Strictly speaking, these equations may be false, because the functions on the left-hand side might have a larger domain than those on the right. Nevertheless, this is hardly worth worrying about. If $f$ and $g$ are differentiable everywhere in their domains, then these equations, and others like them, *are* true, and this is the only case any one cares about.

## PROBLEMS

**1.** As a warm up exercise, find $f'(x)$ for each of the following $f$. (Don't worry about the domain of $f$ or $f'$; just get a formula for $f'(x)$ that gives the right answer when it makes sense.)

(i) $f(x) = \sin(x + x^2)$.
(ii) $f(x) = \sin x + \sin x^2$.
(iii) $f(x) = \sin(\cos x)$.
(iv) $f(x) = \sin(\sin x)$.
(v) $f(x) = \sin\left(\dfrac{\cos x}{x}\right)$.
(vi) $f(x) = \dfrac{\sin(\cos x)}{x}$.
(vii) $f(x) = \sin(x + \sin x)$.
(viii) $f(x) = \sin(\cos(\sin x))$.

**2.** Find $f'(x)$ for each of the following functions $f$. (It took the author 20 minutes to compute the derivatives for the answer section, and it should not take you much longer. Although rapid calculation is not the goal of mathematics, if you hope to treat theoretical applications of the Chain Rule with aplomb, these concrete applications should be child's play—mathematicians like to pretend that they can't even add, but most of them can when they have to.)

(i) $f(x) = \sin((x + 1)^2(x + 2))$.
(ii) $f(x) = \sin^3(x^2 + \sin x)$.
(iii) $f(x) = \sin^2((x + \sin x)^2)$.
(iv) $f(x) = \sin\left(\dfrac{x^3}{\cos x^3}\right)$.
(v) $f(x) = \sin(x \sin x) + \sin(\sin x^2)$.
(vi) $f(x) = (\cos x)^{31^2}$.
(vii) $f(x) = \sin^2 x \sin x^2 \sin^2 x^2$.
(viii) $f(x) = \sin^3(\sin^2(\sin x))$.
(ix) $f(x) = (x + \sin^5 x)^6$.
(x) $f(x) = \sin(\sin(\sin(\sin(\sin x))))$.
(xi) $f(x) = \sin((\sin^7 x^7 + 1)^7)$.
(xii) $f(x) = (((x^2 + x)^3 + x)^4 + x)^5$.
(xiii) $f(x) = \sin(x^2 + \sin(x^2 + \sin x^2))$.
(xiv) $f(x) = \sin(6\cos(6\sin(6\cos 6x)))$.

(xv) $f(x) = \dfrac{\sin x^2 \sin^2 x}{1 + \sin x}$.

(xvi) $f(x) = \dfrac{1}{x - \dfrac{2}{x + \sin x}}$.

(xvii) $f(x) = \sin\left(\dfrac{x^3}{\sin\left(\dfrac{x^3}{\sin x}\right)}\right)$.

(xviii) $f(x) = \sin\left(\dfrac{x}{x - \sin\left(\dfrac{x}{x - \sin x}\right)}\right)$.

3. Find the derivatives of the functions tan, cotan, sec, cosec. (You don't have to memorize these formulas, although they will be needed once in a while; if you express your answers in the right way, they will be simple and somewhat symmetrical.)

4. For each of the following functions $f$, find $f'(f(x))$ (*not* $(f \circ f)'(x)$).

   (i) $f(x) = \dfrac{1}{1+x}$.
   (ii) $f(x) = \sin x$.
   (iii) $f(x) = x^2$.
   (iv) $f(x) = 17$.

5. For each of the following functions $f$, find $f(f'(x))$.

   (i) $f(x) = \dfrac{1}{x}$.
   (ii) $f(x) = x^2$.
   (iii) $f(x) = 17$.
   (iv) $f(x) = 17x$.

6. Find $f'$ in terms of $g'$ if

   (i) $f(x) = g(x + g(a))$.
   (ii) $f(x) = g(x \cdot g(a))$.
   (iii) $f(x) = g(x + g(x))$.
   (iv) $f(x) = g(x)(x - a)$.
   (v) $f(x) = g(a)(x - a)$.
   (vi) $f(x + 3) = g(x^2)$.

7. (a) A circular object is increasing in size in some unspecified manner, but it is known that when the radius is 6, the rate of change of the radius is 4. Find the rate of change of the area when the radius is 6. (If $r(t)$ and $A(t)$ represent the radius and the area at time $t$, then the functions $r$ and $A$ satisfy $A = \pi r^2$; a straightforward use of the Chain Rule is called for.)

(b) Suppose that we are now informed that the circular object we have been watching is really the cross section of a spherical object. Find the rate of change of the *volume* when the radius is 6. (You will clearly need to know a formula for the volume of a sphere; in case you have forgotten, the volume is $\frac{4}{3}\pi$ times the cube of the radius.)

(c) Now suppose that the rate of change of the area of the circular cross section is 5 when the radius is 3. Find the rate of change of the volume when the radius is 3. You should be able to do this problem in two ways: first, by using the formulas for the area and volume in terms of the radius; and then by expressing the volume in terms of the area (to use this method you will need Problem 9-3).

