1 Kaplanski conjectures

1.1 Group algebras and the statements of Kaplanski's conjectures

Suppose that Γ is a group and K is a field. The group algebra $K\Gamma$ is the K-algebra of formal finite linear combinations

$$k_1\gamma_1 + \ldots + k_n\gamma_n$$

of elements of Γ with coefficients in K. A typical element a of $K\Gamma$ can be denoted by

$$\sum_\gamma a_\gamma \gamma$$

where the coefficients $a_\gamma \in K$ are zero for all but finitely many $\gamma \in \Gamma$. The operations on $K\Gamma$ are defined by

$$\left(\sum_\gamma a_\gamma \gamma\right) + \left(\sum_\gamma b_\gamma \gamma\right) = \sum_\gamma \left(a_\gamma + b_\gamma\right)\gamma$$

and

$$\left(\sum_\gamma a_\gamma \gamma\right)\left(\sum_\gamma b_\gamma \gamma\right) = \sum_\gamma \left(\sum_{\rho\rho'=\gamma} a_\rho b_{\rho'}\right)\gamma$$

Several conjectures concerning $K\Gamma$ are attributed to Kaplanski:

- Zero divisors conjecture: $K\Gamma$ has no zero divisors;
- Nilpotent elements conjecture: $K\Gamma$ has no nilpotent elements;
- Idempotent elements conjecture: the only idempotent elements of $K\Gamma$ are 0 and 1;
- Units conjecture: the only units of $K\Gamma$ are $k\gamma$ for $k \in K\setminus\{0\}$ and $\gamma \in \Gamma$;
- Finiteness conjecture: $K\Gamma$ is directly finite, i.e. $ab = 1$ for $a, b \in K\Gamma$ implies $ba = 1$;
- Values of traces on idempotent elements: if τ_0 is the trace on $K\Gamma$ defined by

  $$\tau_0(a) = a_{1_\Gamma}$$

  and b is an idempotent element of $K\Gamma$ then $\tau_0(b)$ belongs to the prime field K_0 of K.

Recall that a **trace** on a K-algebra A is a K-linear functional $\tau : A \to K$ such that $\tau(ab) = \tau(ba)$ for $a, b \in A$.

1.2 Zalesskii's theorem

Among the conjectures mentioned, only the one concerning values of traces on idempotent elemements has been establihed in full generality. This is the content of a theorem of Zalesskii from [3].

Theorem 1 *If $p \in K\Gamma$ is an idempotent element and τ is a trace on $K\Gamma$, then $\tau(p)$ belongs to the prime field K_0 of K.*

Let us consider the particular case when K is a finite field of characteristic p. If $n \geq 0$ and $a \in K\Gamma$, define $\tau_n(a)$ to be the sum of the coefficients of a corresponding to elements of Γ of order p^n. In particular $\tau_0(a)$ is the coefficient of a corresponding to the identity element 1_Γ of Γ.

Exercise 2 *Show that τ_n is a trace on $K\Gamma$ for every $n \geq 0$, i.e. τ_n is a K-linear map such that $\tau(ab) = \tau(ba)$.*

Lemma 3 *Recall that K is supposed to be a finite field of characteristic p. Show that if τ is any trace on $K\Gamma$ then*

$$\tau((a+b)^p) = \tau(a^p) + \tau(b^p)$$

for every $a, b \in K\Gamma$. Thus by induction

$$\tau(a^p) = \sum_\gamma a_\gamma^p \tau(\gamma^p).$$

The latter identities can be referred to as "Frobenius under trace", in analogy with the corresponding identity for elements of a field of characteristic p. Suppose now that $e \in K\Gamma$ is an idempotent element. We want to show that $\tau(e)$ belongs to the prime field K_0 of K. To this purpose it is enough to show that $\tau(e)^p = \tau(e)$. For $n \geq 1$ we have

$$\begin{aligned} \tau_n(e) &= \tau_n(e^p) \\ &= \sum_\gamma e_\gamma^p \tau_n(\gamma^p) \\ &= \sum_{|\gamma| = p^{n+1}} e_\gamma^p \\ &= \left(\sum_{|\gamma| = p^{n+1}} e_\gamma \right)^p \\ &= \tau_{n+1}(e)^p. \end{aligned}$$

On the other hand

$$\begin{aligned}\tau_0(e) &= \tau_0(e^p)\\ &= \sum_{\gamma} e_\gamma^p \tau_0(\gamma^p)\\ &= \sum_{|\gamma|=1} e_\gamma^p + \sum_{|\gamma|=p} e_\gamma^p\\ &= \tau_0(e)^p + \tau_1(e)^p .\end{aligned}$$

From these identities it is easy to prove by induction that

$$\tau_0(e) = \tau_0(e)^p + \tau_n(e)^{p^n}$$

for every $n \in \mathbb{N}$. Since e has finite support, there is $n \in \mathbb{N}$ such that $\tau_n(e) = 0$. This implies that $\tau_0(e) = \tau_0(e)^p$ and hence $\tau_0(e) \in K_0$.

The proof of the general case of Zalesskii's theorem can be inferred from this particular case. The details can be found in [2].

1.3 The complex case of Kaplanski's finiteness conjecture

The particular instance of Kaplanski's finiteness conjecture for the field of complex numbers $\mathbb{C}$ asserts that for any group Γ the complex group algebra $\mathbb{C}\Gamma$ is directly finite. This case can be treated by means of functional analysis and operator algebras. Recall that the complex group algebra $\mathbb{C}\Gamma$ can be embedded into the group von Neumann algebra $L\Gamma$ of Γ. Moreover the trace τ_0 on $\mathbb{C}\Gamma$ defined by $\tau_0(a) = a_{1_\Gamma}$ can be extended to a faithful normalized trace τ_0 on $L\Gamma$. Thus the complex case of Kaplanski's finiteness conjecture is a consequence of the following result.

