Geometria Quantitativa II

Lista 3

1) Mostre as igualdades:

i) $\dfrac{1+sen\,x}{1+cos\,x}\dfrac{1+sec\,x}{1+cosec\,x}=tg\,x$

ii) $\dfrac{cos\,x}{1+sen\,x}+\dfrac{1+sen\,x}{cos\,x}=2sec\,x$

iii) $\dfrac{1+cos\,x+cos\,2x}{sen\,x+sen\,2x}=cotg\,x$

iv) $\dfrac{1+sen\,x-cos\,x}{1+sen\,x+cos\,x}+\dfrac{1+sen\,x+cos\,x}{1+sen\,x-cos\,x}=2cosec\,x$

v) $\dfrac{1+tg^2x}{(1+tg\,x)^2}=\dfrac{1}{1+sen\,2x}$

vi) $\dfrac{1+tg\,x-sec\,x}{sec\,x+tg\,x-1}=\dfrac{1+sec\,x-tg\,x}{sec\,x+tg\,x+1}$

vii) $tg\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)=\dfrac{cos\,x}{1+sen\,x}=\dfrac{1-sen\,x}{cos\,x}$

viii) $tg^2\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)=\dfrac{1-sen\,x}{1+sen\,x}=(sec\,x-tg\,x)^2$

ix) $cotg^2x=\dfrac{1+cos\,x}{1-cos\,x}=\dfrac{sec\,x+1}{sec\,x-1}$

x) $(cosec\,x-cotg\,x)(cosec\,x+cotg\,x)=1$

2) Mostre as igualdades:

i) $cotg\,x+tg\,x=sec\,x\,cosec\,x$

ii) $cotg\,x-tg\,x=2cotg\,2x$

iii) $tg\,2x-tg\,x=tg\,x\,sec\,2x$

iv) $tg^2x-sen^2x=tg^2x\,sen^2x$

v) $1+tg^2x+cotg^2x=(sec^2x+tg\,x)(cosec^2x-cotg\,x)$

vi) Se θ não é um múltiplo de $\pi/2$ e

$$\begin{aligned} x &= 1+cos^2\theta+cos^4\theta+\dots \\ y &= 1+sen^2\theta+sen^4\theta+\dots \\ z &= 1+cos^2\theta\,sen^2\theta+cos^4\theta\,sen^4\theta+\dots \end{aligned}$$

prove que

$$x+y=xy \quad \text{e} \quad x+y+z=xyz$$

3) Mostre as igualdades:

i) $sen(x+y)sen(x-y) = sen^2 x - sen^2 y$

ii) $cos(x+y)cos(x-y) = cos^2 x - sen^2 y$

iii) $sen^2(x+y) - sen^2(x-y) = sen\,2x\,sen\,2y$

iv) $cos^2(x+y) + cos^2(x-y) = 1 + cos\,2x\,cos\,2y$

v) $tg(x+y) = \dfrac{sen\,x + \mu cos\,x}{cos\,x - \mu sen\,x}$, em que $\mu = tg\,y$

vi) $tg\left(\dfrac{x+y}{2}\right) = \dfrac{sen\,x + sen\,y}{cos\,x + cos\,y} = \dfrac{cos\,x - cos\,y}{sen\,y - sen\,x}$

vii) $tg\,x + tg\,y = \dfrac{sen(x+y)}{cos\,x\,cos\,y}$

viii) $\dfrac{sen\,x\,sen\,y}{cos\,x + cos\,y} = \dfrac{2\,tg\frac{x}{2}tg\frac{y}{2}}{1 - tg^2\frac{x}{2}tg^2\frac{y}{2}}$

ix) $\dfrac{cos\,x\,cos\,y}{cos\,x + cos\,y} = \dfrac{\left(1 - tg^2\frac{x}{2}\right)\left(1 - tg^2\frac{y}{2}\right)}{2\left(1 - tg^2\frac{x}{2}tg^2\frac{y}{2}\right)}$