**8.** The area between two varying concentric circles is at all times $9\pi$ in$^2$. The rate of change of the area of the larger circle is $10\pi$ in$^2$/sec. How fast is the circumference of the smaller circle changing when it has area $16\pi$ in$^2$?

**9.** Particle $A$ moves along the positive horizontal axis, and particle $B$ along the graph of $f(x) = -\sqrt{3}x$, $x \le 0$. At a certain time, $A$ is at the point (5,0) and moving with speed 3 units/sec; and $B$ is at a distance of 3 units from the origin and moving with speed 4 units/sec. At what rate is the distance between $A$ and $B$ changing?

**10.** Let $f(x) = x^2 \sin 1/x$ for $x \ne 0$, and let $f(0) = 0$. Suppose also that $h$ and $k$ are two functions such that

$$\begin{aligned} &h'(x) = \sin^2(\sin(x+1)) \qquad && k'(x) = f(x+1) \\ &h(0) = 3 && k(0) = 0. \end{aligned}$$

Find

(i) $(f \circ h)'(0)$.
(ii) $(k \circ f)'(0)$.
(iii) $\alpha'(x^2)$, where $\alpha(x) = h(x^2)$. Exercise great care.

**11.** Find $f'(0)$ if

$$f(x) = \begin{cases} g(x) \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0, \end{cases}$$

and

$$g(0) = g'(0) = 0.$$

**12.** Using the derivative of $f(x) = 1/x$, as found in Problem 9-1, find $(1/g)'(x)$ by the Chain Rule.

**13.** (a) Using Problem 9-3, find $f'(x)$ for $-1 < x < 1$, if $f(x) = \sqrt{1-x^2}$.

(b) Prove that the tangent line to the graph of $f$ at $(a, \sqrt{1-a^2})$ intersects the graph only at that point (and thus show that the elementary geometry definition of the tangent line coincides with ours).

**14.** Prove similarly that the tangent lines to an ellipse or hyperbola intersect these sets only once.

**15.** If $f + g$ is differentiable at $a$, are $f$ and $g$ necessarily differentiable at $a$? If $f \cdot g$ and $f$ are differentiable at $a$, what conditions on $f$ imply that $g$ is differentiable at $a$?

**16.** (a) Prove that if $f$ is differentiable at $a$, then $|f|$ is also differentiable at $a$, provided that $f(a) \neq 0$.
(b) Give a counterexample if $f(a) = 0$.
(c) Prove that if $f$ and $g$ are differentiable at $a$, then the functions $\max(f, g)$ and $\min(f, g)$ are differentiable at $a$, provided that $f(a) \neq g(a)$.
(d) Give a counterexample if $f(a) = g(a)$.

**17.** If $f$ is three times differentiable and $f'(x) \neq 0$, the *Schwarzian derivative* of $f$ at $x$ is defined to be

$$\mathfrak{D}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

(a) Show that

$$\mathfrak{D}(f \circ g) = [\mathfrak{D}f \circ g] \cdot g'^2 + \mathfrak{D}g.$$

(b) Show that if $f(x) = \dfrac{ax+b}{cx+d}$, with $ad - bc \neq 0$, then $\mathfrak{D}f = 0$. Consequently, $\mathfrak{D}(f \circ g) = \mathfrak{D}g$.

**18.** Suppose that $f^{(n)}(a)$ and $g^{(n)}(a)$ exist. Prove *Leibniz's formula*:

$$(f \cdot g)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) \cdot g^{n-k}(a).$$

***19.** Prove that if $f^{(n)}(g(a))$ and $g^{(n)}(a)$ both exist, then $(f \circ g)^{(n)}(a)$ exists. A little experimentation should convince you that it is unwise to seek a formula for $(f \circ g)^{(n)}(a)$. In order to prove that $(f \circ g)^{(n)}(a)$ exists you will therefore have to devise a reasonable assertion about $(f \circ g)^{(n)}(a)$ which can be proved by induction. Try something like: "$(f \circ g)^{(n)}(a)$ exists and is a sum of terms each of which is a product of terms of the form ... ."

**20.** (a) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, find a function $g$ such that $g' = f$. Find another.
(b) If

$$f(x) = \frac{b_2}{x^2} + \frac{b_3}{x^3} + \cdots + \frac{b_m}{x^m},$$

find a function $g$ with $g' = f$.
(c) Is there a function

$$f(x) = a_n x^n + \cdots + a_0 + \frac{b_1}{x} + \cdots + \frac{b_m}{x^m}$$

such that $f'(x) = 1/x$?

**21.** Show that there is a polynomial function $f$ of degree $n$ such that

(a) $f'(x) = 0$ for precisely $n - 1$ numbers $x$.
(b) $f'(x) = 0$ for no $x$, if $n$ is odd.
(c) $f'(x) = 0$ for exactly one $x$, if $n$ is even.
(d) $f'(x) = 0$ for exactly $k$ numbers $x$, if $n - k$ is odd.

