

# REPRESENTATIVE FUNCTIONS ON DISCRETE GROUPOIDS AND DUALITY WITH HOPF ALGEBROIDS.

LAIACHI EL KAOUTIT

ABSTRACT. The aim of this paper is to establish a duality between the category of discrete groupoids and the category of geometrically transitive commutative Hopf algebroids in the sense of P. Deligne and A. Bruguières. In one direction we have the usual contravariant functor which assigns to each Hopf algebroid its characters groupoid (the fiber groupoid at the ground field). In the other direction we construct the contravariant functor which associated to each discrete groupoid its Hopf algebroid of representative functions. This duality extends the well known duality between discrete groups and commutative Hopf algebras, and also sheds light on a new approach to Tannaka-Krein duality for compact topological groupoids. Our results are supported by several illustrative examples including topological ones.

## INTRODUCTION

The notion of smooth representative functions on a Lie group is a powerful tool whose usefulness is too remarkable in the theory of Lie groups. One of the aspects, perhaps the most impressive, where this notion is evidently indispensable, is Chevalley's reformulation of Tannaka's duality theorem which is itself a generalization of the classical Pontryagin duality theorem. Namely, it was shown by Chevalley in [9, Theorem 5, page 211] that Tannaka's duality theorem for a compact Lie group can be formulated as a theorem on the characters group of its Hopf algebra of (complex valued) representative functions. This was a very elegant way to provide any compact Lie group with a structure of algebraic real linear group.

These in fact are particular aspects of a general theory on compact topological groups and more general theory on discrete groups. Specifically, it is well known from Hochschild's result [14, Theorem 3.5, page 30] that the contravariant functor which assigns to each compact topological group its algebra of (real valued) continuous representative functions, establishes an anti-equivalence between the category of compact topological groups and the category of commutative real Hopf algebras with gauge (a Hopf integral coming from the Haar measure) and with dense characters group in the linear dual. When restrict to finitely generated real Hopf algebras one obtains an anti-equivalence with the category of compact Lie groups. Since any real Hopf algebra is a filtrated limit of its finitely generated Hopf subalgebras, one deduces from the previous anti-equivalence that any compact topological group is a projective limit of compact Lie groups, and so it is isomorphic to a closed subgroup of a product of algebraic real linear groups.

In the more abstract case of discrete groups, one only have a duality between groups and commutative Hopf algebras. Precisely, denote by  $\mathbf{Grp}$  the category of groups and by  $\mathbf{CHAlg}_{\mathbb{k}}$  the category of commutative Hopf algebras over a ground field  $\mathbb{k}$ . Consider the contravariant functors  $\mathcal{R}_{\mathbb{k}} : \mathbf{Grp} \rightarrow \mathbf{CHAlg}_{\mathbb{k}}$  (sending any group to its  $\mathbb{k}$ -valued representative functions) and  $\chi_{\mathbb{k}} : \mathbf{CHAlg}_{\mathbb{k}} \rightarrow \mathbf{Grp}$  (associating to any Hopf algebra its characters group). Then the pair  $(\mathcal{R}_{\mathbb{k}}, \chi_{\mathbb{k}})$  establishes a duality between the categories  $\mathbf{Grp}$  and  $\mathbf{CHAlg}_{\mathbb{k}}$ . This means that there are natural transformations  $id_{\mathbf{Grp}} \rightarrow \chi_{\mathbb{k}} \mathcal{R}_{\mathbb{k}}$  and  $id_{\mathbf{CHAlg}_{\mathbb{k}}} \rightarrow \mathcal{R}_{\mathbb{k}} \chi_{\mathbb{k}}$  satisfying the usual triangles. In other words, we have a natural isomorphism  $\mathrm{Hom}_{\mathbf{CHAlg}_{\mathbb{k}}}(-; \mathcal{R}_{\mathbb{k}}(+)) \cong \mathrm{Hom}_{\mathbf{Grp}}(+; \chi_{\mathbb{k}}(-))$ .

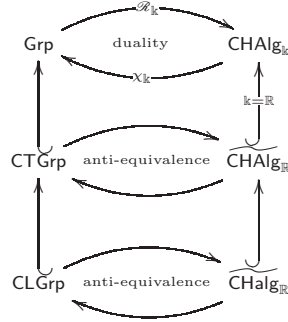
*Date:* April 3, 2019.

*2010 Mathematics Subject Classification.* Primary 18B40, 20L05, 20L15; Secondary 22A22, 14R20.

*Key words and phrases.* Topological Groupoids; Geometrically Transitive Hopf algebroids; Representative Functions; Monoidal Categories; Tannakian Categories; Tannaka-Krein Duality; Equivariant Vector Bundles.

Partially supported by grants MTM2010-20940-C02-01 from the Ministerio de Educación y Ciencia of Spain and FQM-266, P11-FQM-7156 from Junta de Andalucía.

With the pertinent notations, we can see that all the above aspects are then captured in the following diagram:



which in fact provides a kind of dictionary between the geometric properties of groups and algebraic properties of their associated commutative Hopf algebras. For instance, given a compact topological group  $G$  and takes the complex numbers as a base field, then there is a Galois correspondence between closed subgroups of  $G$  and subalgebras of  $\mathcal{R}_C(G)$  that contain the constants and are stable under conjugation and translations, see [13, §5].

