PROOF The hypothesis that $\lim_{x \to a} f'(x)/g'(x)$ exists contains two implicit assumptions:

(1) there is an interval $(a - \delta, a + \delta)$ such that $f'(x)$ and $g'(x)$ exist for all x in $(a - \delta, a + \delta)$ except, perhaps, for $x = a$,
(2) in this interval $g'(x) \neq 0$ with, once again, the possible exception of $x = a$.

On the other hand, f and g are not even assumed to be defined at a. If we define $f(a) = g(a) = 0$ (changing the previous values of $f(a)$ and $g(a)$, if necessary), then f and g are continuous at a. If $a < x < a + \delta$, then the Mean Value Theorem and the Cauchy Mean Value Theorem apply to f and g on the interval $[a, x]$ (and a similar statement holds for $a - \delta < x < a$). First applying the Mean Value Theorem to g, we see that $g(x) \neq 0$, for if $g(x) = 0$ there would be some x_1 in (a, x) with $g'(x_1) = 0$, contradicting (2). Now applying the Cauchy Mean Value Theorem to f and g, we see that there is a number α_x in (a, x) such that

$$[f(x) - 0]g'(\alpha_x) = [g(x) - 0]f'(\alpha_x)$$

or

$$\frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)}.$$

Now α_x approaches a as x approaches a, because α_x is in (a, x); since $\lim_{y \to a} f'(y)/g'(y)$ exists, it follows that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\alpha_x)}{g'(\alpha_x)} = \lim_{y \to a} \frac{f'(y)}{g'(y)}.$$

(Once again, the reader is invited to supply the details of this part of the argument.) ▮

PROBLEMS

1. For each of the following functions, find the maximum and minimum values on the indicated intervals, by finding the points in the interval where the derivative is 0, and comparing the values at these points with the values at the end points.

(i) $f(x) = x^3 - x^2 - 8x + 1$ on $[-2, 2]$.
(ii) $f(x) = x^5 + x + 1$ on $[-1, 1]$.
(iii) $f(x) = 3x^4 - 8x^3 + 6x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.
(iv) $f(x) = \dfrac{1}{x^5 + x + 1}$ on $[-\frac{1}{2}, 1]$.
(v) $f(x) = \dfrac{x + 1}{x^2 + 1}$ on $[-1, \frac{1}{2}]$.
(vi) $f(x) = \dfrac{x}{x^2 - 1}$ on $[0, 5]$.

2. Now sketch the graph of each of the functions in Problem 1, and find all local maximum and minimum points.

3. Sketch the graphs of the following functions.

(i) $f(x) = x + \dfrac{1}{x}$.

(ii) $f(x) = x + \dfrac{3}{x^2}$.

(iii) $f(x) = \dfrac{x^2}{x^2 - 1}$.

(iv) $f(x) = \dfrac{1}{1 + x^2}$.

4. (a) If $a_1 < \cdots < a_n$, find the minimum value of $f(x) = \sum_{i=1}^{n} (x - a_i)^2$.

*(b) Now find the minimum value of $f(x) = \sum_{i=1}^{n} |x - a_i|$. This is a problem where calculus won't help at all: on the intervals between the a_i's the function f is linear, so that the minimum clearly occurs at one of the a_i, and these are precisely the points where f is not differentiable. However, the answer is easy to find if you consider how $f(x)$ changes as you pass from one such interval to another.

*(c) Let $a > 0$. Show that the maximum value of

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - a|}$$

is $(2 + a)/(1 + a)$. (The derivative can be found on each of the intervals $(-\infty, 0)$, $(0, a)$, and (a, ∞) separately.)

5. For each of the following functions, find all local maximum and minimum points.

(i) $$f(x) = \begin{cases} x, & x \neq 3, 5, 7, 9 \\ 5, & x = 3 \\ -3, & x = 5 \\ 9, & x = 7 \\ 7, & x = 9. \end{cases}$$

(ii) $$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/q, & x = p/q \text{ in lowest terms.} \end{cases}$$

(iii) $$f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$$

(iv) $$f(x) = \begin{cases} 1, & x = 1/n \text{ for some } n \text{ in } \mathbf{N} \\ 0, & \text{otherwise.} \end{cases}$$

(v) $$f(x) = \begin{cases} 1, & \text{if the decimal expansion of } x \text{ contains a 5} \\ 0, & \text{otherwise.} \end{cases}$$

