LECTURE II. HYPERLINEAR GROUPS

1. Rădulescu's Theorem

As we have seen in the previous lecture, one can associate a separable II_1-factor to every countable i.c.c. group. So we can consider the following restricted variant of Connes' embedding conjecture, called Connes' embedding conjecture for groups.

Conjecture 1.1. *For every countable i.c.c. group G, the group von Neumann factor $L(G)$ embeds into $R^{\mathcal{U}}$.*

One may hope to reformulate this conjecture as a purely algebraic condition on the underlying group G. This is basically the content of Rădulescu's theorem, which also led to find connections with other conjecture in Group Theory, as Gromov's reformulation of Gottshalk surjunctivity conjecture [8], [5]. Rădulescu's theorem was first proved for countable i.c.c. groups in [11] and later extended to any group in [2].

Theorem 1.2. *For any group G, the following conditions are equivalent:*

(1) G admits a group monomorphism into some $U\left(R^{\mathcal{U}}\right)$;
(2) The group von Neumann algebra $L(G)$ embeds into some $R^{\mathcal{V}}$.

We mentioned in the previous lecture that, if G is countable, one can choose $\mathcal{V}=\mathcal{U}$. This is no longer true for uncountable groups without assuming Continuum Hypothesis. Indeed, by Ge-Hadwin's theorem [7], Continuum Hypothesis implies that $R^{\mathcal{U}}\cong R^{\mathcal{V}}$ and so one can certainly work with the same ultrafilter. On the other hand, without Continuum Hypothesis, one can find ultrafilters $\mathcal{U}$ and $\mathcal{V}$ such that $U\left(R^{\mathcal{U}}\right)$ does not embed into $U\left(R^{\mathcal{V}}\right)$.

We now move towards the proof of Rădulescu's theorem. To this end, we need some preparation.

Definition 1.3. *Given two free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$, their tensor product $\mathcal{U}\otimes\mathcal{V}$ is the free ultrafilter on $\mathbb{N}\times\mathbb{N}$ defined as follows:*

$$B\in\mathcal{U}\otimes\mathcal{V}\qquad \textit{iff}\qquad \{k\in\mathbb{N}:\{n\in\mathbb{N}:(k,n)\in B\}\in\mathcal{V}\}\in\mathcal{U}$$

Exercise 1.4. Prove that $\mathcal{U}\otimes\mathcal{V}$ is a free ultrafilter.

Proposition 1.5. *For any pair of free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$, one has*

$$\left(R^{\mathcal{U}}\right)^{\mathcal{V}}\cong R^{\mathcal{U}\otimes\mathcal{V}}$$

Proof. Let $x\in\left(R^{\mathcal{U}}\right)^{\mathcal{V}}$ and lift it to a sequence $(x_n^k)\in\ell^\infty(R,\mathbb{N}\times\mathbb{N})$ and then map it into $R^{\mathcal{U}\otimes\mathcal{V}}$ using the canonical projection onto the quotient. Denote by x' the element obtained.

To show that this procedure gives rise to an isomorphism, it suffices to show that the trace of x is equal to the trace of x'. Now, by definition, one has

$$\tau(x) = \lim_{k\to\mathcal{U}} \lim_{n\to\mathcal{V}} \tau(x_n^k),$$

and

$$\tau(x') = \lim_{(k,n)\to\mathcal{U}\otimes\mathcal{V}} \tau(x_n^k).$$

Therefore, it is enough to show that for every bounded sequence a_n^k of real numbers, one has

$$\lim_{k\to\mathcal{U}} \lim_{n\to\mathcal{V}} a_n^k = \lim_{(k,n)\in\mathcal{U}\otimes\mathcal{V}} a_n^k.$$

To show this, let $a = \lim_{k\to\mathcal{U}} \lim_{n\to\mathcal{V}} a_n^k$ and fix $\varepsilon > 0$. Denote

$$A = \left\{ k \in \mathbb{N} : \left| \lim_{n\to\mathcal{V}} a_n^k - a \right| < \frac{\varepsilon}{2} \right\}.$$