As was expounded in [5, 27], the theory of groupoids in all its facets is, in some sense, a modern replacement of the theory of groups, which in fact supplies new techniques and tools for studying the symmetry and the structure of fairly complicated objects whose analysis through groups theory does not provide enough information. Under this point of view, a modern replacement of the algebraic counterpart of groups, i.e., commutative Hopf algebras, should be then commutative Hopf algebroids. In this way the above functor of characters group, should be then substituted by the functor of *characters groupoid*, which assigns to each Hopf algebroid its characters groupoid (i.e., the fiber groupoid at the ground field).

Employing the functor of characters groupoid, it is then natural and reasonable to propose a suitable generalization of the duality and anti-equivalences in the above diagram, to the contexts of groupoids and commutative Hopf algebroids. Thus, in the left hand-side column of an analogue diagram for groupoids, it is clear what the categories should be: discrete groupoids, compact topological groupoids and compact Lie groupoids. The main problem then lies on finding the adequate full subcategories of the category of commutative Hopf algebroids, in the right hand-side column, over which the functor of characters establishes a duality or an anti-equivalence (depending on the codomain category of groupoids). In the case of compact topological groupoids and specially for compact Lie groupoid, the search of such a subcategory is up to now more truncated and too difficult. Nevertheless, we solve hereby this problem for discrete groupoids.

The chief concerns of this paper is then to establish a duality between the category of discrete groupoids  $\text{Grpd}$  and a full subcategory of commutative Hopf algebroids. Namely, the category of geometrically transitive Hopf algebroids  $\text{GTCHAlgd}_k$ , see the forthcoming subsection for definitions. Explicitly, our main theorem sated below as Theorem 3.13 which we quote here, says:

**THEOREM I.** *Let  $k$  be a ground field. Then the contravariant functors of  $k$ -characters groupoid and  $k$ -valued representative functions*

$$\mathcal{X}_k : \text{GTCHAlgd}_k \rightleftarrows \text{Grpd} : \mathcal{R}_k$$

*establish a duality between the category of geometrically transitive commutative Hopf algebroids and the category of discrete groupoids. That is, there is a natural isomorphism:*

$$\text{Hom}_{\text{GTCHAlgd}_k} \left( (R, \mathcal{H}) ; (M_k(\mathcal{G}_0), \mathcal{R}_k(\mathcal{G})) \right) \cong \text{Hom}_{\text{Grpd}} \left( \mathcal{G} ; \mathcal{X}_k(R, \mathcal{H}) \right)$$

*for every  $(R, \mathcal{H}) \in \text{GTCHAlgd}_k$  and  $\mathcal{G} \in \text{Grpd}$  with base algebra  $M_k(\mathcal{G}_0) = \text{Maps}(\mathcal{G}_0, k)$ .*

The case of discrete groups, that is, the duality in the top of the above diagram, is then evidently recovered from Theorem I, since we already know that any commutative Hopf  $k$ -algebra is geometrically transitive.

Our techniques of proofs realise heavily on the theory of Tannakian  $\mathbb{k}$ -linear categories. Namely, we first show in Proposition 1.10 (perhaps a well known result) that the category of representations of a given discrete groupoid in finite dimensional  $\mathbb{k}$ -vector spaces is in fact a Tannakian  $\mathbb{k}$ -linear category, see the Appendix where we collect the essential definitions and basic properties of this notion. The resulting commutative Hopf algebroid from the so called the reconstruction process, is what hereby referred to as *the algebra of representative functions* on discrete groupoid, Corollary and Definition 2.1. The terminology is justified by showing that this algebra is in fact isomorphic to a subalgebra of the total algebra attached to this groupoid, Proposition 2.2.

This is mostly done in Sections 1 and 2, where we also treat the case of topological groupoids with compact Hausdorff base space, by showing how to construct its Hopf algebroids of continuous representative functions, Remark 2.6. Contrary to the discrete case, the (geometric) nature of this Hopf algebroid still unknown, at least for us. Nevertheless, we think that its construction surely is the first step through a correct way to establish an anti-equivalence between compact topological groupoids and certain full subcategory of commutative Hopf algebroids, Remark 3.14.

Several examples were expounded along the paper, highlighting the case of an algebraic action  $\mathbb{k}$ -groupoids ( $\mathbb{k}$  is an infinite field), where we were able to relate the *Equivariant Serre Problem* concerning the triviality of algebraic equivariant bundles [19], with some problems of its Hopf algebroid of polynomial representative functions, see Example 3.5 (and also Remark 1.9).

**Basic notions and notations.** We fix a ground field  $\mathbb{k}$ , all rings are considered to be  $\mathbb{k}$ -algebras. The set of all  $\mathbb{k}$ -algebra maps  $R \rightarrow S$  is denoted by  $R(S)$ . Since we are interested in (small) groupoids with set of objects in some cases is of the form  $R(\mathbb{k})$  for a commutative  $\mathbb{k}$ -algebra  $R$ . It is reasonable to assume that the set of object is a non empty set and hence to assume that all handled commutative  $\mathbb{k}$ -algebras  $R$  have the property that  $R(\mathbb{k}) \neq \emptyset$ .

Let  $\theta : R \rightarrow S$  be a morphism of commutative  $\mathbb{k}$ -algebras. We denote by  $\theta^* : \text{Mod}_R \rightarrow \text{Mod}_S$  the extension functor, i.e.,  $\theta^*(-) = - \otimes_R S$  and by  $\theta_* : \text{Mod}_S \rightarrow \text{Mod}_R$  the restriction functor.