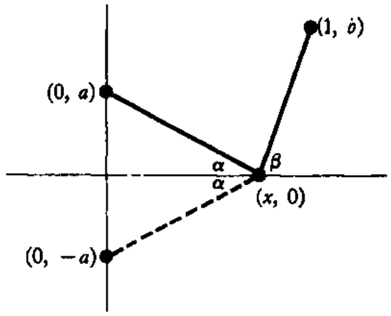

FIGURE 23

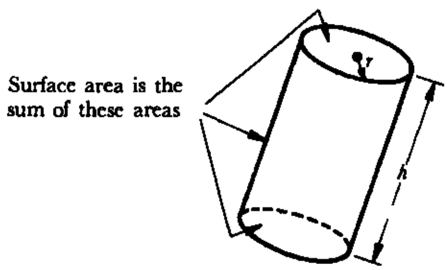

FIGURE 24

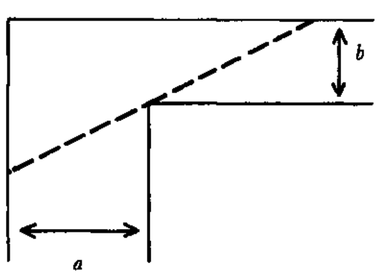

FIGURE 25

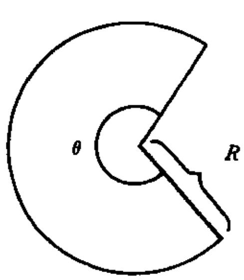

FIGURE 26

6. (a) Let (x_0, y_0) be a point of the plane, and let L be the graph of the function $f(x) = mx + b$. Find the point $\bar{x}$ such that the distance from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$ is smallest. [Notice that minimizing this distance is the same as minimizing its square. This may simplify the computations somewhat.]

(b) Also find $\bar{x}$ by noting that the line from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$ is perpendicular to L.

(c) Find the distance from (x_0, y_0) to L, i.e., the distance from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$. [It will make the computations easier if you first assume that $b = 0$; then apply the result to the graph of $f(x) = mx$ and the point $(x_0, y_0 - b)$.] Compare with Problem 4-22.

(d) Consider a straight line described by the equation $Ax + By + C = 0$ (Problem 4-7). Show that the distance from (x_0, y_0) to this line is $(Ax_0 + By_0 + C)/\sqrt{A^2 + B^2}$.

7. The previous Problem suggests the following question: What is the relationship between the critical points of f and those of f^2?

8. A straight line is drawn from the point $(0, a)$ to the horizontal axis, and then back to $(1, b)$, as in Figure 23. Prove that the total length is shortest when the angles α and β are equal. (Naturally you must bring a function into the picture: express the length in terms of x, where $(x, 0)$ is the point on the horizontal axis. The dashed line in Figure 23 suggests an alternative geometric proof; in either case the problem can be solved without actually finding the point $(x, 0)$.)

9. Prove that of all rectangles with given perimeter, the square has the greatest area.

10. Find, among all right circular cylinders of fixed volume V, the one with smallest surface area (counting the areas of the faces at top and bottom, as in Figure 24).

11. A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.

12. Two hallways, of widths a and b, meet at right angles (Figure 25). What is the greatest possible length of a ladder which can be carried horizontally around the corner?

13. A garden is to be designed in the shape of a circular sector (Figure 26), with radius R and central angle θ. The garden is to have a fixed area A. For what value of R and θ (in radians) will the length of the fencing around the perimeter be minimized?

14. Show that the sum of a positive number and its reciprocal is at least 2.

15. Find the trapezoid of largest area that can be inscribed in a semicircle of radius a, with one base lying along the diameter.

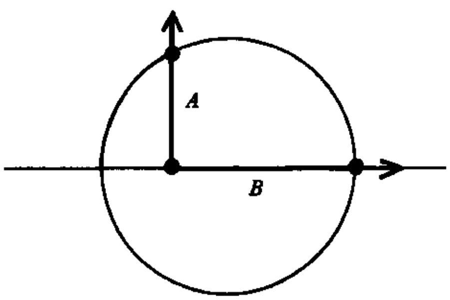

FIGURE 27

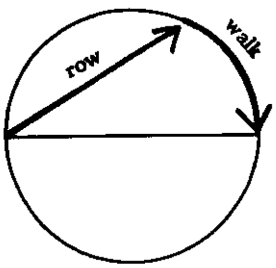

FIGURE 28

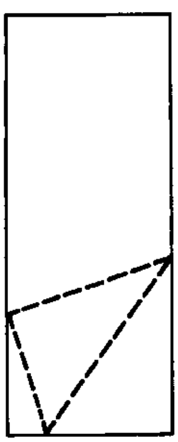

FIGURE 29

16. A right angle is moved along the diameter of a circle of radius a, as shown in Figure 27. What is the greatest possible length $(A + B)$ intercepted on it by the circle?

