1. Kirchberg's theorem

The main purpose of this section is to prove the following theorem.

Theorem 1.1 (**Kirchberg** [11] **).** *The following statements are equivalent:*

(1) Connes' embedding conjecture holds true,

(2)

$$C^*(\mathbb{F}_\infty) \otimes_{max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{min} C^*(\mathbb{F}_\infty).$$

We are not going to present the original proof, but a more recent and completely different one provided by Haagerup and Winslow [7], [8]. This is fundamentally a topological proof that relies on the use of the Effros-Marechal topology on the space of von Neumann algebras.

This section is organized as follows. In Subsection 1.1 to 1.3 we recall standard definitions and constructions, as the minimal and maximal tensor product of C^*-algebras, the full C^*-algebra associated to a locally compact group, and the Effros-Marechal topology. In Subsection 1.4 we prove Kirchberg's theorem.

1.1. **Tensor product of C^*-algebras.** A normed *algebra is an algebra A (over $\mathbb{C}$) equipped with:

(1) an involution $*$ such that
- $(x+y)^* = x^* + y^*$,
- $(xy)^* = x^* y^*$,
- $(\lambda x)^* = \overline{\lambda} x^*$.

(2) a norm $||\cdot||$ such that $||xy|| \leq ||x|| ||y||$.

A Banach *algebra is a normed *algebra that is complete.

Definition 1.2. *A C^*-algebra is a Banach *algebra verifying the following additional property, called C^*-*identity*:*

$$||x^*x|| = ||x||^2, \qquad \forall a \in A$$

Example of C^*-algebras include the (commutative) algebra of complex valued functions on a compact space equipped with the sup norm and $B(H)$ itself.

Remark 1.3. The C^*-identity is a relation between algebraic and topological properties. It indeed implies that sometimes algebraic properties imply topological properties. A classical example of this interplay will be often used in this section:

Fact. A *homomorphism from a normed *algebra to a C^*-algebra is always a contraction.

The algebraic tensor product of two C^*-algebras is a *algebra in a natural way, by setting

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = x_1 x_2 \otimes y_1 y_2,$$

$$(x_1 \otimes x_2)^* = x_1^* \otimes x_2^*.$$

Nevertheless it is not clear how to define a norm to obtain a C^*-algebra.

Definition 1.4. *Let A_1, A_2 be two C^*-algebras and $A_1 \odot A_2$ their algebraic tensor product. A norm $||\cdot||_\beta$ on $A_1 \otimes A_2$ is called C^*-norm if the following properties are satisfied:*

(1) $||xy||_\beta \leq ||x||_\beta ||y||_\beta$, *for all* $x, y \in A_1 \odot A_2$;
(2) $||x^*x||_\beta = ||x||_\beta^2$, *for all* $x \in A_1 \odot A_2$.

If $||\cdot||_\beta$ is a C^*-norm on $A_1 \odot A_2$, then $A_1 \otimes_\beta A_2$ denotes the completion of $A_1 \odot A_2$ with respect to $||\cdot||_\beta$. It is a C^*-algebra.

Exercise 1.5. Prove that every C^*-norm β is multiplicative on elementary tensors; that is, $||x_1 \otimes x_2||_\beta = ||x_1||_{A_1} ||x_2||_{A_2}$.

One can construct at least two C^*-norms, a minimal one and a maximal one, and they are different in general. To define these norms, let us first recall that a representation of a *algebra A is a *preserving algebra-morphism from A to some $B(H)$. We denote $\mathrm{Rep}(A)$ the set of representation of A.

Definition 1.6.

$$||x||_{\max} = \sup \{||\pi(x)|| : \pi \in \mathit{Rep}(A_1 \odot A_2)\} \tag{1}$$

Exercise 1.7. Prove that $||\cdot||_{\max}$ is indeed a C^*-norm. (Hint: to prove that $||x||_{\max} < \infty$, for all x, take inspiration from Lemma 11.3.3(iii) in [10]).

The norm $||\cdot||_{\max}$ has several names: maximal norm, projective norm, but also Turumaru's norm, being first introduced in [16]. The completion of $A_1 \odot A_2$ with respect to it is denoted by $A_1 \otimes_{\max} A_2$.

