1 Algebraic eigenvalues conjectures

1.1 Algebraic eigenvalues conjecture: the statement

Suppose that Γ is a (countable discrete) group. Considering the particular case of the group algebra construction for the field $\mathbb{C}$ of complex numbers one obtains the complex group algebra $\mathbb{C}\Gamma$ of formal finite linear combinations

$$\lambda_1\gamma_1 + \cdots + \lambda_l\gamma_l$$

where $\lambda_i \in \mathbb{C}$ and $\gamma_i \in \Gamma$. The natural action of $\mathbb{C}\Gamma$ on the Hilbert space $\ell^2\Gamma$ defines an inclusion of $\mathbb{C}\Gamma$ into $B\left(\ell^2\Gamma\right)$. The group ring $\mathbb{Z}\Gamma$ is the subring of $\mathbb{C}\Gamma$ of finite linear combinations

$$n_1\gamma_1 + \cdots + n_l\gamma_l$$

where $n_i \in \mathbb{Z}$ and $\gamma_i \in \Gamma$. A conjecture due to Dodziuk, Linnell, Mathai, Schick, and Yates known as *algebraic eigenvalues conjectures* asserts that the eigenvalues of an element $x \in \mathbb{Z}\Gamma$ are algebraic integeres. Recall that a complex number is called an algebraic integer if it is the root of a monic polinomial with integer coefficients. This conjecture has been settled for sofic groups by Andreas Thom in [3]. The proof involves the notion of ultraproduct of tracial von Neumann algebras and can be naturally presented within the framework of logic for metric structures.

1.2 Logic for tracial von Neumann algebras

In the context of tracial von Neumann algebras a *term* $p\left(x_1, \ldots, x_n\right)$ in the variables $x_1, \ldots, x_n$ is a *-polynomial in $x_1, \ldots, x_n$, i.e. a polynomial in the variables $x_1, \ldots, x_n$ and $x_1^*, \ldots, x_n^*$. A basic formula is an expression of the form

$$\tau\left(p\left(x_1, \ldots, x_n\right)\right)$$

where $p\left(x_1, \ldots, x_n\right)$ is a *-polynomial. General formulas can be obtained from basic formulas composing with continuous complex valued complex functions or taking the real part and then the sup or inf over norm bounded subsets of the von Neumann algebra or of the scalars. More formally if $\varphi_1, \ldots, \varphi_m$ are fomulas and $f : \mathbb{C}^n \to \mathbb{C}$ is a continuous function then

$$f\left(\varphi_1, \ldots, \varphi_m\right)$$

is a formula. Analogously if $\varphi\left(x_1, \ldots, x_n, y\right)$ is a formula then

$$\sup_{\|y\|\leq 1} \operatorname{Re}\left(\varphi\left(x_1, \ldots, x_n, y\right)\right)$$
$$\inf_{\|y\|\leq 1} \operatorname{Re}\left(\varphi\left(x_1, \ldots, x_n, y\right)\right)$$
$$\sup_{|\lambda|\leq 1} \operatorname{Re}\left(\varphi\left(x_1, \ldots, x_n, \lambda\right)\right)$$
$$\inf_{|\lambda|\leq 1} \operatorname{Re}\left(\varphi\left(x_1, \ldots, x_n, \lambda\right)\right)$$

are formulas. The interpretation of a formula in a tracial von Neumann algebra is defined in the obvious way by recusion on the complexity. By definition a sentence (i.e. a formula with no free variables) holds in a given tracial von Neumann algebra if its evaluation is zero. For example

$$(\tau(x^*x))^{\frac{1}{2}}$$

is a formula usually abbreviated by $\|x\|_2$ whose interpretation in a tracial von Neumann algebra (M,τ) is the 2-norm on M associated with the trace τ. Analogously

$$\sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} \|x-y\|_2$$

is a sentence (i.e. a formula without free variables) that holds in a tracial von Neumann algebra (M,τ) iff M is abelian. Analogously

$$\sup_{\|x\|\leq 1} \inf_{|\lambda|\leq 1} \|x-\lambda\|_2$$

is a sentence which holds in (M,τ) iff M is one-dimensional (i.e. isomorphic to $\mathbb{C}$).

The notion of ultraproduct of a sequence $(M_n,\tau_n)_{n\in\mathbb{N}}$ of tracial von Neumann algebra has already been defined. The fact that Łos theorem on ultraproducts holds in this context can be seen as an indication that the notion of formulas just introducts is the right one.