By definition of limit along an ultrafilter, one has $A \in \mathcal{U}$. Now, for all $k \in \mathbb{N}$, define

$$A_k = \left\{ n \in \mathbb{N} : \left| a_n^k - \lim_{n\to\mathcal{V}} a_n^k \right| < \frac{\varepsilon}{2} \right\}.$$

Also in this case, by definition of limit along an ultrafilter, one has $A_k \in \mathcal{V}$, for all k. It follows that

$$\{(k,n) : k \in A, n \in A_k\} \in \mathcal{U} \otimes \mathcal{V}.$$

Now observe that this set is contained in $\left\{(k,n) : |a_n^k - a| < \varepsilon\right\}$, which then must be in $\mathcal{U}\otimes\mathcal{V}$ (put reference to the exercise in the previous lecture). This shows that $\lim_{(k,n)\to\mathcal{U}\otimes\mathcal{V}} a_n^k = a$, as desired. ☐

To prove the next lemma, we recall that in a II_1-factor any element x has a polar decomposition $x = u(x^*x)^{\frac{1}{2}}$, where u is a unitary.

Lemma 1.6. *Let M_n be a sequence of II_1 factors, $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and M be the tracial ultraproduct of the M_n's with regard to $\mathcal{U}$. Then every unitary $u \in U(M)$ has a lift (u_n), where each u_n is a unitary in M_n.*

Proof. Let (v_n) be a lift of u. Applying the polar decomposition to any v_n, we may write $v_n = u_n(v_n^* v_n)$, where the u_n's are unitaries. Therefore it is enough to show that the sequence (u_n) is another lift of the unitary u. Given a sequence $(x_n) \in \Pi M_n$, we denote by $(x_n)_\mathcal{U}$ the image of this sequence in the ultraproduct. Recalling that the operations in the ultraproduct are defined component-wise, we have

$$\begin{aligned}(v_n)_{\mathcal{U}} \\ &= (u_n)_{\mathcal{U}}((v_n^*v_n)^{\frac{1}{2}})_{\mathcal{U}} \\ &= (u_n)_{\mathcal{U}}((v_n^*v_n)_{\mathcal{U}})^{\frac{1}{2}} \\ &= (u_n)_{\mathcal{U}}((v_n)_{\mathcal{U}}^*(v_n)_{\mathcal{U}}))^{\frac{1}{2}} \\ &= (u_n)_{\mathcal{U}}(u^*u)^{\frac{1}{2}} \\ &= (u_n)_{\mathcal{U}}\end{aligned}$$

□

To prove the next lemma, recall that there is an isomorphism $R\bar{\otimes}R \to R$, whose existence is made essentially trivial by the description of R as the infinite tensor product of 2 by 2 matrices.

Lemma 1.7. *One has*

$$U\left(R^{\mathcal{U}} \odot R^{\mathcal{U}}\right) \subseteq U\left(R^{\mathcal{U}}\right),$$

where $\odot$ stands for the algebraic tensor product.

Proof. Let $\theta : R\bar{\otimes}R \to R$ be an isomorphism. This isomorphism clearly induces a component-wise isomorphism $(R\bar{\otimes}R)^{\mathcal{U}} \to R^{\mathcal{U}}$. Now we embed $R^{\mathcal{U}} \odot R^{\mathcal{U}}$ into $(R\bar{\otimes}R)^{\mathcal{U}}$ by the mapping

$$(x_n)_n \otimes (y_n)_n \to (x_n \otimes y_n)_n$$

to get an embedding from the algebraic tensor product $R^{\mathcal{U}} \odot R^{\mathcal{U}}$ to $R^{\mathcal{U}}$, which clearly maps unitaries to unitaries. □

Remark 1.8. Observe that the proof of Lemma 1.7 does not give an embedding $R^{\mathcal{U}}\bar{\otimes}R^{\mathcal{U}} \to R^{\mathcal{U}}$. The existence of such an embedding seems to be an open problem.