Let A be a C^*-algebra and $S \subseteq A$. Denote by $C^*(S)$ the C^*-subalgebra of A generated by S. The projective norm has the following universal property (see [14], IV.4.7).

Proposition 1.8. *Given C^*-algebras A_1, A_2, B. Assume $\pi_i : A_i \to B$ are *homomorphisms with commuting ranges, that is for all $x \in \pi_1(A_1)$ and $y \in \pi_2(A_2)$, one has $xy = yx$. Then there exists a unique *homomorphism $\pi : A_1 \otimes_{\max} A_2 \to B$ such that*

(1) $\pi(x_1 \otimes x_2) = \pi_1(x_1)\pi_2(x_2)$
(2) $\pi(A_1 \otimes_{\max} A_2) = C^*(\pi_1(A_1), \pi_2(A_2))$

Exercise 1.9. Prove Proposition 1.8.

We now turn to the definition of the minimal C^*-norm.

Definition 1.10.

$$||x||_{\min} = \sup \{||(\pi_1 \otimes \pi_2)(x)|| : \pi_i \in \mathit{Rep}(A_i)\} \tag{2}$$

Exercise 1.11. *Prove that $||\cdot||_{\min}$ is a C^*-norm on $A_1 \odot A_2$.*

Also this norm has several names: minimal norm, injective norm, but also Guichardet's norm, being first introduced in [6]. The completion of $A_1 \odot A_2$ with respect to it is denoted by $A_1 \otimes_{\min} A_2$.

Remark 1.12. Clearly $||\cdot||_{\min} \leq ||\cdot||_{\max}$, since representations of the form $\pi_1 \otimes \pi_2$ are particular *representation of the algebraic tensor product $A_1 \odot A_2$. These norms are different, in general, as Takesaki showed in [15]. Notation $||\cdot||_{\max}$ reflects the obvious fact that there are no C^*-norms greater than the projective norm. Notation $||\cdot||_{\min}$ has the same justification, but it is much harder to prove:

Theorem 1.13. (Takesaki, [15]**)** $||\cdot||_{\min}$ *is the smallest C^*-norm on $A_1 \odot A_2$.*

1.2. **The full C^*-algebra of a group.** Let G be a locally compact group. Fix a left-Haar measure μ and construct the convolution *algebra $L^1(G)$ as follows: the elements of $L^1(G)$ are μ-integrable complex-valued functions on G; the convolution is defined by $(f*g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y)$ and the involution is defined by $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$, where Δ is the modular function; i.e. the (unique) function $\Delta : G \to [0,\infty)$ such that for all Borel subsets A of G one has $\mu(Ax^{-1}) = \Delta(x)\mu(A)$. The universal C^*-algebra of G, denoted by $C^*(G)$ is the envelopping C^*-algebra of $L^1(G)$, i.e. the completion of $L^1(G)$ with respect to the norm $||f|| = sup_\pi ||\pi(f)||$, where π runs over all non-degenerate *representations of $L^1(G)$ in a Hilbert space[1].

Exercise 1.14. Show that in fact $||\cdot||$ is a norm on $L^1(G)$.

In particular, we can make this construction for the free group on countably many generators, usually denoted by $\mathbb{F}_\infty$. Observe that this group is countable and so its Haar measure is the counting measure which is bi-invariant. Consequently, the modular function is constantly equal to 1.

Exercise 1.15. Let $\delta_g : G \to \mathbb{R}$ be the characteristic function of the point g. Show that $g \to \delta_g$ is an embedding $\mathbb{F}_\infty \hookrightarrow U(C^*(\mathbb{F}_\infty))$.

Exercise 1.16. Prove that the unitaries in Exercise 1.15 form a norm totat sequence in $C^*(\mathbb{F}_\infty)$.

Let $g_1, g_2 \ldots$ be a free set of generators of $\mathbb{F}_\infty$. The corresponding unitaries δ_{g_n} are called universal.