17. Ecological Ed must cross a circular lake of radius 1 mile. He can row across at 2 mph or walk around at 4 mph, or he can row part way and walk the rest (Figure 28). What route should he take so as to

(i) see as much scenery as possible?
(ii) cross as quickly as possible?

18. The lower right-hand corner of a page is folded over so that it just touches the left edge of the paper, as in Figure 29. If the width of the paper is α and the page is very long, show that the minimum length of the crease is $3\sqrt{3}\alpha/4$.

19. Figure 30 shows the graph of the *derivative* of f. Find all local maximum and minimum points of f.

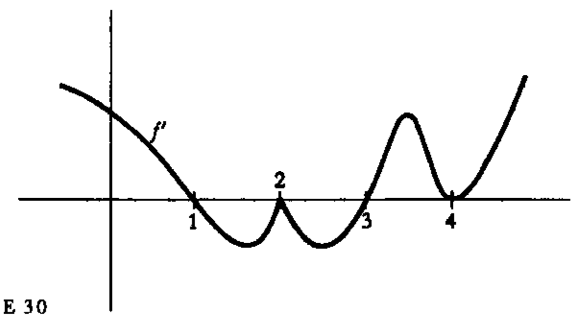

FIGURE 30

***20.** Suppose that f is a polynomial function, $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with critical points -1, 1, 2, 3, 4, and corresponding critical values 6, 1, 2, 4, 3. Sketch the graph of f, distinguishing the cases n even and n odd.

***21.** (a) Suppose that the critical points of the polynomial function $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ are $-1, 1, 2, 3$, and $f''(-1) = 0$, $f''(1) > 0$, $f''(2) < 0$, $f''(3) = 0$. Sketch the graph of f as accurately as possible on the basis of this information.

(b) Does there exist a polynomial function with the above properties, except that 3 is not a critical point?

22. Describe the graph of a rational function (in very general terms, similar to the text's description of the graph of a polynomial function).

23. (a) Prove that two polynomial functions of degree m and n, respectively, intersect in at most $\max(m, n)$ points.

(b) For each m and n exhibit two polynomial functions of degree m and n which intersect $\max(m, n)$ times.

***24.** (a) Suppose that the polynomial function $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has exactly k critical points and $f''(x) \neq 0$ for all critical points x. Show that $n - k$ is odd.

(b) For each n, show that there is a polynomial function f of degree n with k critical points if $n-k$ is odd.

(c) Suppose that the polynomial function $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has k_1 local maximum points and k_2 local minimum points. Show that $k_2 = k_1 + 1$ if n is even, and $k_2 = k_1$ if n is odd.

(d) Let n, k_1, k_2 be three integers with $k_2 = k_1 + 1$ if n is even, and $k_2 = k_1$ if n is odd, and $k_1 + k_2 < n$. Show that there is a polynomial function f of degree n, with k_1 local maximum points and k_2 local minimum points. Hint: Pick $a_1 < a_2 < \cdots < a_{k_1+k_2}$ and try $f'(x) = \prod_{i=1}^{k_1+k_2}(x-a_i)\cdot(1+x^2)^l$ for an appropriate number l.

25. (a) Prove that if $f'(x) \geq M$ for all x in $[a, b]$, then $f(b) \geq f(a) + M(b-a)$.

(b) Prove that if $f'(x) \leq M$ for all x in $[a, b]$, then $f(b) \leq f(a) + M(b-a)$.

(c) Formulate a similar theorem when $|f'(x)| \leq M$ for all x in $[a, b]$.

***26.** Suppose that $f'(x) \geq M > 0$ for all x in $[0, 1]$. Show that there is an interval of length $\frac{1}{4}$ on which $|f| \geq M/4$.

27. (a) Suppose that $f'(x) > g'(x)$ for all x, and that $f(a) = g(a)$. Show that $f(x) > g(x)$ for $x > a$ and $f(x) < g(x)$ for $x < a$.

(b) Show by an example that these conclusions do not follow without the hypothesis $f(a) = g(a)$.

28. Find all functions f such that

(a) $f'(x) = \sin x$.

(b) $f''(x) = x^3$.

(c) $f'''(x) = x + x^2$.

29. Although it is true that a weight dropped from rest will fall $s(t) = 16t^2$ feet after t seconds, this experimental fact does not mention the behavior of weights which are thrown upwards or downwards. On the other hand, the law $s''(t) = 32$ is always true and has just enough ambiguity to account for the behavior of a weight released from any height, with any initial velocity. For simplicity let us agree to measure heights upwards from ground level; in this case velocities are positive for rising bodies and negative for falling bodies, and all bodies fall according to the law $s''(t) = -32$.

(a) Show that s is of the form $s(t) = -16t^2 + \alpha t + \beta$.

