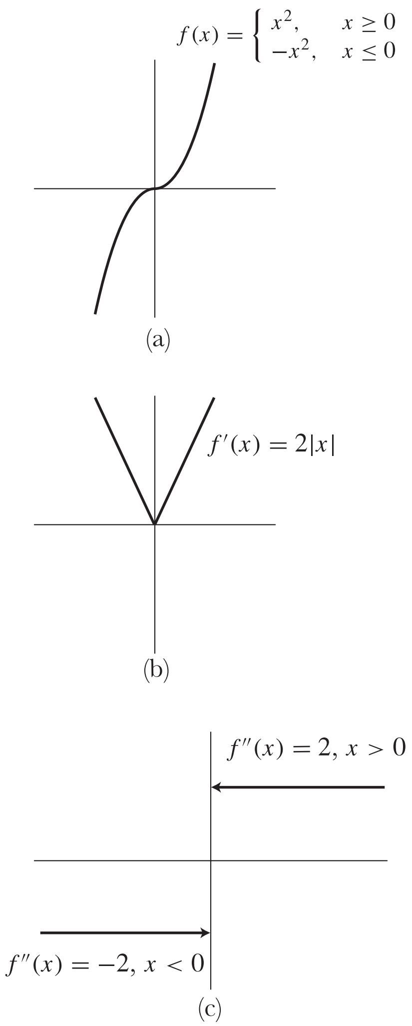

FIGURE 21

It follows that $f''(0)$ does not exist! Existence of the second derivative is thus a rather strong criterion for a function to satisfy. Even a "smooth looking" function like f reveals some irregularity when examined with the second derivative. This suggests that the irregular behavior of the function

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

might also be revealed by the second derivative. At the moment we know that $g'(0) = 0$, but we do not know $g'(a)$ for any $a \neq 0$, so it is hopeless to begin computing $g''(0)$. We will return to this question at the end of the next chapter, after we have perfected the technique of finding derivatives.

PROBLEMS

1. (a) Prove, working directly from the definition, that if $f(x) = 1/x$, then $f'(a) = -1/a^2$, for $a \neq 0$.
(b) Prove that the tangent line to the graph of f at $(a, 1/a)$ does not intersect the graph of f, except at $(a, 1/a)$.

2. (a) Prove that if $f(x) = 1/x^2$, then $f'(a) = -2/a^3$ for $a \neq 0$.
(b) Prove that the tangent line to f at $(a, 1/a^2)$ intersects f at one other point, which lies on the opposite side of the vertical axis.

3. Prove that if $f(x) = \sqrt{x}$, then $f'(a) = 1/(2\sqrt{a})$, for $a > 0$. (The expression you obtain for $[f(a+h) - f(a)]/h$ will require some algebraic face lifting, but the answer should suggest the right trick.)

4. For each natural number n, let $S_n(x) = x^n$. Remembering that $S_1{}'(x) = 1$, $S_2{}'(x) = 2x$, and $S_3{}'(x) = 3x^2$, conjecture a formula for $S_n{}'(x)$. Prove your conjecture. (The expression $(x + h)^n$ may be expanded by the binomial theorem.)

5. Find f' if $f(x) = [x]$.

6. Prove, starting from the definition (and drawing a picture to illustrate):

(a) if $g(x) = f(x) + c$, then $g'(x) = f'(x)$;
(b) if $g(x) = cf(x)$, then $g'(x) = cf'(x)$.

7. Suppose that $f(x) = x^3$.

(a) What is $f'(9)$, $f'(25)$, $f'(36)$?
(b) What is $f'(3^2)$, $f'(5^2)$, $f'(6^2)$?
(c) What is $f'(a^2)$, $f'(x^2)$?

If you do not find this problem silly, you are missing a very important point: $f'(x^2)$ means the derivative of f at the number which we happen to be calling x^2; it is *not* the derivative at x of the function $g(x) = f(x^2)$. Just to drive the point home:

(d) For $f(x) = x^3$, compare $f'(x^2)$ and $g'(x)$ where $g(x) = f(x^2)$.

8. (a) Suppose $g(x) = f(x+c)$. Prove (starting from the definition) that $g'(x) = f'(x+c)$. Draw a picture to illustrate this. To do this problem you must write out the definitions of $g'(x)$ and $f'(x+c)$ correctly. The purpose of Problem 7 was to convince you that although this problem is easy, it is not an utter triviality, and there is something to prove: you cannot simply put prime marks into the equation $g(x) = f(x+c)$. To emphasize this point:

(b) Prove that if $g(x) = f(cx)$, then $g'(x) = c \cdot f'(cx)$. Try to see pictorially why this should be true, also.