All Hopf algebroids which will be considered here are commutative. Recall from [20, A.I] that a *commutative Hopf algebroid* is a pair of commutative  $\mathbb{k}$ -algebras  $(R, \mathcal{H})$  with  $R \neq 0$  and  $\mathbb{k}$ -algebra maps

$$R \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{\varepsilon} \\ \xrightarrow{t} \end{array} \mathcal{H}$$

where  $s$  is the *source* map,  $t$  is the *target* and  $\varepsilon$  is the counit. Together with a *comultiplication*  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes_R \mathcal{H}$ , where  $\mathcal{H}$  is considered as an  $R$ -bimodule with  $t$  acting on the left and  $s$  on the right, as well as an *antipode*  $\mathcal{S} : \mathcal{H} \rightarrow \mathcal{H}$ . All these maps are required to satisfy the usual compatibilities conditions in the Hopf context (dual in some sense to a groupoid axioms). In this way Hopf algebroids are then presheaves of groupoids on affine schemes. Under this point of view, there is a relation between the category of comodules and the category of quasi-coherent sheaves with a groupoid action, as was detailed in [16] and [10].

For a given Hopf algebroid  $(R, \mathcal{H})$  we denote by  $\text{Comod}_{\mathcal{H}}$  its category of right comodules and by  $\mathcal{U}_{\mathcal{H}} : \text{Comod}_{\mathcal{H}} \rightarrow \text{Mod}_R$  the forgetful functor. This is a symmetric monoidal category with  $\mathcal{U}_{\mathcal{H}}$  a strict monoidal functor. The category of left  $\mathcal{H}$ -comodules is monoidally isomorphic to right  $\mathcal{H}$ -comodules, and so enjoys similar properties. The full subcategory of  $\text{Comod}_{\mathcal{H}}$  whose objects are finitely generated as  $R$ -modules is denoted by  $\text{comod}_{\mathcal{H}}$ . It is well known that  $\mathcal{H}_R$  (or  ${}_R\mathcal{H}$ ) is flat iff  $\text{Comod}_{\mathcal{H}}$  is a Grothendieck category and  $\mathcal{U}_{\mathcal{H}}$  is an exact functor. Under any one of these equivalent conditions  $\mathcal{H}$  becomes faithfully flat simultaneously as left and right  $R$ -module.

A *morphism of Hopf algebroids* is a pair  $\alpha = (\alpha_0, \alpha_1) : (R, \mathcal{H}) \rightarrow (S, \mathcal{K})$  of  $\mathbb{k}$ -algebra maps which are in a canonical way compatible with both Hopf algebroids structures. The *extended morphism* of  $\alpha$ , is given by  $(id_S, \widetilde{\alpha}_1) : (S, S \otimes_R \mathcal{H} \otimes_R S) \rightarrow (S, \mathcal{K})$  sending  $p \otimes_R h \otimes_R q \mapsto s(p)\alpha_1(h)t(q)$ , where  $(S, S \otimes_R \mathcal{H} \otimes_R S)$  is endowed within a canonical structure of Hopf algebroid.

The morphism  $\alpha$  clearly induces a functor called *the induction functor*  $\alpha^* := - \otimes_R S : \text{Comod}_{\mathcal{H}} \rightarrow \text{Comod}_{\mathcal{K}}$  such that  $\mathcal{U}_{\mathcal{K}} \circ \alpha^* = \alpha_0^* \circ \mathcal{U}_{\mathcal{H}}$ . In other words, the functor  $\alpha^*$  is a *lifted functor* of the functor  $\alpha_0^*$ . Whether in general a functor  $F : \text{Comod}_{\mathcal{H}} \rightarrow \text{Comod}_{\mathcal{K}}$  is lifted from some functor  $\theta^*$ , where  $\theta : R \rightarrow S$  is an algebra map, is a quite interesting question from the Morita theory point of view, chiefly when  $F$  is asked to be monoidal.

In this paper we will often use *geometrically transitive Hopf algebroids*, a notion which was introduced by A. Bruguières in [6], see also [10]. For sake of completeness and audience convenience, we include here the definition of this notion. Although, as was mentioned above we will restrict our self here to the commutative case. Recall from [6, §5 Definition in page 5838] that a given commutative Hopf algebroid  $(R, \mathcal{H})$  with a base ring  $R$ , is said to be *semi-transitive* if the category of (say right) comodules over the underlying  $R$ -coring  $\mathcal{H}$  satisfies the following three conditions:

ST1 Each object in  $\text{comod}_{\mathcal{H}}$  is projective as  $R$ -module.

ST2 Each object in  $\text{Comod}_{\mathcal{H}}$  is a filtrate inductive limit of objects in  $\text{comod}_{\mathcal{H}}$ .

ST3 The  $\mathbb{k}$ -linear category  $\text{comod}_{\mathcal{H}}$  is locally of finite type.

