Prova 1 - MTM3422

① $u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$

(a) verificar que $\{u_1, u_2, u_3\}$ é ortogonal.

$$\langle u_1, u_2 \rangle = u_1^T u_2$$

$$= (1 \ -1 \ 0 \ 2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 1.1 + (-1).1 + 0.1 + 2.0 = 0$$

$$u_1^T u_3 = (1 \ -1 \ 0 \ 2) \begin{pmatrix} -1 \\ -1 \\ 2 \\ 0 \end{pmatrix} = 1.(-1) + (-1).(-1) + 0.2 + 2.0 = 0$$

$$u_2^T u_3 = (1 \ 1 \ 1 \ 0) \begin{pmatrix} -1 \\ -1 \\ 2 \\ 0 \end{pmatrix} = 1(-1) + 1.(-1) + 1.2 + 0.0 = 0$$

(b) encontre $u_4 \overset{\neq 0}{} \in \mathbb{R}^4$, $u_4 \perp \{u_1, u_2, u_3\}$

$$u_4 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}, \quad 0 = u_4^T u_i, \quad i = 1, 2, 3$$

$$\begin{cases} 0 = u_4^T u_1 = \alpha - \beta + 2\delta & (1) \\ 0 = u_4^T u_2 = \alpha + \beta + \gamma & (2) \\ 0 = u_4^T u_3 = -\alpha - \beta + 2\gamma & (3) \end{cases}$$

$$(3) \Rightarrow 2\gamma = \alpha + \beta \overset{(2)}{=} -\gamma \Rightarrow \boxed{\gamma = 0}$$

$$(2) \Rightarrow \underline{\beta = -\alpha}, \quad (1) \Rightarrow 2\alpha + 2\delta = 0 \Rightarrow \underline{\delta = -\alpha}$$

$$u_4 = \begin{pmatrix} \alpha \\ -\alpha \\ 0 \\ -\alpha \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}}, \quad \alpha \neq 0$$

(c) transforme $\{u_1, \dots, u_4\}$ em um conjunto ortonormal: $\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}, \frac{u_4}{\|u_4\|}$

$$\|u_1\|_2 = (1^2 + (-1)^2 + 0^2 + 2^2)^{1/2} = \sqrt{6}$$

$$\|u_2\|_2 = (1^2 + 1^2 + 1^2 + 0^2)^{1/2} = \sqrt{3}$$

$$\|u_3\|_2 = \dots = \sqrt{6}$$

$$\|u_4\|_2 = \dots = \sqrt{3}$$

(2) $\langle x,y\rangle_A := x^T A y$,

$A \in \mathbb{R}^{n\times n}$, $A^T = A$

positiva definida $x^T A x > 0, \ \forall x \neq 0$

▷ Mostre que $\langle x,y\rangle_A$ é um produto interno.

Propriedades de $\langle \cdot,\cdot\rangle$:

(i) $\langle x,x\rangle \geq 0, \ \forall x \in \mathbb{R}^n$

(ii) $\langle x,y\rangle = \langle y,x\rangle, \ \forall x,y \in \mathbb{R}^n$

(iii) $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$ $\quad \forall x,y \in \mathbb{R}^n$

$\langle x,\alpha y\rangle = \alpha\langle x,y\rangle$ $\quad \alpha \in \mathbb{R}$

(iv) $\langle x+z,y\rangle = \langle x,y\rangle + \langle z,y\rangle$ $\quad \forall x,y,z \in \mathbb{R}^n$

$\langle x,y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

(i) $\langle x,x\rangle_A = x^T A x \geq 0, \ \forall x \in \mathbb{R}^n$

(ii) $\langle x,y\rangle_A = x^T A y = x^T A^T y = (Ax)^T y$

$= y^T (Ax) = y^T A x = \langle y,x\rangle_A$

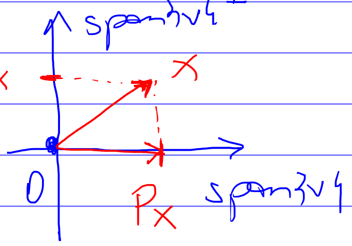
(iii) $\langle \alpha x,y\rangle_A = (\alpha x)^T A y = \alpha\, x^T A y = \alpha\langle x,y\rangle_A$