**22.** (a) The number $a$ is called a **double root** of the polynomial function $f$ if $f(x) = (x - a)^2 g(x)$ for some polynomial function $g$. Prove that $a$ is a double root of $f$ if and only if $a$ is a root of both $f$ and $f'$.
(b) When does $f(x) = ax^2 + bx + c$ $(a \neq 0)$ have a double root? What does the condition say geometrically?

**23.** If $f$ is differentiable at $a$, let $d(x) = f(x) - f'(a)(x - a) - f(a)$. Find $d'(a)$. In connection with Problem 22, this gives another solution for Problem 9-20.

***24.** This problem is a companion to Problem 3-6. Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be given numbers.

(a) If $x_1, \dots, x_n$ are distinct numbers, prove that there is a polynomial function $f$ of degree $2n - 1$, such that $f(x_j) = f'(x_j) = 0$ for $j \neq i$, and $f(x_i) = a_i$ and $f'(x_i) = b_i$. Hint: Remember Problem 22.
(b) Prove that there is a polynomial function $f$ of degree $2n - 1$ with $f(x_i) = a_i$ and $f'(x_i) = b_i$ for all $i$.

***25.** Suppose that $a$ and $b$ are two consecutive roots of a polynomial function $f$, but that $a$ and $b$ are not double roots, so that we can write $f(x) = (x - a)(x - b)g(x)$ where $g(a) \neq 0$ and $g(b) \neq 0$.

(a) Prove that $g(a)$ and $g(b)$ have the same sign. (Remember that $a$ and $b$ are consecutive roots.)
(b) Prove that there is some number $x$ with $a < x < b$ and $f'(x) = 0$. (Also draw a picture to illustrate this fact.) Hint: Compare the sign of $f'(a)$ and $f'(b)$.
(c) Now prove the same fact, even if $a$ and $b$ are multiple roots. Hint: If $f(x) = (x - a)^m (x - b)^n g(x)$ where $g(a) \neq 0$ and $g(b) \neq 0$, consider the polynomial function $h(x) = f'(x)/(x - a)^{m-1}(x - b)^{n-1}$.

This theorem was proved by the French mathematician Rolle, in connection with the problem of approximating roots of polynomials, but the result was not originally stated in terms of derivatives. In fact, Rolle was one of the mathematicians who never accepted the new notions of calculus. This was not such a pigheaded attitude, in view of the fact that for one hundred years no one could define limits in terms that did not verge on the mystic, but on the whole history has been particularly kind to Rolle; his name has become attached to a much more general result, to appear in the next chapter, which forms the basis for the most important theoretical results of calculus.

**26.** Suppose that $f(x) = xg(x)$ for some function $g$ which is continuous at 0. Prove that $f$ is differentiable at 0, and find $f'(0)$ in terms of $g$.

***27.** Suppose $f$ is differentiable at 0, and that $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function $g$ which is continuous at 0. Hint: What happens if you try to write $g(x) = f(x)/x$?

**28.** If $f(x) = x^{-n}$ for $n$ in $\mathbf{N}$, prove that

$$\begin{aligned} f^{(k)}(x) &= (-1)^k \frac{(n+k-1)!}{(n-1)!} x^{-n-k} \\ &= (-1)^k k! \binom{n+k-1}{k-1} x^{-n-k}, \qquad \text{for } x \neq 0. \end{aligned}$$

***29.** Prove that it is impossible to write $x = f(x)g(x)$ where $f$ and $g$ are differentiable and $f(0) = g(0) = 0$. Hint: Differentiate.

**30.** What is $f^{(k)}(x)$ if

(a) $f(x) = 1/(x-a)^n$?
*(b) $f(x) = 1/(x^2-1)$?

***31.** Let $f(x) = x^{2n} \sin 1/x$ if $x \neq 0$, and let $f(0) = 0$. Prove that $f'(0), \ldots, f^{(n)}(0)$ exist, and that $f^{(n)}$ is not continuous at 0. (You will encounter the same basic difficulty as that in Problem 19.)

***32.** Let $f(x) = x^{2n+1} \sin 1/x$ if $x \neq 0$, and let $f(0) = 0$. Prove that $f'(0), \ldots, f^{(n)}(0)$ exist, that $f^{(n)}$ is continuous at 0, and that $f^{(n)}$ is not differentiable at 0.

**33.** In Leibnizian notation the Chain Rule ought to read:

$$\frac{df(g(x))}{dx} = \left.\frac{df(y)}{dy}\right|_{y=g(x)} \cdot \frac{dg(x)}{dx}.$$

Instead, one usually finds the following statement: "Let $y = g(x)$ and $z = f(y)$. Then

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.\text{"}$$

Notice that the $z$ in $dz/dx$ denotes the composite function $f \circ g$, while the $z$ in $dz/dy$ denotes the function $f$; it is also understood that $dz/dy$ will be "an expression involving $y$," and that in the final answer $g(x)$ must be substituted for $y$. In each of the following cases, find $dz/dx$ by using this formula; then compare with Problem 1.

(i) $z = \sin y, \quad y = x + x^2.$
(ii) $z = \sin y, \quad y = \cos x.$
(iii) $z = \cos u, \quad u = \sin x.$
(iv) $z = \sin v, \quad v = \cos u, \quad u = \sin x.$