(b) By setting $t = 0$ in the formula for s, and then in the formula for s', show that $s(t) = -16t^2 + v_0 t + s_0$, where s_0 is the height from which the body is released at time 0, and v_0 is the velocity with which it is released.

(c) A weight is thrown upwards with velocity v feet per second, at ground level. How high will it go? ("How high" means "what is the maximum height for all times".) What is its velocity at the moment it achieves its greatest height? What is its acceleration at that moment? When will it hit the ground again? What will its velocity be when it hits the ground again?

30. A cannon ball is shot from the ground with velocity v at an angle α (Figure 31) so that it has a vertical component of velocity $v \sin\alpha$ and a horizontal component $v\cos\alpha$. Its distance $s(t)$ above the ground obeys the law $s(t) = -16t^2 + (v\sin\alpha)t$, while its horizontal velocity remains constantly $v\cos\alpha$.

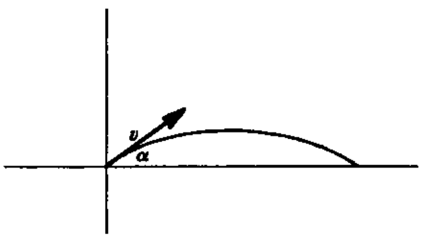

FIGURE 31

(a) Show that the path of the cannon ball is a parabola (find the position at each time t, and show that these points lie on a parabola).

(b) Find the angle α which will maximize the horizontal distance traveled by the cannon ball before striking the ground.

31. (a) Give an example of a function f for which $\lim_{x\to\infty} f(x)$ exists, but $\lim_{x\to\infty} f'(x)$ does not exist.

(b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

(c) Prove that if $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f''(x)$ exists, then $\lim_{x\to\infty} f''(x) = 0$. (See also Problem 20-15.)

32. Suppose that f and g are two differentiable functions which satisfy $fg' - f'g = 0$. Prove that if a and b are adjacent zeros of f, and $g(a)$ and $g(b)$ are not both 0, then $g(x) = 0$ for some x between a and b. (Naturally the same result holds with f and g interchanged; thus, the zeros of f and g separate each other.) Hint: Derive a contradiction from the assumption that $g(x) \neq 0$ for all x between a and b: if a number is not 0, there is a natural thing to do with it.

33. Suppose that $|f(x) - f(y)| \le |x-y|^n$ for $n > 1$. Prove that f is constant by considering f'. Compare with Problem 3-20.

34. A function f is *Lipschitz of order* α at x if there is a constant C such that

$$(*) \qquad |f(x) - f(y)| \le C|x-y|^\alpha$$

for all y in an interval around x. The function f is *Lipschitz of order* α *on an interval* if $(*)$ holds for all x and y in the interval.

(a) If f is Lipschitz of order $\alpha > 0$ at x, then f is continuous at x.

(b) If f is Lipschitz of order $\alpha > 0$ on an interval, then f is uniformly continuous on this interval (see Chapter 8, Appendix).

(c) If f is differentiable at x, then f is Lipschitz of order 1 at x. Is the converse true?

(d) If f is differentiable on $[a, b]$, is f Lipschitz of order 1 on $[a, b]$?

(e) If f is Lipschitz of order $\alpha > 1$ on $[a, b]$, then f is constant on $[a, b]$.

35. Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

then

$$a_0 + a_1 x + \cdots + a_n x^n = 0$$

for some x in $[0, 1]$.

36. Prove that the polynomial function $f_m(x) = x^3 - 3x + m$ never has two roots in $[0, 1]$, no matter what m may be. (This is an easy consequence of Rolle's Theorem. It is instructive, after giving an analytic proof, to graph f_0 and f_2, and consider where the graph of f_m lies in relation to them.)

37. Suppose that f is continuous and differentiable on $[0, 1]$, that $f(x)$ is in $[0, 1]$ for each x, and that $f'(x) \neq 1$ for all x in $[0, 1]$. Show that there is exactly one number x in $[0, 1]$ such that $f(x) = x$. (Half of this problem has been done already, in Problem 7-11.)

38. (a) Prove that the function $f(x) = x^2 - \cos x$ satisfies $f(x) = 0$ for precisely two numbers x.

(b) Prove the same for the function $f(x) = 2x^2 - x\sin x - \cos^2 x$. (Some preliminary estimates will be useful to restrict the possible location of the zeros of f.)

***39.** (a) Prove that if f is a twice differentiable function with $f(0) = 0$ and $f(1) = 1$ and $f'(0) = f'(1) = 0$, then $|f''(x)| \geq 4$ for some x in $[0, 1]$. In more picturesque terms: A particle which travels a unit distance in a unit time, and starts and ends with velocity 0, has at some time an acceleration ≥ 4. Hint: Prove that either $f''(x) > 4$ for some x in $[0, \frac{1}{2}]$, or else $f''(x) < -4$ for some x in $[\frac{1}{2}, 1]$.