(iv) $\langle x+z,y\rangle_A = (x+z)^T A y = x^T A y + z^T A y$

$= \langle x,y\rangle_A + \langle z,y\rangle_A$

③ $u = \begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix}.$

$(I-P)x = x - Px$



$x = Px + x - Px$

$\underline{\underline{Px = \mathrm{proj}_v(x)}}$

(a) $\mathrm{proj}_v(u) = \dfrac{\langle u,v\rangle}{\langle v,v\rangle} v = \dfrac{3}{18}(1\ 4\ 0\ -1) = \dfrac{1}{6}(1\ 4\ 0\ -1)$

$\langle u,v\rangle = u^T v = (-2\ 1\ 3\ -1)\begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} = -2.1 + 1.4 + 3.0 + (-1)(-1) = 3$

$\langle v,v\rangle = v^T v = 1^2 + 4^2 + 0^2 + (-1)^2 = 18$

(b) $\mathrm{proj}_u(v) = \dfrac{\langle v,u\rangle}{\langle u,u\rangle} u = \dfrac{3}{15}(-2\ 1\ 3\ -1) = \dfrac{1}{5}(-2\ 1\ 3\ -1)$

$\langle u,u\rangle = (-2)^2 + 1^2 + 3^2 + (-1)^2 = 15$

(c) $\mathrm{proj}_{v\perp}(u) = u - \mathrm{proj}_v(u)$

$= \dfrac{6}{6}(-2\ 1\ 3\ -1) - \dfrac{1}{6}(1\ 4\ 0\ -1) = \left(\dfrac{-13}{6}, \dfrac{2}{6}, \dfrac{18}{6}, \dfrac{-5}{6}\right)$

(d) $\mathrm{proj}_{u\perp}(v) = v - \mathrm{proj}_u(v)$

$= \dfrac{5}{5}(1\ 4\ 0\ -1) - \dfrac{1}{5}(-2\ 1\ 3\ -1) = \left(\dfrac{7}{5}, \dfrac{19}{5}, \dfrac{-3}{5}, \dfrac{-4}{5}\right)$

(4) $Q = I - uu^T$, $\|u\|_2 = 1$. | $u^Tu = 1$

(a) Prove que Q é projetor ortogonal

$Q^2 = Q$, $Q^T = Q$

- $Q^2 = QQ = (I - uu^T)(I - uu^T)$

$$= I - uu^T - uu^T + u\underbrace{u^T u}_{\|u\|_2^2}u^T$$

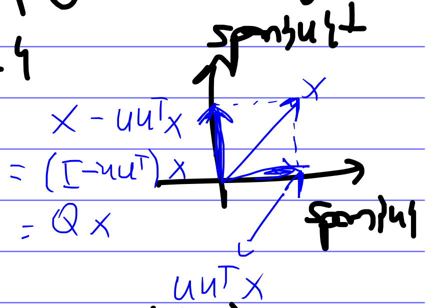
$$= I - 2uu^T + \|u\|_2^2\, uu^T$$

$$= I - 2uu^T + uu^T = I - uu^T = Q$$

- $Q^T = (I - uu^T)^T = I^T - (uu^T)^T = I - uu^T = Q$.

(b) Observe que $P = uu^T$ é projetor ortogonal sobre span{u}

$Q = I - P = I - uu^T$



$R(Q) = \text{span}\{u\}^\perp$

$$Qx = (I - uu^T)x = x - uu^Tx = x - (u^Tx)u$$

$N(Q) = \{z \mid Qz = 0\}$

$$Qz = (I - uu^T)z = z - (u^Tz)u$$

Note que p/ $z = \alpha u$

$$Qz = Q(\alpha u) = \alpha u - \alpha\underbrace{(u^Tu)}_{=1}u$$

$$= \alpha u - \alpha u = 0$$

$N(Q) = \text{span}\{u\}$.

(c) $N(Q) = \text{span}\{u\} \neq \{0\}$
logo Q é singular

(d) $\dim N(Q) = \dim \text{span}\{u\} = 1$

posto$(Q) = \dim R(Q) = n - \dim N(Q) = n - 1$

Teo. núcleo e imagem

$Q: \mathbb{R}^n \to \mathbb{R}^n$

$\dim(\mathbb{R}^n) = n = \dim N(Q) + \dim R(Q)$