Exercise 1.9. *Let $z \in \mathbb{C}$ such that $|z| \leq 1$ and $|z+1| = 2$. Prove that $z = 1$.*

Proof of Theorem 1.2. (2) implies (1) is trivial. Conversely, let G be a group and $\theta : G \to U(R^{\mathcal{U}})$ a group monomorphism. Define a new group monomorphism $\theta' : G \to U(M_2(R^{\mathcal{U}}))$ to be

$$\theta'(g) = \begin{pmatrix} \theta(g) & 0 \\ 0 & 1 \end{pmatrix}$$

By Exercise 1.9, this monomorphism verifies the property that $|\tau(\theta'(g))| < 1$, for all $g \neq 1$. Now, recall that R is, up to isomorphisms, the unique II_1 factor containing an increasing sequence of matrix algebras whose union is weakly dense. This implies that $M_2(R)$ is isomorphic to R and therefore, there is an isomorphism $\phi : M_2\left(R^{\mathcal{U}}\right) \to R^{\mathcal{U}}$. Define $\theta_1 = \phi \circ \theta'$. Therefore, θ_0 is a group monomorphism from G to $U\left(R^{\mathcal{U}}\right)$ having the additional property that $|\tau(\theta_1(g))| < 1$, for all $g \neq 1$. Now define θ_n to be the n-fold tensor product of θ_1. Namely, $\theta_n : G \to U\left(R^{\mathcal{U}} \odot \ldots \odot R^{\mathcal{U}}\right)$ is defined by $\theta_n(g) = \theta_1(g) \otimes \ldots \otimes \theta_1(g)$. By Lemma 1.7, θ_n is still a group monomorphism from G to $U\left(R^{\mathcal{U}}\right)$. Observe that $|\tau(\theta_n(g))| =$

$|\tau(\theta_1(g))|^n$. Next define $\theta : G \to U\left(\left(R^{\mathcal{U}}\right)^{\mathcal{U}}\right)$ by $\theta(g) = (\theta_n(g))_n$ and using Lemma 1.6. Applying Proposition 1.5 we have gotten a group monomorphism $\theta : G \to U\left(R^{\mathcal{U}\otimes\mathcal{U}}\right)$ verifying the property that $\tau(\theta(g)) = 0$, for all $g \neq 1$. Now, $\mathcal{U}\otimes\mathcal{U}$ can be seen, using a bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$, as a new ultrafilter $\mathcal{V}$ on $\mathbb{N}$. Therefore, we have gotten a group monomorphism $\theta : G \to U\left(R^{\mathcal{V}}\right)$ such that $\tau(\theta(g)) = 0$, for all $g \neq 1$. It is clear that this monomorphism can be extended by linearity and continuity to the group von Neumann algebra of G obtaining a von Neumann algebra embedding $L(G) \to R^{\mathcal{V}}$, as desired. □

2. Classes of Hyperlinear and Sofic groups

In this section we describe a large class of sofic and hyperlinear groups, the class of initially subamenable groups, introduced by Gromov in [8]. This class contains amenable and residually finite groups. Examples of sofic (resp. hyperlinear) groups not belonging to this class have been first constructed by de Cornulier in [4] (resp. Andreas Thom in [12]).

We recall the following criterion.

Theorem 2.1. (Elek-Szabo [5]**)** *A group G is sofic if and only if for all finite subsets $F \subseteq G$ and for all $\varepsilon > 0$, there exist a natural n and a map $\theta = \theta_{F,n} : F \to S_{k(F,n)}$ such that:*

(1) if $g, h, gh \in F$, then $d_{hamm}(\theta(g)\theta(h), \theta(gh)) < \varepsilon$,
(2) if $e \in F$, then $d_{hamm}(\theta(e), Id) < \varepsilon$,
(3) there exists $\alpha > 0$, independent of F and ε, such that for all distinct $x, y \in F$, $d_{hamm}(\theta(x), \theta(y)) \geq \alpha$.