The Hopf algebroid  $(R, \mathcal{H})$  is said to be *geometrically semi-transitive* if  $\mathcal{H}$  is projective as an  $(R \otimes_{\mathbb{k}} R)$ -module and satisfies condition ST3. By [6, Proposition 6.2], each geometrically semi-transitive  $R$ -coring is semi-transitive. In particular [6, Proposition 6.2(i)] implies that  $\mathcal{H}_R$  is a projective module whenever so is the module  $\mathcal{H}_{R \otimes_{\mathbb{k}} R}$ . Lastly,  $(R, \mathcal{H})$  is said to be *geometrically transitive*<sup>(1)</sup> if it is geometrically semi-transitive and  $\text{End}_{\text{comod}_{\mathcal{H}}}(R) \cong \mathbb{k}$  where  $R$  is considered as an  $\mathcal{H}$ -comodule using the grouplike element  $1_{\mathcal{H}}$ <sup>(2)</sup>. Notice that condition ST2 follows from the assumption that  $\mathcal{H}_R$  (or  ${}_R\mathcal{H}$ ) is a projective module, either by using the theory of rational modules over the convolution  $R$ -algebra of  $\mathcal{H}$  or directly by applying [6, Proposition 3.3].

Summing up, the result [6, Proposition 7.3] says that a commutative Hopf algebroid  $(R, \mathcal{H})$  is geometrically transitive iff  $\mathcal{H}$  is projective and faithfully flat  $(R \otimes_{\mathbb{k}} R)$ -module, iff  $(R_{\mathbb{k}}, \mathcal{H}_{\mathbb{k}})$  is transitive for any field extension  $\mathbb{k} \rightarrow \mathbb{K}$ , where  $R_{\mathbb{k}} = R \otimes_{\mathbb{k}} \mathbb{K}$  and  $\mathcal{H}_{\mathbb{k}} = \mathcal{H} \otimes_{\mathbb{k}} \mathbb{K}$ . Finally, it is noteworthy to mention that it is implicitly shown in [6, §8] that in our case, i.e., for commutative Hopf algebroid, a necessary and sufficient condition for  $(R, \mathcal{H})$  to be geometrically transitive is to be faithfully flat over  $R \otimes_{\mathbb{k}} R$ .

## 1. THE CATEGORY OF REPRESENTATIONS OF DISCRETE GROUPOIDS.

We will recall here the definition of the category of representations of discrete groupoid in  $\mathbb{k}$ -vector spaces. This is a formal adaptation of the category of sheaves over a presheaf of groupoids on schemes [10, 4, 16] or that of the category of equivariant sheaves over a topological action groupoids [3, 24]. In what follows all statements will be given for discrete groupoids. In the topological case we limit our self to make some remarks which illustrate where the difficulty lies in this case and perhaps offer a well understanding of this situation.

Our goal is to show that the category of representations of a discrete groupoid in finite dimensional  $\mathbb{k}$ -vector spaces, is a Tannakian  $\mathbb{k}$ -linear category, see the Appendix for definitions. For topological groupoids we will see that this category is instate a pseudo-Tannakian  $\mathbb{k}$ -linear category.

Recall that a groupoid  $\mathcal{G}$  is a small category where each arrow is an isomorphism. The source and target will be denoted by  $s, t : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ , and the map which assigns to each object its identity arrow is denoted by  $\iota : \mathcal{G}_0 \rightarrow \mathcal{G}_1$ . The inverse map and the composition as well as the rest of the axioms are supposed to be understood. In all what follows, a groupoid  $\mathcal{G}$  is implicitly assumed to have a non empty set objects  $\mathcal{G}_0 \neq \emptyset$ . A *morphism* between two groupoids is a functor between their underlying categories.

DEFINITION 1.1. Let  $\mathcal{G} : \mathcal{G}_1 \begin{matrix} \xrightarrow{s} \\ \xleftarrow{t} \\ \xrightarrow{\iota} \end{matrix} \mathcal{G}_0$  be a groupoid. A *representation* of  $\mathcal{G}$  or an *n-dimensional  $\mathcal{G}$ -representation* consists on the following data:

- 1)  $\mathcal{E} = \bigcup_{x \in \mathcal{G}_0} E_x$  a disjoint union of finite dimensional  $\mathbb{k}$ -vector spaces  $E_x$  such that there exists an  $n$ -dimensional  $\mathbb{k}$ -vector space  $V$  and linear isomorphisms  $\varphi_x : V \rightarrow E_x$ , for every  $x \in \mathcal{G}_0$ .
- 2) A family of linear isomorphisms  $\{\varrho_g^{\mathcal{E}}\}_{g \in \mathcal{G}_1}$  given by  $\varrho_g^{\mathcal{E}} : E_{s(g)} \rightarrow E_{t(g)}$ , for every  $g \in \mathcal{G}_1$  satisfying the cocycle condition. That is, for every  $x \in \mathcal{G}_0$  and every pair of composable arrows

<sup>(1)</sup> As was shown in [6, Théorème 8.2] this is equivalent to say that  $(\text{Spec}(\mathcal{H}), \text{Spec}(R))$  is a transitive affine  $\mathbb{k}$ -groupoid, in the sense of Deligne. That is,  $(s, t) : \text{Spec}(\mathcal{H}) \rightarrow \text{Spec}(R) \times_{\mathbb{k}} \text{Spec}(R)$  is a cover in the *fpqc* topology.

<sup>(2)</sup> Semi-transitive Hopf algebroid with this property is called *transitive*.