1.3. **Effros-Marechal topology on the space of von Neumann algebras.** In the previous two sections we have introduced all the necessary notations and definitions in order to understand the statement of Kirchberg's theorem. We now move towards Haagerup-Winslow's proof of it. As we mentioned, this is fundamentally a topological proof, whose main tool is the use of the Effros-Marechal topology on the space of von Neumann algebras.

Let H be a Hilbert space and $vN(H)$ be the set of von Neumann algebras acting on H, that is the set of von Neumann subalgebras of $B(H)$. The Effros-Marechal topology on $vN(H)$, first introduced in Effros in [4] and Marechal in [12], can be defined is several different equivalent ways. For pur purposes, we need just two of them.

[1] A representation $\pi : L^1(G) \to B(H)$ is said to be non-degenerate if the set $\{\pi(f)\xi : f \in L^1(G), \xi \in H\}$ is dense in H.

Terminology. Given a net $\{C_a\}_{a\in A}$ on a direct set A, we say that a property P is *eventually* true if there is $a \in A$ such that for all $b \geq a$, C_b has property P; and we say that P is *frequently* true, if for all $a \in A$, there is $b \geq a$, such that C_b has the property P.

First definition of the Effros-Marechal topology. Let X be a compact Hausdorff space, $c(X)$ be the set of closed subsets of X and $\mathcal{N}(x)$ be the filter of neighborhoods of a point $x \in X$. $\mathcal{N}(x)$ is a direct set, ordered by inclusion. Let $\{C_a\}$ be a net of subsets of $c(X)$, define

$$\underline{\lim} C_a = \{x \in X : \forall N \in \mathcal{N}(x), N \cap C_a \neq \emptyset \text{ eventually}\}, \tag{3}$$

$$\overline{\lim} C_a = \{x \in X : \forall N \in \mathcal{N}(x), N \cap C_a \neq \emptyset \text{ frequently}\}. \tag{4}$$

It is clear that $\underline{\lim} C_\alpha \subseteq \overline{\lim} C_\alpha$, but the other inclusion is far from being true in general.

Exercise 1.17. Find an explicit example of a sequence C_n of closed subsets of the real interval $[0,1]$ such that $\underline{\lim} C_n \subsetneq \overline{\lim} C_n$.

Effros proved in [5] that there is only one topology on $c(X)$, whose convergence is described by the conditions

$$C_a \to C \qquad \text{if and only if} \qquad \overline{\lim} C_a = \underline{\lim} C_a = C. \tag{5}$$

Exercise 1.18. Let M be a von Neumann subalgebra of $B(H)$. Denote by $Ball(M)$ the unit ball of M. Prove that $Ball(M)$ is weakly compact in $Ball(B(H))$.

By Exercise 1.18, we can use Effros' convergence in our setting.

Definition 1.19. *Let $\{M_a\} \subseteq vN(H)$ be a net. The Effros-Marechal topology is described by the following notion of convergence:*

$$M_a \to M \qquad \textit{if and only if} \qquad \overline{\lim} Ball(M_a) = \underline{\lim} Ball(M_a) = Ball(M). \tag{6}$$

Second definition of the Effros-Marechal topology. Let $x \in B(H)$ and let $so^*(x)$ denote the filter of neighborhoods of x with respect to the strong* topology[2].

Definition 1.20. *Let $\{M_a\} \subseteq vN(H)$ be a net. We set*

$$\liminf M_a = \{x \in B(H) : \forall U \in so^*(x), U \cap M_a \neq \emptyset \textit{ eventually}\}. \tag{7}$$

Observe that $\liminf M_a$ is obviously so*-closed, contains the identity and it is closed under involution, scalar product and sum, since these operations are so*-continuous. Haagerup and Winslow proved in [7], Lemma 2.2 and Theorem 2.3, that $\liminf M_a$ is also closed under multiplication. Therefore $\liminf M_a$ is a von Neumann algebra. Moreover, by Theorem 2.6 in [7], $\liminf M_a$ can be seen as the largest element in $vN(H)$ whose unit ball is contained in $\underline{\lim} Ball(M_a)$. This suggests to define $\limsup M_a$ as the smallest element in $vN(H)$ whose unit ball contains $\overline{\lim} Ball(M_a)$, that is clearly $(\overline{\lim} Ball(M_a))''$. Indeed, the double