4) Em todos os itens abaixo, considere um triângulo ΔABC:

i) $cos\,A + cos(B-C) = 2sen\,B\,sen\,C$

ii) $cos\dfrac{A}{2} + sen\dfrac{B-C}{2} = 2sen\dfrac{B}{2}cos\dfrac{C}{2}$

iii) $sen\,A + sen\,B + sen\,C = 4cos\dfrac{A}{2}cos\dfrac{B}{2}cos\dfrac{C}{2}$

iv) $sen\,2A + sen\,2B + sen\,2C = 4sen\,A\,sen\,B\,sen\,C$

v) $tg\,A + tg\,B + tg\,C = tg\,A\,tg\,B\,tg\,C$

vi) $cotg\dfrac{A}{2} + cotg\dfrac{B}{2} + cotg\dfrac{C}{2} = cotg\dfrac{A}{2}\,cotg\dfrac{B}{2}\,cotg\dfrac{C}{2}$

vii) $cotg\,A\,cotg\,B + cotg\,B\,cotg\,C + cotg\,C\,cotg\,A = 1$

viii) $AB \cdot sen(A-B) + BC \cdot sen(B-C) + CA \cdot sen(C-A) = 0$

5) Teorema de Ptolomeu: para um quadrilátero inscrito em uma circunferência, o produto das diagonais é igual à soma dos produtos dos lados opostos.

Demonstre o Teorema de Ptolomeu provando que

$$sen(x+y)\,sen(y+z) = sen\,x\,sen\,z + sen\,y\,sen(x+y+z)$$

6) Prove as igualdades:

i) $$\sum_{k=1}^{n} sen\, kx = \frac{sen\frac{n+1}{2}x}{sen\frac{x}{2}} sen\frac{nx}{2}$$

ii) $$\sum_{k=1}^{n} cos\, kx = \frac{cos\frac{n+1}{2}x}{sen\frac{x}{2}} sen\frac{nx}{2}$$

iii) $$\sum_{k=1}^{n} ksen\, kx = \frac{(n+1)sen\, nx - n\, sen(n+1)x}{4sen^2\frac{x}{2}}$$

iv) $$\sum_{k=1}^{n} kcos\, kx = \frac{(n+1)cos\, nx - n\, cos(n+1)x - 1}{4sen^2\frac{x}{2}}$$

v) $$\prod_{k=0}^{n} cos\, 2^k x = \frac{sen\, 2^{n+1}x}{2^{n+1}sen\, x}$$

vi) $$\sum_{k=1}^{n} \frac{1}{2^k} tg\, \frac{x}{2^k} = \frac{1}{2^n} cotg \frac{x}{2^n} - cotg\, x \;\; (x \neq m\pi)$$

vii) $$\sum_{k=1}^{n} arc\, cotg(2n+1) = -n\, arc\, tg\, 1 + \sum_{k=1}^{n} arc\, tg \frac{n+1}{n}$$

7) Resolva as equações e desigualdades:

i) $3cos\, x + 4sen\, x = 5$

ii) $cos\, x + sen\, x + 1 = 0$

iii) $1 + tg\, x/2 = 0$

iv) $tg\, x + sec\, x = 2$

v) $3sen\, 2x = 7 - 8cos^2 x$

vi) $sen\, x + sen\, 2x = cos\, x + cos\, 2x$

vii) $sen\, 2x > sen\, x \quad (0 < x < 2\pi)$

viii) $cos\, x + cos\, 2x > 3cos\, x - 4cos^3 x \quad (0 < x < 2\pi)$

ix) $2tg\, x + tg\, 2x > 3cotg\, x \quad (0 < x < 2\pi)$

8) i) Se $(a^2 - b^2)sen\, x + 2ab\, cos\, x = a^2 + b^2$, mostre que $tg\, x = \dfrac{a^2 - b^2}{2ab}$.

ii) Se $\dfrac{sen\, x}{a} = \dfrac{cos\, x}{b}$, mostre que $sen\, x - cos\, x = \dfrac{a - b}{\sqrt{a^2 + b^2}}$.