Theorem 1 (Łos for tracial von Neumann algebras) *Suppose that $(M_n,\tau_n)_{n\in\mathbb{N}}$ is a sequence of tracial von Neumann algebras, $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$, and $\prod_{\mathcal{U}} M_n$ is the ultraproduct of the sequence $(M_n,\tau_n)_{n\in\mathbb{N}}$. If φ is a sentence (i.e. a formulas with no free variables) in the language of tracial von Neumann algebras, then the evaluation of φ in $\prod_{\mathcal{U}} M_n$ is the limit according to $\mathcal{U}$ of the sequence of evaluations of φ in the structures M_n.More generally if $\varphi(x_1,\ldots,x_k)$ is a formula with free variables $x_1,\ldots,x_l$ and $a^{(1)},\ldots,a^{(k)}$ are elements of $\prod_{\mathcal{U}} M_n$ then*

$$\varphi^{\prod_{\mathcal{U}} M_n}\left(a^{(1)},\ldots,a^{(k)}\right) = \lim_{n\to\mathcal{U}} \varphi^{M_n}\left(a_n^{(1)},\ldots,a_n^{(k)}\right)$$

where $\left(a_n^{(j)}\right)_{n\in\mathbb{N}}$ is a representative sequence for $a^{(j)}$.

The notion of elementary property and axiomatizable class of tracial von Neumann algebras are defined as in the case of length groups or rank rings. In particular the previous examples shows that the property of being abelian and the property of being one-dimensional are elementary. Exercise 5 shows that the property of being a factor is elementary.

A sequence $(\varphi_n(x_1,\ldots,x_k))_{n\in\mathbb{N}}$ of formulas with paramters from a tracial von Neumann algebra (M,τ) is

- approximately realized in (M,τ) if for every $n\in\mathbb{N}$ there are $b_n^{(1)},\ldots,b_n^{(k)}\in M$ such that for $i\leq m$

$$\varphi_i\left(b_n^{(1)},\ldots,b_n^{(k)}\right) < \frac{1}{n};$$

- realized in (M, τ) if there are $b^{(1)}, \ldots, b^{(k)} \in M$ such that for every $i \in \mathbb{N}$

$$\varphi_i \left(b_n^{(1)}, \ldots, b_n^{(k)} \right) < \frac{1}{n}.$$

Exercise 2 *If $(M_n, \tau_n)_{n \in \mathbb{N}}$ is a sequence of von Neumann algebras and $\mathcal{U}$ is a free ultrafilter over $\mathbb{N}$, then the tracial ultraproduct $\prod_{\mathcal{U}} M_n$ is* ***countably saturated****. This means that any approximately realized sequence is realized.*

Hint. For every $n \in \mathbb{N}$ the set

$$\left\{ n \in \mathbb{N} \,\middle|\, \forall i \leq n,\ \varphi_i \left(b^{(1,n)}, \ldots, b^{(k,n)} \right) < \frac{1}{n} \right\}$$

belongs to $\mathcal{U}$. Use this fact to build sequences $\left(b_n^{(1)} \right)_{n \in \mathbb{N}}, \ldots, \left(b_n^{(k)} \right)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \mathcal{U}} \varphi_i \left(b_n^{(1)}, \ldots, \varphi_i^{(k)} \right) = 0$$

for every $i \in \mathbb{N}$. ■

Exercise 3 *Deduce from Exercise 2 that the unit ball of $\prod_{\mathcal{U}} M_n$ is complete with respect to the 2-norm.*

Hint. Suppose that $(x_n)_{n \in \mathbb{N}}$ is a sequence in the unit ball of $\prod_{\mathcal{U}} M_n$ which is Cauchy with respect to the 2-norm. Consider the sequence of formulas

$$\inf_{\|y\| \leq 1} \max \left\{ \|x_1 - y\|_2, \ldots, \|x_n - y\| \right\}$$

and argue that it is approximately realized, and hence realized.

It is clear that $\prod_{\mathcal{U}} M_n$ is a C*-algebra and τ is a trace on $\prod_{\mathcal{U}} M_n$. Moreover the unit ball of $\prod_{\mathcal{U}} M_n$ is complete in 2-norm. This shows that $\prod_{\mathcal{U}} M_n$ coincides with the weak closure of its GNS representation and hence it is a von Neumann algebra. ■

2 Dixmier averaging trick and elementarity of factors

Recall that the *center* $Z(M)$ of a von Neumann algebra M, i.e. the set of elements that commute with any other elmenent of M, is a closed subalgebra of M and hence it is itself a von Neumann algebra. A von Neumann algebra M is called a *factor* if its center contains only the scalar multiples of the identity. The unitary group $U(M)$ of M is the mutliplicative group of unitary elements of M, i.e. elements u satisfying $uu^* = u^*u = 1$.