[2] The strong* operator topology on $B(H)$ is the weakest locally convex topology such that, for all $\xi \in H$, the mappings $B(H) \ni x \to ||x\xi||$ and $B(H) \ni x \to ||x^*\xi||$ are continuous.

commutant of a *subset[3] of $B(H)$ is always a *algebra and the double commutant theorem of von Neumann states that this is the smallest von Neumann algebra containing the set. So we are led to the following

Definition 1.21. *Let $\{M_a\} \subseteq vN(H)$ be a net. We set*

$$\limsup M_a := \left(\overline{\lim} Ball(M_a)\right)''. \tag{8}$$

Definition 1.22. *The Effros-Marechal topology on $vN(H)$ is described by the following notion of convergence:*

$$M_a \to M \qquad \textit{if and only if} \qquad \liminf M_a = \limsup M_a = M.$$

These two definitions of the Effros-Marechal topology are shown to be equivalent in [7], Theorem 2.8.

Connes' embedding conjecture regards separable II_1-factors, that is, factors with a faithful representation in $B(H)$, with H is separable. Assuming separability on H one gets that the Effros-Marechal topology on $vN(H)$ is metrizable, separable and complete (i.e. $vN(H)$ is a Polish space). Moreover, one possible distance is given by the Hausdorff distance between the unit balls:

$$d(M,N) = \max\left\{\sup_{x\in Ball(M)}\left\{\inf_{y\in Ball(N)} d(x,y)\right\}, \sup_{x\in Ball(N)}\left\{\inf_{y\in Ball(M)} d(x,y)\right\}\right\}. \tag{9}$$

where d is a metric on the unit ball of $B(H)$ which induces the weak topology [12]. Moreover, if $\{M_a\} = \{M_n\}$ is a sequence, the definition of the Effros-Marechal topology may be simplified by making use of the second definition. In particular we have

$$\liminf M_n = \left\{x \in B(H) : \exists\{x_n\} \in \prod M_n \text{ such that } x_n \to^{s^*} x\right\}. \tag{10}$$

We now state the main theorem of [8]. A part from being of intrinsic interest, it allows to reformulate Kirchberg's theorem in the form that we are going to prove eventually.

Let us fix some notation: $\mathfrak{F}_{I_{\text{fin}}}$ is the set of finite factors of type I acting on H, that is the set of von Neumann factors that are isomorphic to a matrix algebra; $\mathfrak{F}_I$ is the set of type I factors acting on H, that is the set of von Neumann factors having a non-zero minimal projections; $\mathfrak{F}_{\text{AFD}}$ is the set of approximately finite dimensional factors acting on H, that is the set of factors containing an increasing sequence of matrix algebras whose union is weakly dense; finally, $\mathfrak{F}_{\text{inj}}$ is the set of injective factors acting on H, that is the set of factors that are the image of a bounded linear projection of norm 1, $P : B(H) \to M$.

Theorem 1.23. (Haagerup-Winslow) *The following statements are equivalent:*

(1) $\mathfrak{F}_{I_{fin}}$ is dense in $vN(H)$,

[3] A *subset A of $B(H)$ is one such that $a \in A$ if and only if $a^* \in A$. Clearly the unit ball of a von Neumann algebra is a *subset.

(2) $\mathfrak{F}_I$ *is dense in* $vN(H)$*,*
(3) $\mathfrak{F}_{AFD}$ *is dense in* $vN(H)$*,*
(4) $\mathfrak{F}_{inj}$ *is dense in* $vN(H)$*,*
(5) Connes' embedding conjecture is true.

Moreover, a separable II_1*-factor* M *is embeddable into* R^ω *if and only if* $M \in \overline{\mathfrak{F}_{inj}}$*.*

Since $\mathfrak{F}_{I_{\text{fin}}} \subseteq \mathfrak{F}_I \subseteq \mathfrak{F}_{\text{AFD}}$, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. The implication (3) $\Rightarrow$ (1) follows from the fact that AFD factors contain, by mere definition, an increasing chain of type I_{fin} factors, whose union is weakly dense and from the first definition of the Effros-Marechal topology (Definition 1.19). The equivalence between (3) and (4) is a theorem by Alain Connes proved in [2]. What is really new and important in Theorem 1.23 is the equivalence between (4) and (5), proved in [8], Corollary 5.9, and the proof of the *last sentence*, proved in [8], Theorem 5.8.