(c) Suppose that f is differentiable and periodic, with period a (i.e., $f(x+a) = f(x)$ for all x). Prove that f' is also periodic.

9. Find $f'(x)$ and also $f'(x+3)$ in the following cases. Be very methodical, or you will surely slip up somewhere. Consult the answers (after you do the problem, naturally).

(i) $f(x) = (x+3)^5$.
(ii) $f(x+3) = x^5$.
(iii) $f(x+3) = (x+5)^7$.

10. Find $f'(x)$ if $f(x) = g(t+x)$, and if $f(t) = g(t+x)$. The answers will *not* be the same.

11. (a) Prove that Galileo was wrong: if a body falls a distance $s(t)$ in t seconds, and s' is proportional to s, then s cannot be a function of the form $s(t) = ct^2$.

(b) Prove that the following facts are true about s if $s(t) = (a/2)t^2$ (the first fact will show why we switched from c to $a/2$):

(i) $s''(t) = a$ (the acceleration is constant).
(ii) $[s'(t)]^2 = 2as(t)$.

(c) If s is measured in feet, the value of a is 32. How many seconds do you have to get out of the way of a chandelier which falls from a 400-foot ceiling? If you don't make it, how fast will the chandelier be going when it hits you? Where was the chandelier when it was moving with half that speed?

12. Imagine a road on which the speed limit is specified at every single point. In other words, there is a certain function L such that the speed limit x miles from the beginning of the road is $L(x)$. Two cars, A and B, are driving along this road; car A's position at time t is $a(t)$, and car B's is $b(t)$.

(a) What equation expresses the fact that car A always travels at the speed limit? (The answer is *not* $a'(t) = L(t)$.)

(b) Suppose that A always goes at the speed limit, and that B's position at time t is A's position at time $t-1$. Show that B is also going at the speed limit at all times.

(c) Suppose B always stays a constant distance behind A. Under what conditions will B still always travel at the speed limit?

13. Suppose that $f(a) = g(a)$ and that the left-hand derivative of f at a equals the right-hand derivative of g at a. Define $h(x) = f(x)$ for $x \leq a$, and $h(x) = g(x)$ for $x \geq a$. Prove that h is differentiable at a.

14. Let $f(x) = x^2$ if x is rational, and $f(x) = 0$ if x is irrational. Prove that f is differentiable at 0. (Don't be scared by this function. Just write out the definition of $f'(0)$.)

***15.** (a) Let f be a function such at $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0. (If you have done Problem 14 you should be able to do this.)

(b) This result can be generalized if x^2 is replaced by $|g(x)|$, where g has what property?

16. Let $\alpha > 1$. If f satisfies $|f(x)| \leq |x|^\alpha$, prove that f is differentiable at 0.

17. Let $0 < \beta < 1$. Prove that if f satisfies $|f(x)| \geq |x|^\beta$ and $f(0) = 0$, then f is not differentiable at 0.

***18.** Let $f(x) = 0$ for irrational x, and $1/q$ for $x = p/q$ in lowest terms. Prove that f is not differentiable at a for any a. Hint: It obviously suffices to prove this for irrational a. Why? If $a = m.a_1a_2a_3\ldots$ is the decimal expansion of a, consider $[f(a+h) - f(a)]/h$ for h rational, and also for

$$h = -0.00\ldots0a_{n+1}a_{n+2}\ldots.$$

19. (a) Suppose that $f(a) = g(a) = h(a)$, that $f(x) \leq g(x) \leq h(x)$ for all x, and that $f'(a) = h'(a)$. Prove that g is differentiable at a, and that $f'(a) = g'(a) = h'(a)$. (Begin with the definition of $g'(a)$.)

(b) Show that the conclusion does not follow if we omit the hypothesis $f(a) = g(a) = h(a)$.

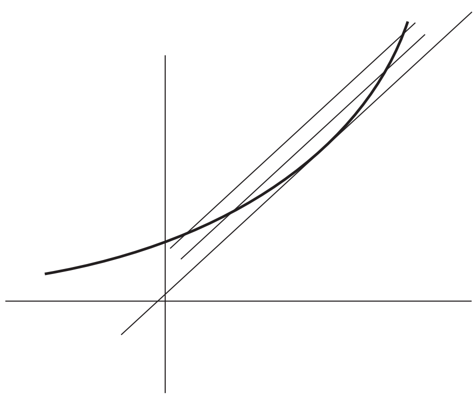

FIGURE 22

20. Let f be any polynomial function; we will see in the next chapter that f is differentiable. The tangent line to f at $(a, f(a))$ is the graph of $g(x) = f'(a)(x-a) + f(a)$. Thus $f(x) - g(x)$ is the polynomial function $d(x) = f(x) - f'(a)(x-a) - f(a)$. We have already seen that if $f(x) = x^2$, then $d(x) = (x-a)^2$, and if $f(x) = x^3$, then $d(x) = (x-a)^2(x+2a)$.