$g, h \in \mathcal{G}_1$ , we have

$$\varrho_{\iota(x)}^{\mathcal{E}} = id_{E_x}, \quad \varrho_{gh}^{\mathcal{E}} = \varrho_g^{\mathcal{E}} \circ \varrho_h^{\mathcal{E}}. \quad (1)$$

A  $\mathcal{G}$ -representation will be denoted by a pair  $(\mathcal{E}, \varrho^{\mathcal{E}})$  (the corresponding vector space  $V$  is implicitly understood) or simply by  $\varrho^{\mathcal{E}}$  if there is no danger of misunderstanding.

Given a family of finite dimensional  $\mathbb{k}$ -vector spaces  $\{E_x\}_{x \in \mathcal{G}_0}$  which satisfies condition 1) in Definition 1.1, we can consider the so called *frame groupoid*  $\Phi(\mathcal{E})$  whose set of objects is  $\mathcal{G}_0$  and the set of arrows is the set of linear isomorphisms between the (fibers)  $E_x$ 's. In this way, a  $\mathcal{G}$ -representation  $(\mathcal{E}, \varrho^{\mathcal{E}})$  is nothing but a morphism of groupoids (functor)  $\varrho : \mathcal{G} \rightarrow \Phi(\mathcal{E})$ , where  $\varrho(x) = x$  and  $\varrho(g) = \varrho_g^{\mathcal{E}}$ .

Take  $\mathcal{I} = \bigcup_{x \in \mathcal{G}_0} \mathcal{I}_x$ , with  $\mathcal{I}_x = \mathbb{k}$ , for every  $x \in \mathcal{G}_0$  and  $\varrho_g^{\mathcal{I}} = id_{\mathbb{k}}$ , for every  $g \in \mathcal{G}_1$ . Then obviously,  $\mathcal{I}$  is a 1-dimensional  $\mathcal{G}$ -representation. The zero dimensional  $\mathcal{G}$ -representation, is the representation with fibers the zero  $\mathbb{k}$ -vector space.

Consider two  $\mathcal{G}$ -representations  $(\mathcal{E}, \varrho^{\mathcal{E}})$  and  $(\mathcal{F}, \varrho^{\mathcal{F}})$ . A *morphism of representation* from  $(\mathcal{E}, \varrho^{\mathcal{E}})$  to  $(\mathcal{F}, \varrho^{\mathcal{F}})$  is a family of linear maps  $\{\alpha_x\}_{x \in \mathcal{G}_0}$ ,  $\alpha_x : E_x \rightarrow F_x$  rendering commutative the following diagrams

$$\begin{array}{ccc} E_{s(g)} & \xrightarrow{\varrho_g^{\mathcal{E}}} & E_{t(g)} \\ \alpha_{s(g)} \downarrow & & \downarrow \alpha_{t(g)} \\ F_{s(g)} & \xrightarrow{\varrho_g^{\mathcal{F}}} & F_{t(g)} \end{array} \quad (2)$$

We denote by  $\mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$  the category of  $\mathcal{G}$ -representations in  $\mathbb{k}$ -vector spaces. The  $\mathbb{k}$ -vector space of morphisms in this category will be denoted as usual by  $\text{Hom}_{\mathbf{Rep}_{\mathbb{k}}(\mathcal{G})}(\varrho^{\mathcal{E}}, \varrho^{\mathcal{F}})$ . Before going on given the properties of this category, let us discuss the topological case and give some elementary examples.

REMARK 1.2. Up to now we could perfectly define the category of representations of topological groupoids by taking the category of locally trivial vector bundles over  $\mathcal{G}_0$ . Precisely, given a topological groupoid  $\mathcal{G}$  (see for instance [21, Definition 2.1.]), a  $\mathcal{G}$ -representation is a three-tuple  $(\mathcal{E}, \pi_{\mathcal{E}}, \varrho^{\mathcal{E}})$  where  $\pi_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{G}_0$  is a locally trivial real (or complex) vector bundle and  $\varrho^{\mathcal{E}} : s^*\mathcal{E} \rightarrow t^*\mathcal{E}$  is an isomorphism between the induced (or pull-back) vector bundles which satisfies the cocycle conditions. A morphism between two  $\mathcal{G}$ -representations, is then a morphism of vector bundles  $\alpha : \mathcal{E} \rightarrow \mathcal{F}$  compatible with the actions, that is, satisfies  $t^*\alpha \circ \varrho^{\mathcal{F}} = \varrho^{\mathcal{E}} \circ s^*\alpha$ .

This category is denoted by  $\mathbf{Rep}_{\mathbb{k}}^{\text{top}}(\mathcal{G})$ , and clearly we have a functor  $\mathbf{Rep}_{\mathbb{k}}^{\text{top}}(\mathcal{G}) \rightarrow \mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$ . Of course if we consider our discrete groupoid as a discrete topological groupoids and the base field a discrete field (i.e. each finite dimensional vector space is discrete), then clearly Definition 1.1 is a particular instance of this one. That is, the previous functor is an equality.

There are also in the literature others specific notions of representations on some classes of topological groupoids. For example in the case of locally compact or Lie groupoids, one can speak about continuous representations, see [28, 22, 2, 7].