Proposition 4 (Dixmier averaging trick) *If $x \in M$ then the closure of the convex hull of the set*

$$\{ uxu^* \mid u \in U(A) \}$$

contains a central element.

Proof. (as in Blackadar) Suppose $x \in M_{sa}$ has norm 1. I claim that there are $u \in U(M)$ and $z \in Z(M)$ such that

$$\left\| \frac{x + uxu^*}{2} - y \right\| \leq \frac{3}{4} \|x\| .$$

Suppose that p is the support projection of x_+ and choose a central projection z sucht hat

$$pz \precsim (1-p) z$$

and

$$(1-p)(1-z) \precsim p(1-z)$$

and partial isometries v, w such that

$$v^* v = pz$$

and

$$vv^* \leq (1-p) z$$

and

$$w^* w = (1-p)(1-z)$$

and

$$ww^* \leq p(1-z) .$$

Then

$$u = v + v^* + w + w^* + ((1-p) z - vv^*) + (p(1-z) - ww^*)$$

is unitary and

$$y = \frac{1-2z}{4}$$

works. Finish the proof by iteration. ■

Exercise 5 *Suppose that M is a von Neumann algebra endowed with a faithful trace τ. Dixmier averaging theorem ([1], III.2.5.18.) asserts that for every $x \in M$ the closure of the convex Use this fact together with the fact that a von Neumann algebra is generated by its projections to prove that M is a factor if and only if for every $x \in M$*

$$\|y - \tau(y)\|_2 \leq \sup_{y \in M_1} \|xy - yx\|_2 .$$

Conclude that the property of being a factor is elementary.

Hint. If M is not a factor then a nontrivial projection p in $Z(M)$ violates 5. If M is a factor then 5 is a consequence of Dixmier averagin theorem. ■

A consequence of Exercise 5 and Łos theorem for ultraproducts is that an ultraproduct of factors is itself a factor. The type classification of factors asserts that a factor endowed with a faithful normalized trace is either a II_1 factor or it is isomorphic to $\mathbb{M}_n$ for some $n \in \mathbb{N}$. Exercise 6 infers from this that the property of being a II_1 factor is elementary. In particular an ultraproduct of II_1 factors is a II_1 factor.

Exercise 6 *Fix an irrational number $\alpha \in (0,1)$. Using the type classification for (finite) factors prove that a factor M is II_1 if and only there is a projection of trace r. Deduce that the property of being a II_1 factor is elementary.*

Hint. Recall that the trace in a II_1 factor attains on projections all the values between 0 and 1. ■

3 The complex group algebra of a sofic group

Suppose that Γ is a discrete group. The complex group algebra $\mathbb{C}\Gamma$ can be endowed with a linear involutive map $x \mapsto x^*$ such that

$$(\lambda\gamma)^* = \overline{\lambda}\gamma^{-1}.$$

Recall that the trace τ on $\mathbb{C}\Gamma$ is defined by

$$\tau\left(\sum_\gamma \lambda_\gamma \gamma\right) = \lambda_{1_\Gamma}.$$

The weak closure $L\Gamma$ of $\mathbb{C}\Gamma$ in $B\left(\ell^2\Gamma\right)$ is a von Neumann algebra containing $\mathbb{C}\Gamma$ as a *-subalgebra. The trace of $\mathbb{C}\Gamma$ admits a unique extension to a faithful normalized trace τ on $L\Gamma$. Moreover $\mathbb{C}\Gamma$ is dense in the unit ball of $L\Gamma$ with respect to the 2-norm of $L\Gamma$ defined by $\|x\|_2 = \tau(x^*x)$.

In the rest of the section the matrix algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with complex coefficients is regarded as a tracial von Nemann algebra endowed with the (unique) canonical normalized trace τ_n. If $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$ then $\prod_\mathcal{U} M_n(\mathbb{C})$ denotes here the ultraproduct of $M_n(\mathbb{C})$ as tracial von Neumann algebras. Observe that this is different from the ultraproduct of $M_n(\mathbb{C})$ as rank rings, for example. Denote by $\prod_\mathcal{U} M_n(\mathbb{Z})$ the closed self-adjoint subalgebra of $\prod_\mathcal{U} M_n(\mathbb{C})$ consisting of elements admitting representative sequences of matrices with integer coefficients. Recall that U_n denotes the group of unitary elements of $M_n(\mathbb{C})$. If σ is a permutation over n, then the associated permutation matrix P_σ is a unitary element of $M_n(\mathbb{C})$ such that $\tau(P_\sigma) = 1 - \ell(\sigma)$. This fact has been used in ... to prove that any sofic group is hyperlinear. An analogous argument can be used to solve Exercise 7.