(a) Find $d(x)$ when $f(x) = x^4$, and show that it is divisible by $(x-a)^2$.

(b) There certainly seems to be some evidence that $d(x)$ is always divisible by $(x-a)^2$. Figure 22 provides an intuitive argument: usually, lines parallel to the tangent line will intersect the graph at two points; the tangent line intersects the graph only once near the point, so the intersection should be a "double intersection." To give a rigorous proof, first note that

$$\frac{d(x)}{x-a} = \frac{f(x) - f(a)}{x-a} - f'(a).$$

Now answer the following questions. Why is $f(x) - f(a)$ divisible by $(x-a)$? Why is there a polynomial function h such that $h(x) =$

$d(x)/(x-a)$ for $x \neq a$? Why is $\lim_{x \to a} h(x) = 0$? Why is $h(a) = 0$? Why does this solve the problem?

21. (a) Show that $f'(a) = \lim_{x \to a} [f(x) - f(a)]/(x-a)$. (Nothing deep here.)
(b) Show that derivatives are a "local property": if $f(x) = g(x)$ for all x in some open interval containing a, then $f'(a) = g'(a)$. (This means that in computing $f'(a)$, you can ignore $f(x)$ for any particular $x \neq a$. Of course you can't ignore $f(x)$ for all such x at once!)

***22.** (a) Suppose that f is differentiable at x. Prove that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

Hint: Remember an old algebraic trick—a number is not changed if the same quantity is added to and then subtracted from it.

**(b) Prove, more generally, that

$$f'(x) = \lim_{h,k \to 0^+} \frac{f(x+h) - f(x-k)}{h+k}.$$

***23.** Prove that if f is even, then $f'(x) = -f'(-x)$. (In order to minimize confusion, let $g(x) = f(-x)$; find $g'(x)$ and *then* remember what other thing g is.) Draw a picture!

***24.** Prove that if f is odd, then $f'(x) = f'(-x)$. Once again, draw a picture.

25. Problems 23 and 24 say that f' is even if f is odd, and odd if f is even. What can therefore be said about $f^{(k)}$?

26. Find $f''(x)$ if

(i) $f(x) = x^3$.
(ii) $f(x) = x^5$.
(iii) $f'(x) = x^4$.
(iv) $f(x+3) = x^5$.

27. If $S_n(x) = x^n$, and $0 \leq k \leq n$, prove that

$$\begin{aligned} S_n{}^{(k)}(x) &= \frac{n!}{(n-k)!} x^{n-k} \\ &= k! \binom{n}{k} x^{n-k}. \end{aligned}$$

***28.** (a) Find $f'(x)$ if $f(x) = |x|^3$. Find $f''(x)$. Does $f'''(x)$ exist for all x?
(b) Analyze f similarly if $f(x) = x^4$ for $x \geq 0$ and $f(x) = -x^4$ for $x \leq 0$.

***29.** Let $f(x) = x^n$ for $x \geq 0$ and let $f(x) = 0$ for $x \leq 0$. Prove that $f^{(n-1)}$ exists (and find a formula for it), but that $f^{(n)}(0)$ does not exist.

30. Interpret the following specimens of Leibnizian notation; each is a restatement of some fact occurring in a previous problem.

(i) $\dfrac{dx^n}{dx} = nx^{n-1}$

(ii) $\dfrac{dz}{dy} = -\dfrac{1}{y^2}$ if $z = \dfrac{1}{y}$.

(iii) $\dfrac{d[f(x)+c]}{dx} = \dfrac{df(x)}{dx}$.

(iv) $\dfrac{d[cf(x)]}{dx} = c\dfrac{df(x)}{dx}$.

(v) $\dfrac{dz}{dx} = \dfrac{dy}{dx}$ if $z = y + c$.

(vi) $\left.\dfrac{dx^3}{dx}\right|_{x=a^2} = 3a^4$.

(vii) $\left.\dfrac{df(x+a)}{dx}\right|_{x=b} = \left.\dfrac{df(x)}{dx}\right|_{x=b+a}$.

(viii) $\left.\dfrac{df(cx)}{dx}\right|_{x=b} = c \cdot \left.\dfrac{df(x)}{dx}\right|_{x=cb}$.

(ix) $\dfrac{df(cx)}{dx} = c \cdot \left.\dfrac{df(y)}{dy}\right|_{y=cx}$.

(x) $\dfrac{d^k x^n}{dx^k} = k!\dbinom{n}{k}x^{n-k}$.