An analogous result holds for hyperlinear groups and was first proved by Rădulescu in [11].

As a first application of Theorem 2.1 we show that amenable groups are sofic. Amenable groups have been introduced by von Neumann in [9] in response to the Tarski paradox and they form a hugely studied class of groups still nowadays. Indeed, there are lots of equivalent conditions to define amenability making this notion one of the most fundamental and tranversal one with large applications to even apparently far fields of research, as game theory [1], [3]. We define amenability through the so-called Følner condition, discovered in [6].

Definition 2.2. *A countable group G is amenable if for all finite $F \subseteq G$ and for all $\varepsilon > 0$ there is a finite subset Φ of G such that for each $g \in F$ one has*

$$|g\Phi\triangle\Phi| < \varepsilon|\Phi|,$$

where $\triangle$ stands for the symmetric difference of two sets.

Finite groups are amenable. Abelian groups are amenable. The easiest example of a non-amenable group is the free group on two generators.

Proposition 2.3. *Amenable groups are sofic.*

Proof. Fix a finite subset F of G and $\varepsilon > 0$. Let Φ be a Følner set for F and ε. Fix $g \in F$ and consider the map $\alpha_g : \Phi \to G$ defined by $\alpha_g(x) = gx$. Følner's condition guarantees that α_g maps a subset Φ_1 of Φ of normalized counting measure $> 1 - \varepsilon$ to itself. Extend $\alpha_g|_{\Phi_1}$ over the rest of Φ so as to get a bijection in a casual manner and denote β_g the map obtained this way. So we have gotten a map $\theta : F \to S_{|\Phi|}$, defined by $\theta(g) = \beta_g$, that clearly verifies the first two conditions in Criterion 2.1. To see the third condition, observe that

$$\begin{aligned} d_{hamm}(\theta(g), \theta(h)) & \\ &= \frac{1}{|\Phi|}\, |\{x \in \Phi : \beta_g(x) \neq \beta_h(x)\}| \\ &\geq \frac{1}{|\Phi|}\, |\{x \in \Phi_1 : \beta_g(x) \neq \beta_h(x)\}| \\ &= \frac{|\Phi_1|}{|\Phi|} \\ &\geq 1 - \varepsilon. \end{aligned}$$

□

Definition 2.4. (Gromov [8]**)** *A group G is called initially subamenable if for all finite subset F of G one can find an amenable group containing a copy of F with the same partial multiplication.*

Proposition 2.5. *Initially subamenable groups are sofic.*

Proof. This is a straightforward application of Propositions 2.1 and 2.3. □

The class of initially subamenable groups contains certainly all amenable groups, but we now show that it is much larger and contains for instance all free groups. To this end, we recall the following terminology: a normal subgroup H of a given group G is said to be an amenable quotient (resp. finite quotient) if the quotient group G/H is amenable (resp. finite). A group is called residually amenable (resp. finite) if the intersection of all amenable (resp. finite) quotients is trivial. Since finite groups are amenable, it follows that residually finite groups are also residually amenable. In particular, free groups are residually amenable.

Proposition 2.6. *Residually amenable groups are initially subamenable.*

Proof. Let G be residually amenable and let $F \subseteq G$ be a finite subset. Denote $\overline{F} = \{xy : x, y \in F \cup F^{-1}\}$. Since $\overline{F}$ is finite and since the intersection of amenable quotients is an amenable quotient, then we can find an amenable quotient H such that $H \cap \overline{F}$ is either empty (if $\overline{F}$ does not contain the identity) or it contains only the identity. In any case, the standard quotient map $G \to G/H$ provides a group homomorphism from G to an amenable group that maps F to a copy of itself with the same partial multiplication inside the amenable group G/H. □

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