1 First class on sofic and hyperlinear groups

1.1 Definition of sofic groups

A *pseudo length function* ℓ on a group G is a function $\ell : G \to [0,1]$ such that for every $x, y \in G$:

- $\ell(xy) \leq \ell(x) + \ell(y)$;
- $\ell(xy) = \ell(yx)$;
- $\ell\left(x^{-1}\right) = \ell(x)$;
- $\ell(1) = 0$.

A pseudo length function is called *length function* if moreover $\ell(x) = 0$ implies $x = 1_G$. A group endowed with a length function is called a *length group*. If G is a length group with length function ℓ, then the function $d : G \times G \to [0,1]$ defined by

$$d(x, y) = \ell\left(xy^{-1}\right)$$

is a bi-invariant metric on G. This means that d is a metric on G, and left and right translations in G are d-isometries. Conversely any bi-invariant metric d on G gives rise to a length function ℓ on G by

$$\ell(x) = d(x, 1_G).$$

This shows that there is a bijective correspondence between length functions and bi-invariant metrics on a group G.

If ℓ_0 is a pseduo length function on a group G, then

$$N_{\ell_0} = \{x \in G \,|\, \ell_0(x) = 1\}$$

is a normal subgroup of G. The quotient G/N_{ℓ_0} endowed with the length function ℓ defined by

$$\ell\left(xN_{\ell_0}\right) = \ell_0(x)$$

is called the length quotient of G induced by the pseudo length function ℓ_0.

If Γ is any group, then the function ℓ_d on Γ defined by $\ell(x) = 1$ if $x \neq 1_G$ and $\ell(1_G) = 0$ is a length function on Γ, called the trivial length function. A (discrete) group can be regarded as a length group endowed with the trivial length function.

Denote for $n \in \mathbb{N}$ by S_n the group of permutations over the set $\{1, \ldots, n\}$. The **Hamming length function** ℓ on S_n is defined by

$$\ell_{S_n}(\sigma) = \frac{1}{n}\left|\{i \in \{1, \ldots, n\} \,|\, \sigma(i) \neq i\}\right|.$$

It is not hard to see that this is indeed a length function on S_n. The corresponding bi-invariant metric on S_n is denoted by d_{S_n}.

Definition 1 *A countable discrete group* Γ *is* ***sofic*** *if for every* $\varepsilon > 0$ *and every finite subset* F *of* $\Gamma \backslash \{1_\Gamma\}$ *there is a natural number* n *and a function* $\Phi : \Gamma \to S_n$ *such that* $\Phi(1_\Gamma) = 1_{S_n}$ *and for every* $g, h \in F \backslash \{1\}$*:*

- $d_{S_n}(\Phi(gh), \Phi(g)\Phi(h)) < \varepsilon$*;*
- $\ell_{S_n}(\Phi(g)) > 1 - \varepsilon$*.*

This *local approximation property* can be reformulated in terms of embedding into (metric) ultraproduct. The product

$$\prod_{n \in \mathbb{N}} S_n$$

is a group with respect to the coordinatewise multiplication. Fix a free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. Define the pseudo length function $\ell_{\mathcal{U}}$ on $\prod_{n\in\mathbb{N}} S_n$ by

$$\ell_{\mathcal{U}}\left((\sigma_n)_{n\in\mathbb{N}}\right) = \lim_{n \to \mathcal{U}} \ell_{S_n}(\sigma_n).$$

It is not hard to check that $\ell_{\mathcal{U}}$ is indeed a pseudo length function. The length quotient of $\prod_{n\in\mathbb{N}} S_n$ induced by $\ell_{\mathcal{U}}$ is denoted by $\prod_{\mathcal{U}} S_n$ and called ultraproduct of the sequence of length groups $(S_n)_{n\in\mathbb{N}}$. It is not difficult to reformulate the notion of sofic group in term of existence of an embedding into $\prod_{\mathcal{U}} S_n$.