EXAMPLE 1.3. Let  $\mathcal{G}$  be an *action groupoid*<sup>(3)</sup>. This means that we are given a group  $G$  and say a left  $G$ -set  $X$  with action  $G \times X \rightarrow X$ ,  $(g, x) \mapsto gx$ , where we set  $\mathcal{G}_1 = G \times X$  and  $\mathcal{G}_0 = X$  with  $s(g, x) = x$ ,  $t(g, x) = gx$  and  $\iota(x) = (e, x)$  ( $e$  is the neutral element of  $G$ ). Thus a  $\mathcal{G}$ -representation is a family of  $n$ -dimensional  $\mathbb{k}$ -vector space  $\{E_x\}_{x \in X}$  (each of them is isomorphic to some fixed  $n$ -dimensional  $\mathbb{k}$ -vector space) and there is linear isomorphisms  $\varrho_g^{\mathcal{E}} : E_x \rightarrow E_{gx}$ , for every  $g \in G$ . In other words the fibers are 'equivariant' under the action of  $G$ .

If  $G$  is a topological group and  $X$  is a topological space such that the  $G$ -action is a continuous map ( $\mathbb{k}$  is the field of real or complex numbers). Then the category  $\mathbf{Rep}_{\mathbb{k}}^{\text{top}}(\mathcal{G})$  described in Remark 1.2, is nothing but the category of  $G$ -equivariant vector bundles (or  $G$ -vector bundles). In particular, if  $X = \{*\}$  is a one point set, then the category  $\mathbf{Rep}_{\mathbb{k}}^{\text{top}}(\mathcal{G})$  coincides with category of continuous  $G$ -representations. Of course, when  $G$  is a discrete group,  $X = \{*\}$  and  $\mathbb{k}$  is any

<sup>(3)</sup> *Semi-direct groupoid* in the terminology of [5].

field, then  $\mathbf{Rep}_{\mathbb{k}}(G)$  obviously coincides with the classical category of  $G$ -representations in finite dimensional  $\mathbb{k}$ -vector spaces.

EXAMPLE 1.4 (The additive and multiplicative  $\mathbb{k}$ -groupoids  $\mathcal{G}^{a,-}$  and  $\mathcal{G}^{m,-}$ ). Fix a commutative  $\mathbb{k}$ -algebra  $A$ . Given another commutative  $\mathbb{k}$ -algebra  $R$  we consider the following small category. The set of objects is  $A(R)$  the set of all  $\mathbb{k}$ -algebras maps from  $A$  to  $R$ . A morphism  $f : \sigma \rightarrow \gamma$  between two objects  $\sigma, \gamma \in A(R)$  is given by an element  $r(f) \in R$ . The identity arrow of an object  $\sigma$  is given by the element  $r(1_\sigma) = 0$ . The composition is the sum  $r(f \circ g) = r(f) + r(g)$ , and each arrow is invertible by taking  $r(f^{-1}) = -r(f)$ . We refer to this groupoid as *the additive groupoid of  $R$  over  $A$*  and denoted by  $\mathcal{G}^{a,A}(R)$ . The *multiplicative groupoid  $\mathcal{G}^{m,A}(R)$  of  $R$  over  $A$*  is constructed by taking the same set of object  $A(R)$  where a morphism  $f : \sigma \rightarrow \gamma$  is given in this case by an element  $s(f) \in \mathcal{U}(R)$  of the unit group of  $R$ . The identity arrow of  $\sigma$  is  $s(1_\sigma) = 1_R$  the identity element of  $R$ , the composition is the multiplication  $s(f \circ g) = s(f)s(g)$ , and each arrow  $f$  have an inverse given by  $s(f^{-1}) = s(f)^{-1}$ .

It is easily seen that  $\mathcal{G}^{a,A}$  and  $\mathcal{G}^{m,A}$  are affine  $\mathbb{k}$ -groupoids, respectively, represented by the geometrically transitive Hopf algebroids  $(A, (A \otimes_{\mathbb{k}} A)[T])$  and  $(A, (A \otimes_{\mathbb{k}} A)[T, T^{-1}])$ . Obviously, when  $A = \mathbb{k}$ , one recover the notions of additive and multiplicative affine  $\mathbb{k}$ -groups  $G_a$  and  $G_m$ .

Assume now that  $R$  and  $A$  are finite dimensional  $\mathbb{k}$ -algebras and consider the multiplicative groupoid  $\mathcal{G}^{m,A}(R)$  of  $R$  over  $A$ . For any object  $\sigma \in A(R)$  we put  $R_\sigma = R$  as a  $\mathbb{k}$ -vector space. Take the 'vector bundle'  $\mathcal{R} := \cup_{\sigma \in R(A)} R_\sigma$  and for any arrow  $f : \sigma \rightarrow \gamma$  in  $\mathcal{G}^{m,A}(R)_1$ , set  $\varrho_f^{\mathcal{R}} : R_\sigma \rightarrow R_\gamma$  by sending  $x \mapsto s(f)x$ . Then it is clear that  $(\mathcal{R}, \varrho^{\mathcal{R}})$  is a  $\mathcal{G}^{m,A}(R)$ -representation.

Now, we come back to the properties of the category of representations of a groupoid. Here are some basic well known facts in this category.