(b) Show that in fact we must have $|f''(x)| > 4$ for some x in $[0, 1]$.

40. Suppose that f is a function such that $f'(x) = 1/x$ for all $x > 0$ and $f(1) = 0$. Prove that $f(xy) = f(x) + f(y)$ for all $x, y > 0$. Hint: Find $g'(x)$ when $g(x) = f(xy)$.

***41.** Suppose that f satisfies

$$f''(x) + f'(x)g(x) - f(x) = 0$$

for some function g. Prove that if f is 0 at two points, then f is 0 on the interval between them. Hint: Use Theorem 6.

42. Suppose that f is n-times differentiable and that $f(x) = 0$ for $n+1$ different x. Prove that $f^{(n)}(x) = 0$ for some x.

43. Let $a_1, \ldots, a_{n+1}$ be arbitrary points in $[a, b]$, and let

$$Q(x) = \prod_{i=1}^{n+1} (x - x_i).$$

Suppose that f is $(n+1)$-times differentiable and that P is a polynomial function of degree $\leq n$ such that $P(x_i) = f(x_i)$ for $i = 1, \ldots, n+1$ (see page 49). Show that for each x in $[a, b]$ there is a number c in (a, b) such that

$$f(x) - P(x) = Q(x) \cdot \frac{f^{(n+1)}(c)}{(n+1)!}.$$

Hint: Consider the function

$$F(t) = Q(x)[f(t) - P(t)] - Q(t)[f(x) - P(x)].$$

Show that F is zero at $n+2$ different points in $[a, b]$, and use Problem 42.

44. Prove that

$$\tfrac{1}{9} < \sqrt{66} - 8 < \tfrac{1}{8}$$

(without computing $\sqrt{66}$ to 2 decimal places!).

45. Prove the following slight generalization of the Mean Value Theorem: If f is continuous and differentiable on (a, b) and $\lim_{y \to a^+} f(y)$ and $\lim_{y \to b^-} f(y)$ exist, then there is some x in (a, b) such that

$$f'(x) = \frac{\lim_{y \to b^-} f(y) - \lim_{y \to a^+} f(y)}{b - a}.$$

(Your proof should begin: "This is a trivial consequence of the Mean Value Theorem because ... ".)

46. Prove that the conclusion of the Cauchy Mean Value Theorem can be written in the form

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)},$$

under the additional assumptions that $g(b) \neq g(a)$ and that $f'(x)$ and $g'(x)$ are never simultaneously 0 on (a, b).

***47.** Prove that if f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for x in (a, b), then there is some x in (a, b) with

$$\frac{f'(x)}{g'(x)} = \frac{f(x) - f(a)}{g(b) - g(x)}.$$

Hint: Multiply out first, to see what this really says.

48. What is wrong with the following use of l'Hôpital's Rule:

$$\lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2} = 3.$$

(The limit is actually -4.)

49. Find the following limits:

(i) $\lim_{x\to 0} \dfrac{x}{\tan x}$.

(ii) $\lim_{x\to 0} \dfrac{\cos^2 x - 1}{x^2}$.

50. Find $f'(0)$ if

$$f(x) = \begin{cases} \dfrac{g(x)}{x}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

and $g(0) = g'(0) = 0$ and $g''(0) = 17$.

51. Prove the following forms of l'Hôpital's Rule (none requiring any essentially new reasoning).

(a) If $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$, and $\lim_{x\to a^+} f'(x)/g'(x) = l$, then $\lim_{x\to a^+} f(x)/g(x) = l$ (and similarly for limits from below).

(b) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, and $\lim_{x\to a} f'(x)/g'(x) = \infty$, then $\lim_{x\to a} f(x)/g(x) = \infty$ (and similarly for $-\infty$, or if $x \to a$ is replaced by $x \to a^+$ or $x \to a^-$).

(c) If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, and $\lim_{x\to\infty} f'(x)/g'(x) = l$, then $\lim_{x\to\infty} f(x)/g(x) = l$ (and similarly for $-\infty$). Hint: Consider $\lim_{x\to 0^+} f(1/x)/g(1/x)$.

(d) If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, and $\lim_{x\to\infty} f'(x)/g'(x) = \infty$, then $\lim_{x\to\infty} f(x)/g(x) = \infty$.

52. There is another form of l'Hôpital's Rule which requires more than algebraic manipulations: If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to\infty} f'(x)/g'(x) = l$, then $\lim_{x\to\infty} f(x)/g(x) = l$. Prove this as follows.