Exercise 1 *Suppose that* Γ *is a countable discrete group regarded as a length group endowed with the trivial length. Show that the following statements are equivalent:*

1. Γ *is sofic;*
2. *there is an length-preserving homomorphism* $\Phi : \Gamma \to \prod_{\mathcal{U}} S_n$ *for every free ultrafilter* $\mathcal{U}$ *over* $\mathbb{N}$*;*
3. *there is an length-preserving homomorphism* $\Phi : \Gamma \to \prod_{\mathcal{U}} S_n$ *for some free ultrafilter* $\mathcal{U}$ *over* $\mathbb{N}$*.*

Hint. For $1 \Rightarrow 2$ observe that the hypothesis implies that there is a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of maps from Γ to S_n such that $\Phi_n(1_\Gamma) = 1_{S_n}$ and for every $g, h \in \Gamma \backslash \{1\}$

$$\lim_{n\to+\infty} d_{S_n}(\Phi_n(gh), \Phi_n(h)\Phi_n(g)) = 0$$

and

$$\lim_{n\to+\infty} \ell_{S_n}(\Phi_n(g)) = 1.$$

Define $\Phi : \Gamma \to \prod_{\mathcal{U}} S_n$ sending g to the element of $\prod_{\mathcal{U}} S_n$ having $(\Phi_n(g))_{n\in\mathbb{N}}$ as representative sequence. For $3 \Rightarrow 1$ observe that if $\Phi : \Gamma \to \prod_{\mathcal{U}} S_n$ is a length preserving homomorphism and for every $g \in G$

$$(\Phi_n(g))_{n\in\mathbb{N}}$$

is a representative sequence of $\Phi(g)$ then the maps $\Psi_n = \Phi_n(1_\Gamma)^{-1}\Phi_n(g)$ satisfy the following properties: $\Psi_n(1_\Gamma) = 1_{S_n}$ and for every $g, h \in \Gamma \setminus \{1\}$

$$\lim_{n\to\mathcal{U}} d_{S_n}(\Psi_n(gh), \Psi_n(g)\Psi_n(h)) = 0$$

and

$$\lim_{n\to\mathcal{U}} \ell_{S_n}(\Psi_n(g)) = 1.$$

If $F \subset \Gamma$ is finite and $\varepsilon > 0$ then the maps Φ_n for n large enough witness the condition of soficity of Γ relative to F and $\varepsilon > 0$. ■

An amplification argument of Elek and Szabo (see [2]) shows that the condition of soficity is equivalent to the an apparently weaker property, which is discussed in Exercise 2.

Exercise 2 *Prove that a countable discrete group Γ is sofic if and only if there is a function $r : \Gamma \to (0,1)$ such that for some $\varepsilon > 0$ and every $F \subset \Gamma \setminus \{1_\Gamma\}$ finite there is a natural number n and a function $\Phi : \Gamma \to S_n$ such that $\Phi(1_\Gamma) = 1_{S_n}$ and for every $g, h \in F$:*

- $d_{S_n}(\Phi(gh), \Phi(g)\Phi(h)) < \varepsilon$;
- $\ell_{S_n}(\Phi(g)) > r(g)$.

Hint. If $n, k \in \mathbb{N}$ and $\sigma \in S_n$ consider the permutation $\sigma^{\otimes k}$ of the set $\{1, \ldots, n\}^k$ of k-sequences of elements of $\{1, \ldots, n\}$ defined by

$$\sigma^{\otimes k}(i_1, \ldots, i_k) = (\sigma(i_1), \ldots, \sigma(i_k)).$$

Identifying the group of permutations of $\{1, \ldots, n\}^k$ with S_{n^k}, the function

$$\sigma \mapsto \sigma^{\otimes k}$$

defines a group homomorphism from S_n to S_{n^k} such that

$$1 - \ell_{S_{n^k}}\left(\sigma^{\otimes k}\right) = (1 - \ell_{S_n}(\sigma))^k.$$

■

Using Exercise 2 one can express the notion of soficity in terms of (not necessarily isometric) embedding into metric ultraproducts of permutations groups.