LEMMA 1.5. *Let  $\mathcal{G}$  be a groupoid. Then its category of representations  $\mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$ , is a  $\mathbb{k}$ -linear abelian symmetric rigid (or autonomous) monoidal category.*

*Proof.* The fact that  $\mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$  is an abelian category, follows directly from  $\mathbb{k}$ -vector spaces. Thus, the construction of kernels and cokernels as well as the reminder axioms of an abelian category can be checked fiberwise. For instance, given a morphism  $\alpha \in \mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$ , then one can easily show that  $\text{Ker}(\alpha) := \cup_{x \in \mathcal{G}_0} \text{Ker}(\alpha_x)$  and  $\text{Coker}(\alpha) := \cup_{x \in \mathcal{G}_0} \text{Coker}(\alpha_x)$  are the underlying sets, respectively, of the kernel and cokernel of  $\alpha$  in  $\mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$ . The rest of the statements follows also from the category of  $\mathbb{k}$ -vector spaces. The tensor product and duals are given as follows:

$$(\mathcal{E}, \varrho^{\mathcal{E}}) \otimes (\mathcal{F}, \varrho^{\mathcal{F}}) := (\mathcal{E} \otimes \mathcal{F}, \varrho^{\mathcal{E}} \otimes \varrho^{\mathcal{F}}) = \left( \bigcup_{x \in \mathcal{G}_0} E_x \otimes_{\mathbb{k}} F_x, \{ \varrho_g^{\mathcal{E}} \otimes_{\mathbb{k}} \varrho_g^{\mathcal{F}} \}_{g \in \mathcal{G}_1} \right) \quad (3)$$

$$(\mathcal{E}, \varrho^{\mathcal{E}})^* := (\mathcal{E}^*, \varrho^{\mathcal{E}^*}) = \left( \bigcup_{x \in \mathcal{G}_0} E_x^*, \{ [(\varrho_g^{\mathcal{E}})^*]^{-1} \}_{g \in \mathcal{G}_1} \right), \quad (4)$$

where we denote by  $X^* = \text{Hom}_{\mathbb{k}}(X, \mathbb{k})$  the linear dual of a vector space  $X$  and similar notation is used for linear maps. The identity object is the 1-dimensional representation  $(\mathcal{I}, \varrho^{\mathcal{I}})$ .  $\square$

As in the case of groups or rings, morphisms between groupoids entail functors between their categories of representations.

LEMMA 1.6. *Let  $\phi : \mathcal{G} \rightarrow \mathcal{H}$  be a morphism of groupoids. Then there is a  $\mathbb{k}$ -linear functor called the restriction functor*

$$\mathcal{R}(\phi) : \mathbf{Rep}_{\mathbb{k}}(\mathcal{H}) \longrightarrow \mathbf{Rep}_{\mathbb{k}}(\mathcal{G}), \quad (5)$$

*sending any representation  $(\mathcal{P}, \varrho^{\mathcal{P}}) \in \mathbf{Rep}_{\mathbb{k}}(\mathcal{H})$  to the representation  $(\phi_0^* \mathcal{P}, \phi_1^* \varrho^{\mathcal{P}})$ , where  $\phi_0^* \mathcal{P} = \bigcup_{x \in \mathcal{G}_0} P_{\phi_0(x)}$  and  $(\phi_1^* \varrho^{\mathcal{P}})_g = \varrho_{\phi_1(g)}^{\mathcal{P}}$ , for every  $x \in \mathcal{G}_0$  and  $g \in \mathcal{G}_1$ , and acting in a obvious way on morphisms. Moreover, the restriction functor  $\mathcal{R}(\phi)$  is strong monoidal.*

*Proof.* Straightforward.  $\square$

EXAMPLE 1.7. Let  $\mathcal{G}$  be an action groupoid as in Example 1.3 with a group  $G$  and a left  $G$ -set  $X$ . Consider  $G$  as a groupoid with one object. Then the projection  $pr_1 : G \times X \rightarrow G$  defines a morphism of groupoids. The associated restriction functor  $\mathcal{R}(pr)$  coincides then with the functor  $- \times X$ , which sends any  $G$ -representation  $(V, \varrho^V)$  to the  $\mathcal{G}$ -representation  $V \times X = \cup_{x \in X} V \times \{x\}$  with  $\mathcal{G}$ -action  $\varrho_{(g,x)}^{V \times X} : V \times \{x\} \rightarrow V \times \{gx\}$  sending  $(v, x) \mapsto (\varrho_g^V(v), gx)$ , for every  $g \in G$ . Of course, we have  $\mathcal{R}(pr)(f) = f \times X$ , for any morphism  $f : (V, \varrho^V) \rightarrow (W, \varrho^W)$  of  $G$ -representations.

REMARK 1.8. When both  $\mathcal{G}$  and  $\mathcal{H}$  are groups in Lemma 1.6, we have (see Example 1.3) that the restriction functor coincides with the usual restriction functor in group theory. Obviously this functor can be extended to all representation (including the infinite dimensional ones). The search of a (left) adjoint functor to  $\mathcal{R}(\phi)$ , that is, the construction of the *induction functor*, is a classical subject in group theory which is strongly related to *Frobenius reciprocity formula*. In our case we think that it is possible to construct *the induction functor for groupoids*, i.e. an adjoint functor to  $\mathcal{R}(\phi)$  of Lemma 1.6. Naturally, this construction could be very interesting in the study of discrete groupoids, as it was for discrete groups, however, we think that this deserves a separate project.

