Then $x_0 = a_0$ and $x_1 = a_1$ and (Problem 4-2) the points x_t all lie between a_0 and a_1, with analogous statements for y_t and z_t. Moreover,

$$x_t < y_t < z_t \qquad \text{for} \qquad 0 \le t \le 1.$$

Now consider the function

$$g(t) = \frac{f(y_t) - f(x_t)}{y_t - x_t} - \frac{f(z_t) - f(x_t)}{z_t - x_t} \qquad \text{for } 0 \le t \le 1.$$

By step (1), $g(t) \neq 0$ for all t in $[0, 1]$. So either $g(t) > 0$ for all t in $[0, 1]$ or $g(t) < 0$ for all t in $[0, 1]$. Thus, either f is convex or f is concave (compare pages 231–232). ▌

PROBLEMS

1. Sketch, indicating regions of convexity and concavity and points of inflection, the functions in Problem 11-1 (consider (iv) as double starred).

2. Figure 30 in Chapter 11 shows the graph of f'. Sketch the graph of f.

3. Find two convex functions f and g such that $f(x) = g(x)$ if and only if x is an integer.

4. Show that f is convex on an interval if and only if for all x and y in the interval we have

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y), \qquad \text{for } 0 < t < 1.$$

(This is just a restatement of the definition, but a useful one.)

5. (a) Prove that if f and g are convex and f is increasing, then $f \circ g$ is convex. (It will be easiest to use Problem 4.)
(b) Give an example where $g \circ f$ is not convex.
(c) Suppose that f and g are twice differentiable. Give another proof of the result of part (a) by considering second derivatives.

6. (a) Suppose that f is differentiable and convex on an interval. Show that either f is increasing, or else f is decreasing, or else there is a number c such that f is decreasing to the left of c and increasing to the right of c.
(b) Use this fact to give another proof of the result in Problem 5(a) when f and g are (one-time) differentiable. (You will have to be a little careful when comparing $f'(g(x))$ and $f'(g(y))$ for $x < y$.)
(c) Prove the result in part (a) without assuming f differentiable. You will have to keep track of several different cases, but no particularly clever ideas are needed. Begin by showing that if $a < b$ and $f(a) < f(b)$, then f is increasing to the right of b; and if $f(a) > f(b)$, then f is decreasing to the left of a.

***7.** Let f be a twice-differentiable function with the following properties: $f(x) > 0$ for $x \ge 0$, and f is decreasing, and $f'(0) = 0$. Prove that

$f''(x) = 0$ for some $x > 0$ (so that in reasonable cases f will have an inflection point at x—an example is given by $f(x) = 1/(1+x^2)$). Every hypothesis in this theorem is essential, as shown by $f(x) = 1 - x^2$, which is not positive for all x; by $f(x) = x^2$, which is not decreasing; and by $f(x) = 1/(x+1)$, which does not satisfy $f'(0) = 0$. Hint: Choose $x_0 > 0$ with $f'(x_0) < 0$. We cannot have $f'(y) \leq f'(x_0)$ for all $y > x_0$. Why not? So $f'(x_1) > f'(x_0)$ for some $x_1 > x_0$. Consider f' on $[0, x_1]$.

***8.** (a) Prove that if f is convex, then $f([x+y]/2) < [f(x) + f(y)]/2$.

(b) Suppose that f satisfies this condition. Show that $f(kx + (1-k)y) < kf(x) + (1-k)f(y)$ whenever k is a rational number, between 0 and 1, of the form $m/2^n$. Hint: Part (a) is the special case $n = 1$. Use induction, employing part (a) at each step.

(c) Suppose that f satisfies the condition in part (a) and f is continuous. Show that f is convex.

***9.** Let $p_1, \dots, p_n$ by *positive* numbers with $\sum_{i=1}^{n} p_i = 1$.

(a) For any numbers $x_1, \dots, x_n$ show that $\sum_{i=1}^{n} p_i x_i$ lies between the smallest and the largest x_i.

(b) Show the same for $(1/t) \sum_{i=1}^{n-1} p_i x_i$, where $t = \sum_{i=1}^{n-1} p_i$.

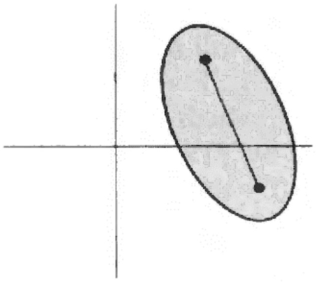
(c) Prove *Jensen's inequality:* If f is convex, then $f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i)$. Hint: Use Problem 4, noting that $p_n = 1 - t$. (Part (b) is needed to show that $(1/t)\sum_{i=1}^{n-1} p_i x_i$ is in the domain of f if $x_1, \dots, x_n$ are.)

***10.** (a) For any function f, the right-hand derivative, $\lim_{h \to 0^+} [f(a+h) - f(a)]/h$, is denoted by $f_+{}'(a)$, and the left-hand derivative is denoted by $f_-{}'(a)$. The proof of Theorem 1 actually shows that $f_+{}'$ and $f_-{}'$ always exist if f is convex. Check this assertion, and also show that $f_+{}'$ and $f_-{}'$ are increasing, and that $f_-{}'(a) \leq f_+{}'(a)$.

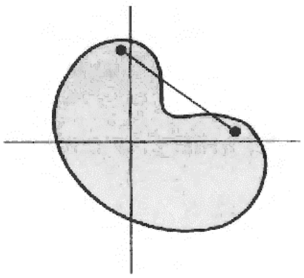
**(b) Show that if f is convex, then $f_+{}'(a) = f_-{}'(a)$ if and only if $f_+{}'$ is continuous at a. (Thus f is differentiable precisely when $f_+{}'$ is continuous.) Hint: $[f(b) - f(a)]/(b-a)$ is close to $f_-{}'(a)$ for $b < a$ close to a, and $f_+{}'(b)$ is less than this quotient.

***11.** (a) Prove that a convex function on $\mathbf{R}$, or on any open interval, must be continuous.

(b) Give an example of a convex function on a closed interval that is *not* continuous, and explain exactly what kinds of discontinuities are possible.



(a) a convex subset of the plane



(b) a nonconvex subset of the plane

FIGURE 14

12. Call a function f *weakly convex* on an interval if for $a < b < c$ in this interval we have
$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}.$$

(a) Show that a weakly convex function is convex if and only if its graph contains no straight line segments. (Sometimes a weakly convex function is simply called "convex," while convex functions in our sense are called "strictly convex".)

(b) Reformulate the theorems of this section for weakly convex functions.

13. A set A of points in the plane is called *convex* if A contains the line segment joining any two points in it (Figure 14). For a function f, let A_f be the set of points (x, y) with $y \geq f(x)$, so that A_f is the set of points on or above the graph of f. Show that A is convex if and only if f is weakly convex, in the terminology of the previous problem. Further information on convex sets will be found in reference [19] of the Suggested Reading.