representable in $B(H)$, with H separable Hilbert space. We recall that if H is separable, then the strong topology on $B(H)$ is separable. Moreover, we recall that the strong topology coincides with the Hilbert-Schmidt topology[5] on bounded sets. So it suffices to prove that $R^{\mathcal{U}}$ contains an uncountable family of unitaries $\{u^{(t)}\}$ such that $||u^{(t)} - u^{(s)}||_2 = \sqrt{2}$ for all $t \neq s$.

It is a simple exercise, using the construction of the group factor, to show that there is a sequence $\{u_n\} \subseteq U(R)$ of distinct unitaries such that $u_n \neq 1$, for all $n \in \mathbb{N}$ and $\tau(u_n^* u_m) = 0$ for all $n \neq m$. Take such a sequence a take a number $t \in \left[\frac{1}{10}, 1\right)$, for instance $t = 0,132471....$ Define

$$I_t = \{1, 13, 132, 1324, 13247, 132471, ...\}$$

i.e. I_t is the sequence of the approximations of t. Clearly, $\{I_t\}_{t \in \left[\frac{1}{10},1\right)}$ is uncountable and $I_t \cap I_s$ is finite for all $t \neq s$ (this property requires the choice of $t \geq \frac{1}{10}$!).

Now define

$$\begin{array}{llll}
u_1^{(t)} = u_1 & u_2^{(t)} = u_2 & \dots & u_{12}^{(t)} = u_{12} \\
u_{13}^{(t)} = u_1 & u_{14}^{(t)} = u_2 & \dots & u_{131}^{(t)} = u_{131-12} \\
u_{132}^{(t)} = u_1 & \dots & \dots & \\
\vdots & & &
\end{array}$$

i.e. any time we find an element of I_t, we start again from u_1. Now define $u^{(t)} = \prod_{\mathcal{U}} u_n^{(t)} \in U(R^{\mathcal{U}})$. Since $I_t \cap I_s$ is finite (for $t \neq s$), then $u^{(t)}$ and $u^{(s)}$ have only a finite

[5]The Hilbert-Schmidt topology on a II_1-factor is the topology induced by the norm $||x||_2 = \tau(x^* x)^{\frac{1}{2}}$.

number of common components. Thus we have

$$||u^{(t)} - u^{(s)}||_2^2 = \lim_{n\to\mathcal{U}} \tau_n((u_n^{(t)} - u_n^{(s)})^*(u_n^{(t)} - u_n^{(s)}))$$

where τ_n is the normalized trace on the n-th copy of R. Now we observe that

$$\tau_n((u_n^{(t)} - u_n^{(s)})^*(u_n^{(t)} - u_n^{(s)})) = \begin{cases} 0 & if \quad u_n^{(t)} = u_n^{(s)} \\ 2 & if \quad u_n^{(t)} \neq u_n^{(s)} \end{cases}$$

Since $u_n^{(t)} = u_n^{(s)}$ holds only for finitely many n's and since $\mathcal{U}$ is free (and thus it does not contain finite sets), it follows that

$$\lim_{n\to\mathcal{U}} \tau_n((u_n^{(t)} - u_n^{(s)})^*(u_n^{(t)} - u_n^{(s)})) = 2$$

and thus $||u^{(t)} - u^{(s)}||_2 = \sqrt{2}$. □

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