(a) For every $\varepsilon > 0$ there is a number a such that

$$\left| \frac{f'(x)}{g'(x)} - l \right| < \varepsilon \quad \text{for } x > a.$$

Apply the Cauchy Mean Value Theorem to f and g on $[a, x]$ to show that

$$\left| \frac{f(x) - f(a)}{g(x) - g(a)} - l \right| < \varepsilon \quad \text{for } x > a.$$

(Why can we assume $g(x) - g(a) \neq 0$?)

(b) Now write

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} \cdot \frac{f(x)}{f(x)-f(a)} \cdot \frac{g(x)-g(a)}{g(x)}$$

(why can we assume that $f(x) - f(a) \neq 0$ for large x?) and conclude that

$$\left| \frac{f(x)}{g(x)} - l \right| < 2\varepsilon \quad \text{for sufficiently large } x.$$

53. To complete the orgy of variations on l'Hôpital's Rule, use Problem 52 to prove a few more cases of the following general statement (there are so many possibilities that you should select just a few, if any, that interest you):

If $\lim_{x\to[\ ]} f(x) = \lim_{x\to[\ ]} g(x) = \{\quad\}$ and $\lim_{x\to[\ ]} f'(x)/g'(x) = (\quad)$, then $\lim_{x\to[\ ]} f(x)/g(x) = (\quad)$. Here $[\ ]$ can be a or a^+ or a^- or ∞ or $-\infty$, and $\{\quad\}$ can be 0 or ∞ or $-\infty$, and $(\quad)$ can be l or ∞ or $-\infty$.

***54.** (a) Suppose that f is differentiable on $[a, b]$. Prove that if the minimum of f on $[a, b]$ is at a, then $f'(a) \geq 0$, and if it is at b, then $f'(b) \leq 0$. (One half of the proof of Theorem 1 will go through.)

(b) Suppose that $f'(a) < 0$ and $f'(b) > 0$. Show that $f'(x) = 0$ for some x in (a, b). Hint: Consider the minimum of f on $[a, b]$; why must it be somewhere in (a, b)?

(c) Prove that if $f'(a) < c < f'(b)$, then $f'(x) = c$ for some x in (a, b). (This result is known as Darboux's Theorem.) Hint: Cook up an appropriate function to which part (b) may be applied.

55. Suppose that f is differentiable in some interval containing a, but that f' is discontinuous at a.

(a) The one-sided limits $\lim_{x\to a^+} f'(x)$ and $\lim_{x\to a^-} f'(x)$ cannot both exist. (This is just a minor variation on Theorem 7.)

(b) Neither of these one-sided limits can exist even in the sense of being $+\infty$ or $-\infty$. Hint: Use Darboux's Theorem (Problem 54).

***56.** It is easy to find a function f such that $|f|$ is differentiable but f is not. For example, we can choose $f(x) = 1$ for x rational and $f(x) = -1$ for x irrational. In this example f is not even continuous, nor is this a mere coincidence: Prove that if $|f|$ is differentiable at a, and f is continuous at a, then f is also differentiable at a. Hint: It suffices to consider only a with $f(a) = 0$. Why? In this case, what must $|f|'(a)$ be?

***57.** (a) Let $y \neq 0$ and let n be even. Prove that $x^n + y^n = (x + y)^n$ only when $x = 0$. Hint: If $x_0{}^n + y^n = (x_0 + y)^n$, apply Rolle's Theorem to $f(x) = x^n + y^n - (x + y)^n$ on $[0, x_0]$.

(b) Prove that if $y \neq 0$ and n is odd, then $x^n + y^n = (x+y)^n$ only if $x = 0$ or $x = -y$.

****58.** Use the method of Problem 57 to prove that if n is even and $f(x) = x^n$, then every tangent line to f intersects f only once.

****59.** Prove even more generally that if f' is increasing, then every tangent line intersects f only once.

***60.** Suppose that $f(0) = 0$ and f' is increasing. Prove that the function $g(x) = f(x)/x$ is increasing on $(0, \infty)$. Hint: Obviously you should look at $g'(x)$. Prove that it is positive by applying the Mean Value Theorem to f on the right interval (it will help to remember that the hypothesis $f(0) = 0$ is essential, as shown by the function $f(x) = 1 + x^2$).

***61.** Use derivatives to prove that if $n \geq 1$, then

$$(1+x)^n > 1 + nx \quad \text{for } -1 < x < 0 \text{ and } 0 < x$$

(notice that equality holds for $x = 0$).