Exercise 7 *Supppose that Γ is a sofic group and $\mathcal{U}$ is a nonprincipal ultrafilter over $\mathbb{N}$. Show that there is a trace preserving homomorphism of *-algebras from $\mathbb{C}\Gamma$ to $\prod_\mathcal{U} M_n(\mathbb{C})$ (ultraproduct as tracial von Neumann algebras) sending $\mathbb{Z}\Gamma$ into $\prod_\mathcal{U} M_n(\mathbb{Z})$. Infer that there is a trace preserving embedding of $L\Gamma$ into $\prod_\mathcal{U} M_n(\mathbb{C})$ sending $\mathbb{Z}\Gamma$ into $\prod_\mathcal{U} M_n(\mathbb{Z})$.*

Suppose that $M \subset B(H)$ is a von Neumann algebra, and x is an operator in M. A complex number λ is an eigenvalue for x if and only if $\operatorname{Ker}(x - \lambda I) \neq \{0\}$. Considering the orthogonal projection $p \in M$ on $\operatorname{Ker}(x - \lambda I)$ shows that this is equivalent to the existence of a nonzero projection $p \in M$ such that $(x - \lambda)p =$

0. In particular if $\Psi : M \to N$ is an embedding of von Neumann algebra, then $x \in M$ has the same eigenvalues as $\Psi(x)$. This observation together with exercise 7 allows one to conclude that in order to establish the algebraic eigenvalues conjecture for sofic groups it is enough to show that the elements of $\prod_{\mathcal{U}} M_n(\mathbb{Z})$ have algebraic eigenvalues. This is shown in section ... using Łos theorem on ultraproducts together with a characterization of algebraic integers due to Thom.

4 Algebraic eigenvalues

Suppose in the following that $\lambda \in \mathbb{C}$ is not an algebraic integer.

Theorem 8 *If ε is a strictly positive real number and N is a natural number, then there is a natural number $M(N,\varepsilon)$ such that for every monic polynomial p with integer coefficients such that all the zeroes of p have absolute value at most N the proportion of zeroes of p at distance at most $\frac{1}{M(N,\varepsilon)}$ from λ is at most ε.*

Corollary 9 *For every positive real number ε and every natural number $N \in \mathbb{N}$ there is a natural number $M(N,\varepsilon)$ such that for every finite rank matrix with integer coefficients A of operator norm at most N there is a complex matrix B of the same size of operator norm at most $M(N,\varepsilon)$ such that*

$$\|B(\lambda - A) - I\|_2 \leq \varepsilon.$$

Observe that M does not depend from the size of the matrix A.

Proof. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Suppose that M is obtained from ε and N applying Theorem 8. If A is a finite rank matrix with integer coefficients of norm at most N, then the minimal polynomial p_A of A is a monic polynomial with integer coefficients whose zeroes have all absolute value at most N. Moreover since λ is not an algebraic integer, p_A does not have λ as a root. In particular $\lambda - A$ is an invertible matrix. Moreover by the choice of M the proportion of zeroes of p at distance at least $\frac{1}{M}$ from λ is at least $1-\varepsilon$. This means that if p is the projection on the eigenspace corresponding to these eigenvalues, then $\tau(p) > 1-\varepsilon$. Define $B = p(\lambda - A)^{-1}$. Observe that B has operator norm at most M and

$$\|B(\lambda - A) - 1\|_2 = \|1-p\|_2 = \tau(1-p) \leq \varepsilon.$$

■

Corollary 9 shows that elements of $M_n(\mathbb{C})$ with integer coefficients having operator norm at most N satisfy the formula $\varphi_{N,\varepsilon}(x)$ defined by

$$\inf_{\|y\| \leq M(N,\varepsilon)} \max\{\|y(\lambda - x) - 1\|_2 - \varepsilon, 0\}$$

By Łos theorem on ultraproducts if $x \in \prod_{\mathcal{U}} M_n(\mathbb{Z})$ and $\varepsilon > 0$ then there is $y \in \prod_{\mathcal{U}} M_n(\mathbb{C})$ such that $\|y(\lambda - x) - 1\|_2 \leq \varepsilon$. This implies that λ is not an eigenvalue of x. In fact if $p \in \prod_{\mathcal{U}} M_n(\mathbb{C})$ is a projection such that $(x-\lambda)p = 0$ then

$$\begin{aligned} \|p\|_2 &\leq \|(y(x-\lambda)-1)p\|_2 + \|y(x-\lambda)p\| \\ &\leq \|y(x-\lambda)-1\|_2 \\ &\leq \varepsilon. \end{aligned}$$

Being this true for every $\varepsilon > 0$, $p = 0$. This shows that λ is not and eigenvalue of x.

References

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