1.4. **Proof of Kirchberg's theorem.** By using Theorem 1.23, it is enough to show the following statements:

(1) If $\mathfrak{F}_{I_{\text{fin}}}$ is dense in $vN(H)$, then $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.
(2) If $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$, then $\mathfrak{F}_{\text{inj}}$ is dense in $vN(H)$.

Proof of (1). Let π be a *representation of the algebraic tensor product $C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty)$ into $B(H)$. Since $C^*(\mathbb{F}_\infty)$ is separable, we may assume that H is separable. In this way

$$A = \pi(C^*(\mathbb{F}_\infty) \odot \mathbb{C}1) \qquad \text{and} \qquad B = \pi(\mathbb{C}1 \odot C^*(\mathbb{F}_\infty)).$$

belong to $B(H)$, with H separable. Let $\{u_n\}$ be the universal unitaries in $C^*(\mathbb{F}_\infty)$, as in Exercise 1.15. Let

$$v_n = \pi(u_n \otimes 1) \in A \qquad \text{and} \qquad w_n = \pi(1 \otimes u_n) \in B.$$

Now, let $M = A'' \in vN(H)$. By hypothesis, there exists a sequence $\{F_m\} \subseteq \mathfrak{F}_{I_{\text{fin}}}$ such that $F_m \to M$ in the Effros-Marechal topology. Therefore, $A \subseteq M = \liminf F_m = \limsup F_m$. Thus, we have

$$A \subseteq \liminf F_m.$$

It follows that

$$\{v_n\} \subseteq U(A) \subseteq U(\liminf F_m) = \underline{\lim} Ball(F_m) \cap U(B(H)),$$

where the equality follows from [7] th.2.6. Let $w(x)$ and $so^*(x)$ respectively the filters of weakly and strong* open neighborhoods of an element $x \in B(H)$. We have just proved that for every $n \in \mathbb{N}$ and $W \in w(v_n)$, one has $W \cap Ball(F_m) \cap U(B(H)) \neq \emptyset$ eventually in m. Let $S \in so^*(v_n)$, by [7] Lemma 2.4, there exists $W \in w(v_n)$ such that $W \cap Ball(F_m) \cap U(B(H)) \subseteq S \cap Ball(F_m) \cap U(B(H))$. Now, since the first set must be eventually non empty, also the second one must be the same. This means that we can approximate in the strong* topology v_n with elements $v_{m,n} \in U(F_m)$.

In a similar way we can find unitaries $w_{m,n}$ in F'_m such that $w_{m,n} \to^{so^*} w_n$. Indeed, by Exercise **??**, we have

$$B \subseteq A' = M' = (\limsup F_m)' = \liminf F'_m,$$

where the last equality follows by the *commutant theorem* ([7], Theorem 3.5). Therefore, we may repeat word by word the previous argument.

Now let m be fixed, $\pi_{m,1}$ a representation of $C^*(\mathbb{F}_\infty)$ mapping u_n to $v_{m,n}$ and $\pi_{m,2}$ a representation of $C^*(\mathbb{F}_\infty)$ mapping u_n to $w_{m,n}$. We can find these representations because the $u'_n s$ are free and because any representation of G extends to a representation of $C^*(G)$. Notice that the ranges of these representations commute, since $v_n \in A$ and $w_n \in B$ and A, B commute. More precisely, the image of $\pi_{m,1}$ belongs into $C^*(F_m)$ and the image of $\pi_{m,2}$ belongs to $C^*(F'_m)$. So, by the universal property in Proposition 1.8, there are unique representations π_m of $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$ such that

$$\pi_m(u_n \otimes 1) = v_{m,n} \qquad \text{and} \qquad \pi_m(1 \otimes u_n) = w_{m,n}, \qquad m, n \in \mathbb{N},$$

whose image lies into $C^*(F_m, F'_m)$.