62. Let $f(x) = x^4 \sin^2 1/x$ for $x \neq 0$, and let $f(0) = 0$ (Figure 32).

(a) Prove that 0 is a local minimum point for f.
(b) Prove that $f'(0) = f''(0) = 0$.

This function thus provides another example to show that Theorem 6 cannot be improved. It also illustrates a subtlety about maxima and minima that often goes unnoticed: a function may not be increasing in any interval to the right of a local minimum point, nor decreasing in any interval to the left.

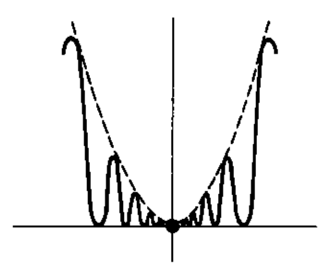

FIGURE 32

***63.** (a) Prove that if $f'(a) > 0$ and f' is continuous at a, then f is increasing in some interval containing a.

The next two parts of this problem show that continuity of f' is essential.

(b) If $g(x) = x^2 \sin 1/x$, show that there are numbers x arbitrarily close to 0 with $g'(x) = 1$ and also with $g'(x) = -1$.

(c) Suppose $0 < \alpha < 1$. Let $f(x) = \alpha x + x^2 \sin 1/x$ for $x \neq 0$, and let $f(0) = 0$ (see Figure 33). Show that f is not increasing in any open interval containing 0, by showing that in any interval there are points x with $f'(x) > 0$ and also points x with $f'(x) < 0$.

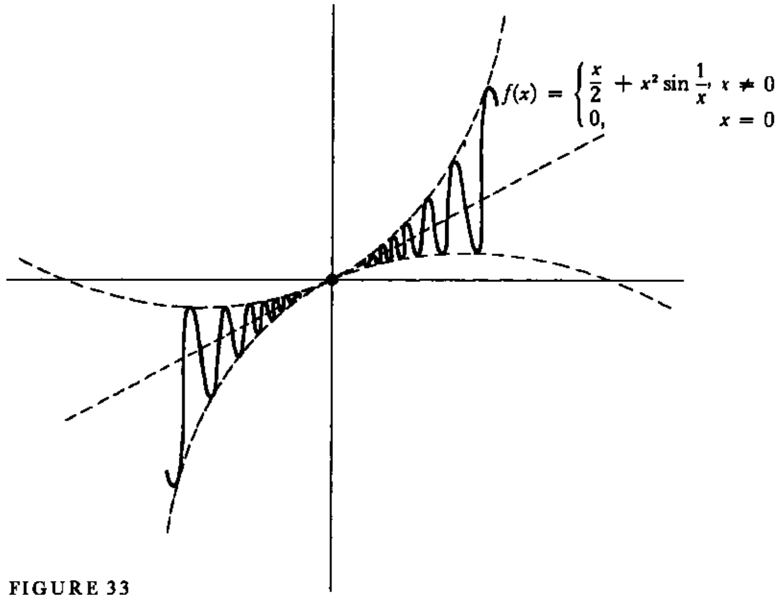

FIGURE 33

The behavior of f for $\alpha \geq 1$, which is much more difficult to analyze, is discussed in the next problem.

****64.** Let $f(x) = \alpha x + x^2 \sin 1/x$ for $x \neq 0$, and let $f(0) = 0$. In order to find the sign of $f'(x)$ when $\alpha \geq 1$ it is necessary to decide if $2x \sin 1/x - \cos 1/x$ is < -1 for any numbers x close to 0. It is a little more convenient to consider the function $g(y) = 2(\sin y)/y - \cos y$ for $y \neq 0$; we want to know if $g(y) < -1$ for large y. This question is quite delicate; the most significant part of $g(y)$ is $-\cos y$, which does reach the value -1, but this happens only when $\sin y = 0$, and it is not at all clear whether g itself can have values < -1. The obvious approach to this problem is to find the local minimum values of g. Unfortunately, it is impossible to solve the equation $g'(y) = 0$ explicitly, so more ingenuity is required.

(a) Show that if $g'(y) = 0$, then

$$\cos y = (\sin y)\left(\frac{2 - y^2}{2y}\right),$$

and conclude that

$$g(y) = (\sin y)\left(\frac{2 + y^2}{2y}\right).$$

(b) Now show that if $g'(y) = 0$, then

$$\sin^2 y = \frac{4y^2}{4+y^4},$$

and conclude that

$$|g(y)| = \frac{2+y^2}{\sqrt{4+y^4}}.$$

(c) Using the fact that $(2+y^2)/\sqrt{4+y^4} > 1$, show that if $\alpha = 1$, then f is not increasing in any interval around 0.

(d) Using the fact that $\lim_{y\to\infty} (2+y^2)/\sqrt{4+y^4} = 1$, show that if $\alpha > 1$, then f *is* increasing in some interval around 0.