Theorem 4 *If M is a von Neumann algebra endowed with a faithful finite trace τ, then M is a directly finite algebra.*

Assume that M is a von Neumann algebra an τ is a faithful normalized trace on M. If $x, y \in M$ are such that $xy = 1$ then $yx \in M$ is an idempotent element such that

$$\tau(yx) = \tau(xy) = \tau(1) = 1.$$

It is thus enough to prove that if $e \in M$ is an idempotent element such that $\tau(e) = 1$ then $e = 1$. This is equivalent to the assertion that if $e \in M$ is an idempotent element such that $\tau(e) = 0$ then $e = 0$. This assertion is proved in Lemma 5 (cf. Lemma 2.1 in [2]).

Lemma 5 *If M is a von Neumann algebra endowed with a faithful finite trace τ and $e \in M$ is an idempotent such that $\tau(e) = 0$, then $e = 0$.*

Proof. The conclusion is obvious if e is a self-adjoint idempotent element (i.e. a projection). In fact in this case

$$\tau(e) = \tau(e^* e) = 0$$

implies $e = 0$ by faithfulness of τ. In order to establish the general case it is enough to show that if $e \in M$ is idempotent, then there is a self-adjoint invertible element z of M such that $f = ee^* z^{-1}$ is a projection and $\tau(e) = \tau(f)$. Define

$$z = 1 + (e^* - e)^* (e^* - e).$$

Observe that z is an invertible element (see [1], II.3.1.4) commuting with e. It is not difficult to check that $f = ee^* z^{-1}$ has the required properties. ■

1.4 Kaplanski's finiteness conjecture for finite fields and Gottschalk's conjecture

Suppose that Γ is a group and A is a finite set. Denote by A^Γ the set of Γ-sequences of elements of A. The product topology on A^Γ with respect to the discrete topology on A is compact metrizable. The *Bernoulli shift* of Γ with alphabet A is the left action of Γ on A^Γ defined by

$$\rho \cdot (a_\gamma)_{\gamma \in \Gamma} = \left(a_{\rho^{-1}\gamma}\right)_{\gamma \in \Gamma}.$$

A continuous function $f : A^\Gamma \to A^\Gamma$ is *equivariant* if it preserves the Bernoulli action, i.e. $f(\rho \cdot x) = \rho \cdot f(x)$ for every $x \in A^\Gamma$. **Gottschalk's surjunctivity conjecture** asserts that if $f : A^\Gamma \to A^\Gamma$ is a continuous injective equivariant function, then f is surjective.

Gottschalk's surjunctivity conjecture implies Kaplanski's finiteness conjecture for finite groups. Suppose that Γ is a group and K is a *finite* field. Consider the Bernoulli action of Γ with alphabet K. Denote the element $(a_\gamma)_{\gamma \in \Gamma}$ of K^Γ by $\sum_\gamma a_\gamma$. Observe that the group algebra $K\Gamma$ can be regarded as a subset of K^Γ. Defining

$$\left(\sum_\gamma a_\gamma \gamma\right) \cdot \left(\sum_\gamma b_\gamma \gamma\right) = \sum_\gamma \left(\sum_{\rho\rho' = \gamma} a_\rho b_{\rho'}\right) \gamma$$

for $\sum_\gamma a_\gamma \gamma \in K^\Gamma$ and $\sum_\gamma b_\gamma \gamma \in K\Gamma$ gives a right action of $K\Gamma$ on K^Γ extending the multiplication operation in $K\Gamma$ and commuting with the left action of Γ on $K\Gamma$. Suppose that a, $b \in K\Gamma$ are such that $ab = 1_\Gamma$. Define the continuous equivariant map $f : K^\Gamma \to K^\Gamma$ by $f(x) = x \cdot a$. Observe that for every $x \in K^\Gamma$

$$x = x \cdot ab = (x \cdot a) \cdot b = f(x) \cdot b.$$

It follows that f is injective. Gottschalk's conjecture for Γ implies that f is also surjective. In particular there is $x_0 \in K^\Gamma$ such that $x_0 \cdot a = f(x_0) = 1_\Gamma$. In particular

$$b = 1_\Gamma \cdot b = (x_0 \cdot a) \cdot b = x_0 \cdot (ab) = x_0 \cdot 1_\Gamma = x_0.$$

Therefore

$$1_\Gamma = x_0 \cdot a = ba.$$

1.5 Kervaire-Laudenbach conjecture

Suppose $\gamma_1, \ldots, \gamma_l \in \Gamma$ and define the monomial

$$w(x) = x^{n_1}\gamma_1 \ldots x^{n_l}\gamma_l$$

where $n_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, l$. Consider the following problem: Determine if the equation

$$w(x) = 1$$

has a solution in some group extending Γ. The answer in general is "no". Consider for example the equation

$$xax^{-1}b^{-1} = 1$$

If a and b are different orders then clearly this equation hsa no solution in any group extending Γ. Assuming that the sum $\sum_{i=1}^{l} n_i$ of the exponents of x in $w(x)$ is nonzero is a way to rule out this obstruction. A conjecture attributed to Kervaire and Laudenbach asserts that this is enough to guarantee the existence of a solution of the equation $w(x) = 1$ in some group extending Γ.

References

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