Exercise 1.24. Prove that $C^*(F_m, F'_m) = F_m \otimes_{\min} F'_m$ (Hint: use Theorem 4.1.8(iii) in [9] and Lemma 11.3.11 in [10]).

By Exercise 1.24, $C^*(F_m, F'_m) = F_m \otimes_{\min} F'_m$ and thus π_m splits: $\pi_m = \sigma_m \otimes \rho_m$, for some σ_m, ρ_m representation of $C^*(\mathbb{F}_\infty)$ in $C^*(F_m, F'_m)$. Consequently, by the very definition of minimal C^*-norm, $||\pi_m(x)|| \leq ||x||_{\min}$ for all $m \in \mathbb{N}$ and $x \in C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty)$. Now, by Exercise 1.16, the sequence $\{u_n\}$ is total and therefore π_m converges to π in the strong* point-wise sense; namely, for all $x \in C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty)$, one has $\pi_m(x) \to^{so^*} \pi(x)$.

Exercise 1.25. *Prove that if $x_n \in B(H)$ converges to $x \in B(H)$ in the strong* topology, then $||x|| \leq \liminf ||x_n||$.*

Therefore

$$||\pi(x)|| \leq \liminf ||\pi_m(x)|| \leq ||x||_{\min}, \qquad \forall x \in C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty).$$

Since π is arbitrary, it follows that $||x||_{\max} \leq ||x||_{\min}$ and the proof of the first implication is complete. □

In order to prove (2) we need a few more definitions and preliminary results. Given two *representation π and ρ of the same C^*-algebra A in $B(H)$, we say that they are unitarily equivalent, and we write $\pi \sim \rho$, if there is $u \in U(B(H))$ such that, for all $x \in A$, one has $u\pi(x)u^* = \rho(x)$. Given a family of representations π_a of the same C^*-algebra in $B(H_a)$, we may define the direct sum $\bigoplus_a \pi_a$ to be a representation of A in $B(\bigoplus H_a)$ in the obvious way; i.e.

$$\left(\bigoplus \pi_a\right)(x)\xi = \bigoplus \left(\pi_a(x)\xi_a\right), \qquad \text{for all } \xi = (\xi_a) \in \bigoplus H_a.$$

A family of representation is called *separating* if their direct sum is faithful (i.e., injective). We state the following unexpected and deep lemma, making use of Voiculescu's Weyl-von Neumann theorem.

Lemma 1.26. (Haagerup-Winslow, [8] **Lemma 4.3)** *Let A be a unital C^*-algebra and λ, ρ representations of A in $B(H)$. Assume ρ is faithful and satisfies $\rho \sim \rho \oplus \rho \oplus ....$ Then there exists a sequence $\{u_n\} \subseteq U(B(H))$ such that*

$$u_n \rho(x) u_n^* \to^{s^*} \lambda(x), \qquad \forall x \in A.$$

Theorem 1.27. (Choi, [1] **th.7)** *Let $\mathbb{F}_2$ be the free group with two generators. Then $C^*(\mathbb{F}_2)$ has a separating family of finite dimensional representations.*

Exercise 1.28. Show that $\mathbb{F}_\infty$ embeds into $\mathbb{F}_2$.

Proof of (2). By using Choi's theorem and Exercise 1.28 we can find a sequence σ_n of finite dimensional representations of $C^*(\mathbb{F}_\infty)$ such that $\sigma = \sigma_1 \oplus \sigma_2 \oplus ...$ is faithful. Replacing σ with the direct sum of countably many copies of itself, we may assume that $\sigma \sim \sigma \oplus \sigma \oplus ....$ Now, $\rho = \sigma \otimes \sigma$ is a faithful (by [14] IV.4.9) representation of $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$ (because ρ is factorizable). This representation still satisfies $\rho \sim \rho \oplus \rho \oplus ....$ Now, given $M \in vN(H)$, let $\{v_n\}, \{w_n\}$ be strong* dense sequences of unitaries in $Ball(M)$ and $Ball(M')$, respectively. Let $\{z_n\}$ be the universal unitaries representing free generators of $\mathbb{F}_\infty$ in $C^*(\mathbb{F}_\infty)$, as in Exercise 1.15. Since the z_n's are free, we may find *representations λ_1 and λ_2 of $C^*(\mathbb{F}_\infty)$ in $B(H)$ such that