65. A function f is **increasing at a if there is some number $\delta > 0$ such that

$$f(x) > f(a) \quad \text{if} \quad a < x < a + \delta$$

and

$$f(x) < f(a) \quad \text{if} \quad a - \delta < x < a.$$

Notice that this does *not* mean that f is increasing in the interval $(a-\delta, a+\delta)$; for example, the function shown in Figure 33 is increasing at 0, but is not an increasing function in any open interval containing 0.

(a) Suppose that f is continuous on $[0, 1]$ and that f is increasing at a for every a in $[0, 1]$. Prove that f is increasing on $[0, 1]$. (First convince yourself that there is something to be proved.) Hint: For $0 < b < 1$, prove that the minimum of f on $[b, 1]$ must be at b.

(b) Prove part (a) without the assumption that f is continuous, by considering for each b in $[0, 1]$ the set $S_b = \{x : f(y) \geq f(b) \text{ for all } y \text{ in } [b, x]\}$. (This part of the problem is not necessary for the other parts.) Hint: Prove that $S_b = \{x : b \leq x \leq 1\}$ by considering $\sup S_b$.

(c) If f is increasing at a and f is differentiable at a, prove that $f'(a) \geq 0$ (this is easy).

(d) If $f'(a) > 0$, prove that f is increasing at a (go right back to the definition of $f'(a)$).

(e) Use parts (a) and (d) to show, without using the Mean Value Theorem, that if f is continuous on $[0, 1]$ and $f'(a) > 0$ for all a in $[0, 1]$, then f is increasing on $[0, 1]$.

(f) Suppose that f is continuous on $[0, 1]$ and $f'(a) = 0$ for all a in $(0, 1)$. Apply part (e) to the function $g(x) = f(x) + \varepsilon x$ to show that $f(1) - f(0) > -\varepsilon$. Similarly, show that $f(1) - f(0) < \varepsilon$ by considering $h(x) = \varepsilon x - f(x)$. Conclude that $f(0) = f(1)$.

This particular proof that a function with zero derivative must be constant has many points in common with a proof of H. A. Schwarz, which may be the first rigorous proof ever given. Its discoverer, at least, seemed to think it was. See his exuberant letter in reference [40] of the Suggested Reading.

****66.** (a) If f is a constant function, then every point is a local maximum point for f. It is quite possible for this to happen even if f is not a constant function: for example, if $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. But prove, using Problem 8-4, that if f is continuous on $[a, b]$ and every point of $[a, b]$ is a local maximum point, then f is a constant function. The same result holds, of course, if every point of $[a, b]$ is a local minimum point.

(b) Suppose now that every point is either a local maximum or a local minimum point for f (but we don't preclude the possibility that some points are local maxima while others are local minima). Prove that f is constant, as follows. Suppose that $f(a_0) < f(b_0)$. We can assume that $f(a_0) < f(x) < f(b_0)$ for $a_0 < x < b_0$. (Why?) Using Theorem 1 of the Appendix to Chapter 8, partition $[a_0, b_0]$ into intervals on which $\sup f - \inf f < (f(b_0) - f(a_0))/2$; also choose the lengths of these intervals to be less than $(b_0 - a_0)/2$. Then there is one such interval $[a_1, b_1]$ with $a_0 < a_1 < b_1 < b_0$ and $f(a_1) < f(b_1)$. (Why?) Continue inductively and use the Nested Interval Theorem (Problem 8-14) to find a point x that cannot be a local maximum or minimum.

****67.** (a) A point x is called a **strict maximum point** for f on A if $f(x) > f(y)$ for all y in A with $y \neq x$ (compare with the definition of an ordinary maximum point). A **local strict maximum point** is defined in the obvious way. Find all local strict maximum points of the function

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \dfrac{1}{q}, & x = \dfrac{p}{q} \text{ in lowest terms.} \end{cases}$$

It seems quite unlikely that a function can have a local strict maximum at *every* point (although the above example might give one pause for thought). Prove this as follows.

(b) Suppose that every point is a local strict maximum point for f. Let x_1 be any number and choose $a_1 < x_1 < b_1$ with $b_1 - a_1 < 1$ such that $f(x_1) > f(x)$ for all x in $[a_1, b_1]$. Let $x_2 \neq x_1$ be any point in (a_1, b_1) and choose $a_1 \leq a_2 < x_2 < b_2 \leq b_1$ with $b_2 - a_2 < \frac{1}{2}$ such that $f(x_2) > f(x)$ for all x in $[a_2, b_2]$. Continue in this way, and use the Nested Interval Theorem (Problem 8-14) to obtain a contradiction.