Exercise 3 *Suppose that Γ is a countable discrete group. Show that the following statements are equivalent:*

- *Γ is sofic;*
- *there is an injective homomorphism $\Phi : \Gamma \to \prod_\mathcal{U} S_n$ for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$;*
- *there is an injective homomorphism $\Phi : \Gamma \to \prod_\mathcal{U} S_n$ for some free ultrafilter $\mathcal{U}$ over $\mathbb{N}$.*

Hint. Follows the same steps as in the proof of Exercise 3, replacing the condition given in the definition of sofic group with the equivalent condition expressed in Exercise 2. ■

1.2 Definition of hyperlinear groups

If $n \in \mathbb{N}$ denote by $\mathbb{M}_n$ the tracial von Neumann algebra of $n \times n$ matrices over the complex numbers. The *normalized* trace τ of $\mathbb{M}_n$ is defined by

$$\tau\left((a_{ij})\right) = \frac{1}{n}\sum_{i=1}^{n} a_{ii}.$$

The operator norm $\|x\|$ of an element x of $\mathbb{M}_2$ is defined by

$$\|x\| = \sup\left\{\|x\xi\| \mid \xi \in \mathbb{C}^n,\ \|\xi\| \leq 1\right\},$$

while the Hilbert-Schmidt norm $\|x\|_2$ is defined by

$$\|x\|_2 = \tau\left(x^* x\right).$$

An element x of $\mathbb{M}_n$ is unitary if $x^* x = x x^* = 1$. The set U_n of unitary elements of $\mathbb{M}_n$ is a group with respect to multiplication. The Hilbert-Schmidt length function on U_n is defined by

$$\ell_{U_n}(u) = \frac{1}{\sqrt{2}}\|u-1\|_2.$$

Observe that

$$\begin{aligned}
\ell_{U_n}(u)^2 &= \frac{1}{2}\|u-1\|_2^2 \\
&= \frac{1}{2}\tau\left((u-1)^*(u-1)\right) \\
&= \frac{1}{2}\tau\left(2-u-u^*\right) \\
&= 1-\mathrm{Re}\tau(u).
\end{aligned}$$

Exercise 4 *Show that ℓ_{U_n} is a length function on U_n.*

Hyperlinear groups are defined exactly as sofic groups, where the permutation groups with the Hamming length function are replaced with the unitary groups with the Hilbert-Schmidt length function.

Definition 2 *A countable discrete group Γ is* ***hyperlinear*** *if for every $\varepsilon > 0$ and every finite subset F of $\Gamma \setminus \{1_\Gamma\}$ there is a natural number n and a function $\Phi : \Gamma \to U_n$ such that $\Phi(1_\Gamma) = 1_{U_n}$ and for every $g, h \in F$:*

- $d_{U_n}\left(\Phi(gh), \Phi(g)\Phi(h)\right) < \varepsilon$;
- $\ell_{U_n}\left(\Phi(g)\right) > 1-\varepsilon$.

As before this notion can be equivalently reformulated in terms of embedding into (metric) ultraproducts. If $\mathcal{U}$ is a free ultrafilter over $\mathbb{N}$ the (metric) ultraproduct $\prod_{\mathcal{U}} U_n$ is the length quotient of $\prod_n U_n$ with respect to the pseudo length function

$$\ell_{\mathcal{U}}\left((u_n)_{n\in\mathbb{N}}\right) = \lim_{n\to\mathcal{U}} \ell_{U_n}(u_n).$$

Exercise 5 *Suppose that Γ is a countable discrete group regarded as a length group with respect to the trivial length function. Show that the following statements are equivalent:*

- *Γ is hyperlinear;*
- *there is a length-preserving homomorphism $\Phi : \Gamma \to \prod_{\mathcal{U}} U_n$ for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$;*
- *there is a length-preserving homomorphism $\Phi : \Gamma \to \prod_{\mathcal{U}} U_n$ for some free ultrafilter $\mathcal{U}$ over $\mathbb{N}$.*

An amplification argument due to Radulescu (see [3]) predating the analogous argument for permutation groups of Elek and Szabo shows that hyperlinearity is equivalent to an apparently weaker property. This is discussed in Exercise 6.