$$\lambda_1(z_n) = v_n \qquad \text{and} \qquad \lambda_2(z_n) = w_n.$$

Since the ranges of these representations commute, we can apply the universal property in Proposition 1.8 to find a representation λ of $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$ such that

$$\lambda(z_n \otimes 1) = v_n \qquad \text{and} \qquad \lambda(1 \otimes z_n) = w_n, \qquad \forall n \in \mathbb{N},$$

where we can use the minimal norm instead of the maximal one thank to the hypothesis of the theorem. This means that λ and ρ satisfy the hypotheses of Lemma 1.26 and therefore there are unitaries $u_n \in U(B(H))$ such that

$$u_n \rho(x) u_n^* \to^{so^*} \lambda(x), \qquad \forall x \in C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty).$$

Define

$$M_n = u_n \rho(C^*(\mathbb{F}_\infty) \otimes \mathbb{C}1)'' u_n^*.$$

Therefore, using also Equation (10), we have

$$\lambda(C^*(\mathbb{F}_\infty) \otimes \mathbb{C}1) = \liminf u_n \rho(C^*(\mathbb{F}_\infty) \otimes \mathbb{C}1) u_n^* \subseteq \liminf M_n. \tag{11}$$

Now, observe that

$$u_n \rho(\mathbb{C}1 \otimes C^*(\mathbb{F}_\infty)) u_n^* \subseteq M_n'$$

and therefore

$$\lambda(\mathbb{C}1 \otimes C^*(\mathbb{F}_\infty)) \subseteq \liminf M_n'. \tag{12}$$

Since $\liminf M_a$ is always a von Neumann algebra, the inclusions in (11) and (12) still hold passing to the strong closure. Therefore, by (11), we obtain

$$M = \lambda(C^*(\mathbb{F}_\infty) \otimes \mathbb{C}1)'' \subseteq \liminf M_n \tag{13}$$

where the equality with M follows from the fact that the v_n's are strong* dense in $Ball(M)$. Analogously, by (12), we obtain

$$M' = \lambda(\mathbb{C}1 \otimes C^*(\mathbb{F}_\infty))'' \subseteq \liminf M'_n, \tag{14}$$

where the equality with M' follows from the fact that the w_n's are strong* dense in $Ball(M')$.

Now, using (13) and (14) and applying the commutant theorem ([7], Theorem 3.5), we get

$$M = M'' \supseteq \left(\liminf M'_n\right)' = \limsup (M_n)'' = \limsup M_n.$$

Therefore

$$\limsup M_n \subseteq M \subseteq \liminf M_n,$$

i.e., $M_n \to M$ in the Effros-Marechal topology. Therefore, we have proved that every von Neumann algebra M can be approximated by von Neumann algebras M_n that are constructed as strong closure of faithful representations that are direct sum of countably many finite dimensional representations. Such von Neumann algebras are injective and therefore, we have proved that $vN_{\text{inj}}(H)$ is dense in $vN(H)$. Now, by [7], Theorem 5.2, vN_{inj} is a G_δ subset of $vN(H)$ and by [7], Theorem 3.11, and [8], Theorem 2.5, the set of all factors $\mathfrak{F}(H)$ is a G_δ-subset of vN(H) and it is dense. On the other hand, $vN(H)$ is a Polish space and therefore we can apply Baire's theorem and conclude that also the intersection $vN_{\text{inj}}(H) \cap \mathfrak{F}(H) = \mathfrak{F}_{\text{inj}}(H)$ must be dense. □

Let K denote the class of groups G satisfying *Kirchberg's property*

$$C^*(G) \otimes_{\min} C^*(G) = C^*(G) \otimes_{\max} C^*(G)$$

The following seems to be an open and interesting problem [13].

Problem 1.29. Does K contain a countable discrete non-amenable group?

Another interesting question comes from the observation that the previous proof as well as Kirchberg's original proof uses free groups in a very strong way.

Problem 1.30. For which groups G, does the implication $G \in K$ implies Connes' embedding conjecture hold?

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