Exercise 6 *Prove that a countable discrete group Γ is hyperlinear if and only if there is a function $r : \Gamma \to (0,1)$ such that for some $\varepsilon > 0$ and every $F \subset \Gamma\backslash\{1_\Gamma\}$ finite there is a natural number n and a function $\Phi : \Gamma \to U_n$ such that $\Phi(1_\Gamma) = 1_{U_n}$ and for every $g, h \in F$:*

- $d_{U_n}(\Phi(gh), \Phi(g)\Phi(h)) < \varepsilon$;
- $\ell_{U_n}(\Phi(g)) > r(g)$.

Hint. If $A = (a_{ij}) \in \mathbb{M}_n$ and $B = (b_{ij}) \in \mathbb{M}_m$ define $A \otimes B \in \mathbb{M}_{nm}$ by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & \dots & \dots & a_{nn}B \end{pmatrix}$$

Prove the following

- $(A \otimes B)(A' \otimes B') = AA' \otimes BB'$;
- $\|A \otimes B\| \leq \|A\| \|B\|$;
- $\tau(A \otimes B) = \tau(A)\tau(B)$;
- $A \otimes B$ is unitary if both A and B are unitary;

If $u \in U_n$ define recursively $u^{\otimes 1} = u \in U_n$ and $u^{\otimes k} = u^{\otimes (k-1)} \otimes u$ for $k \geq 2$. Observe that the function

$$u \mapsto u^{\otimes k}$$

is a group homomorphism from U_n to U_{n^k} such that

$$\tau \left(u^{\otimes k} \right) = \tau \left(u \right)^k .$$

■

As before Exercise 6 entails a characterization of hyperlinear groups in terms of algebraic embeddings into ultraproducts of unitary groups.

Exercise 7 *Suppose that Γ is a countable discrete group. Show that the following statements are equivalent:*

- *Γ is hyperlinear;*
- *there is an injective homomorphism $\Phi : \Gamma \to \prod_{\mathcal{U}} U_n$ for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$;*
- *there is an injective homomorphism $\Phi : \Gamma \to \prod_{\mathcal{U}} U_n$ for some free ultrafilter $\mathcal{U}$ over $\mathbb{N}$.*

If σ is a permutation over n denote by P_σ the permutation matrix associated with σ, acting as σ on the canonical basis of $\mathbb{C}^n$. Observe that P_σ is a unitary matrix and the function

$$\sigma \mapsto P_\sigma$$

is a homomorphism from S_n to U_n. Moreover

$$\tau \left(P_\sigma \right)^2 = 1 - \ell_{S_n} \left(\sigma \right) .$$

It is not difficult to deduce from this that any sofic group is hyperlinear. This is the content of Exercise 8.

Exercise 8 *Fix a free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. Show that the function*

$$\left(\sigma_n \right)_{n \in \mathbb{N}} \mapsto \left(P_{\sigma_n} \right)_{n \in \mathbb{N}}$$

from $\prod_n S_n$ to $\prod_n U_n$ induces an algebraic embedding of $\prod_{\mathcal{U}} S_n$ into $\prod_{\mathcal{U}} U_n$. Infer from this that any sofic groups is hyperlinear.Kervaire-Laudenbach conjecture for hyperlinear groups

Conjecture 3 (Kervaire-Laudenbach) *Suppose that Γ is a group and $a_1, \ldots, a_l \in \Gamma$. Denote by $w \left(t, a_1, \ldots, a_l \right)$ the word*

$$t^{s_1} a_1 \cdots t^{s_l} a_n .$$

If $s = \sum_{i=1}^{l} s_i \neq 0$ then there is an element b in some group extending Γ such that

$$w \left(b, a_1, \ldots, a_n \right) = b^{s_1} a_1 \cdots b^{s_n} a_n = 1$$

In the following we will show that the Kervaire-Laudenbach conjecture holds for hyperlinear groups.

Theorem 4 (Gerstenhaber-Rothaus, 1962) *Suppose that $n \in \mathbb{N}$ and U_n is the group of unitary matrices of rank n. Assume that $a_1, \ldots, , a_n \in U_n$ and $w(t, a_1, \ldots, a_l)$ denotes the word*

$$t^{s_1} a_1 \cdots t^{s_l} a_l.$$

If $s = \sum_{i=1}^{l} s_i \neq 0$ then there is an element b of U_n such that

$$w(b, a_1, \ldots, a_n) = 1.$$

Observe that we are able to find b already in U_n, and not just in some group extending U_n.
Proof. Consider the map

$$f : U_n \to U_n$$

defined by

$$b \mapsto w(b, a_1, \ldots, a_n)$$

We just need to prove that f is onto. Recall that U_n is a compact manifold of dimension n^2. Thus the homology group $H_{n^2}(U_n)$ is an infinite cyclic group. Being continuous (and in fact smooth) f induces a map

$$f_* : H_{n^2}(U_n) \to H_{n^2}(U_n).$$

If e is a generator of $H_{n^2}(U_n)$ then

$$f_*(e) = de$$

for some $d \in \mathbb{Z}$ called the degree of f. In order to show that f is onto, it is enough to show that its degree is nonzero. I claim that $d = s^n$ where $s = \sum_{i=1}^{n} s_i$. Since U_n is connected, the map f is homotopy equivalent to the map

$$f_s : U_n \to U_n$$

defined by

$$b \mapsto b^s.$$

Since the degree of a map is homotopy invariant, f and f_s have the same degree. Therefore we just have to show that f_s has degree s^n. The facts that the generic element of U_n has s^n s-roots of unity, and the degree of a map can be computed locally, shows that the degree of f_s is s^n. ■

Gerstenhaber and Rothaus proved in fact in [1] a more general version of Theorem 4, where U_n is replaced by any compact Lie group. Moreover they consider systems of equations in possibly more than one variable.

Observe that the conclusion of Theorem 4 can be expressed by a formula. The following corollary follows immediately using Łos theorem for ultraproducts.

Corollary 5 *If $\mathcal{V}$ is an ultrafilter over $\mathbb{N}$ then the universal hyperlinear group $U_{\mathcal{V}} = \prod_n^{\mathcal{V}} U_n$ has the followin property: Suppose that $a_1, \ldots, a_l \in U_{\mathcal{V}}$ and $w(t, a_1, \ldots, a_l)$ is the word*

$$t^{s_1} a_1 \cdots t^{s_l} a_l.$$

If $s = \sum_{i=1}^{l} s_i \neq 0$ then there is $b \in U_{\mathcal{V}}$ such that

$$w(b, a_1, \ldots, a_l) = 1.$$

In particular any universal hyperlinear group $U_{\mathcal{V}}$ satisfies the Kervaire-Laudenbach conjecture. Obviously the Kervaire-Laudenbach conjecture holds for any subgroup of a group that satisfies the Kervaire-Laudenbach conjecture. It follows that all countable hyperlinear groups satisfy the Kervaire-Laudenbach conjecture.

References

[1] M. Gerstenhaber, O. S. Rothaus, The solution of sets of equations in groups, Proc. Nat. Acad. Sci. U.S.A. 48 1962 1531–1533.

[2] G. Elek, E. Szabo, Hyperlinearity, essentially free actions and L^2-invariants. The sofic property. Math. Ann. 332 (2005), no. 2, 421–441.

[3] F. Radulescu, The von Neumann algebra of the non-residually finite Baumslag group $\langle a, b \,|\, ab^3a^{-1} = b^2 \rangle$ embeds into R^ω, Hot topics in operator theory, 173–185, Theta Ser. Adv. Math., 9, Theta, Bucharest, 2008.