REMARK 1.9. If we take a topological groupoid  $\mathcal{G}$ . Then it is well know that the category  $\mathbf{Rep}_{\mathbb{k}}^{top}(\mathcal{G})$  of representations described in Remark 1.2, is no longer abelian. Although, it is symmetric rigid monoidal  $\mathbb{k}$ -linear category. In the discrete case the advantage was that for any  $\mathcal{G}$ -representation  $(\mathcal{E}, \varrho^{\mathcal{E}})$  the underlying set  $\mathcal{E}$  can be identified with  $\mathcal{G}_0 \times V$  for some finite dimensional  $\mathbb{k}$ -vector space  $V$  (i.e., it is a trivial bundle over  $\mathcal{G}_0$ ) and there were no topological properties need to be checked. In the topological case we can show, using [17, Theorem 6.3] and diagrams (2), that the category  $\mathbf{Rep}_{\mathbb{k}}^{top}(\mathcal{G})$  is in fact a *pseudo-abelian category* in the sense of M. Karoubi [17, Definition 6.7].

On the other hand, the functor of Lemma 1.6 can be also constructed for topological groupoids. Here of course we are implicitly assuming that the structure maps  $\phi_0, \phi_1$  are continuous, although, one only need to assume that  $\phi_1$  is continuous. In this case, the construction of that functor uses in part the induced vector bundles by  $\phi_0$ . For instance, in the case of a topological action groupoid Example 1.3, we have as in Example 1.7 a morphism of topological groupoids  $pr_1 : \mathcal{G}_1 = G \times X \rightarrow G$  which leads to the functor  $\mathcal{R}^{top}(pr) : \mathbf{Rep}_{\mathbb{k}}^{top}(G) \rightarrow \mathbf{Rep}_{\mathbb{k}}^{top}(\mathcal{G})$  from the category of continuous  $G$ -representations to the category of  $G$ -equivariant bundles. In this direction, assume further that  $X \in \mathbf{Rep}_{\mathbb{k}}^{top}(G)$ , we can address here a similar problem to the *Equivariant Serre Problem* [19] (see also Example 3.5 below). Precisely, we can ask whether the functor  $\mathcal{R}^{top}(pr)$  is 'surjective' and naturally isomorphic to  $\mathcal{R}^{top}(pr) \cong - \times X$ . That is, one can ask whether a given  $G$ -equivariant bundle is trivial, i.e., isomorphic to  $V \times X$  for some continuous  $G$ -representation  $V$ .

The following notations will be used in the sequel. Let  $\mathcal{G}$  be a groupoid, we denote by  $M_{\mathbb{k}}(\mathcal{G}_0)$  and  $M_{\mathbb{k}}(\mathcal{G}_1)$  the commutative  $\mathbb{k}$ -algebras of all maps, respectively, from  $\mathcal{G}_0$  and  $\mathcal{G}_1$  to  $\mathbb{k}$ . The algebra  $M_{\mathbb{k}}(\mathcal{G}_0)$  is referred to as *the base algebra of  $\mathcal{G}$*  and  $M_{\mathbb{k}}(\mathcal{G}_1)$  *the total algebra of  $\mathcal{G}$* . We consider  $M_{\mathbb{k}}(\mathcal{G}_1)$  as an  $(M_{\mathbb{k}}(\mathcal{G}_0) \otimes_{\mathbb{k}} M_{\mathbb{k}}(\mathcal{G}_0))$ -algebra via the algebra map

$$t^* \otimes_{\mathbb{k}} s^* : M_{\mathbb{k}}(\mathcal{G}_0) \otimes_{\mathbb{k}} M_{\mathbb{k}}(\mathcal{G}_0) \longrightarrow M_{\mathbb{k}}(\mathcal{G}_1),$$

where  $s^* := M_{\mathbb{k}}(s)$  and  $t^* := M_{\mathbb{k}}(t)$  are the corresponding  $\mathbb{k}$ -algebra maps of  $s$  and  $t$ . We denote by  $\text{proj}(M_{\mathbb{k}}(\mathcal{G}_0))$  the category of finitely generated and projective  $M_{\mathbb{k}}(\mathcal{G}_0)$ -modules.

Take a  $\mathcal{G}$ -representation  $(\mathcal{E}, \varrho^{\mathcal{E}})$  and let  $\pi_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{G}_0$  be the canonical surjection. We consider the  $M_{\mathbb{k}}(\mathcal{G}_0)$ -modules of ('global') sections. Precisely, we set

$$\Gamma(\mathcal{E}) = \{s : \mathcal{G}_0 \rightarrow \mathcal{E} \mid \pi_{\mathcal{E}} \circ s = id_{\mathcal{G}_0}\}. \quad (6)$$

This is a  $\mathbb{k}$ -vector space endowed with structure of  $M_{\mathbb{k}}(\mathcal{G}_0)$ -modules explicitly given by:

$$(s + s')(x) = s(x) + s'(x), (a.s)(x) = a(x)s(x), \text{ for every } x \in \mathcal{G}_0, a \in M_{\mathbb{k}}(\mathcal{G}_0), \text{ and } s, s' \in \Gamma(\mathcal{E}).$$

In fact  $\Gamma(\mathcal{E})$  is a finitely generated and projective  $M_{\mathbb{k}}(\mathcal{G}_0)$ -module. Namely, this can be checked either by thinking of  $(\mathcal{E}, \pi_{\mathcal{E}})$  as a trivial bundle or by directly computing a dual basis. A precise way of computing a dual basis is given as follows. Consider the  $n$ -dimensional underlying vector space  $V$  (the 'type fiber') together with the linear isomorphisms  $\varphi_x : V \rightarrow E_x$ . Fix,  $\{v_1, \dots, v_n\}$  a basis for  $V$ , and consider, for any  $i = 1, \dots, n$  the following section  $s_i : \mathcal{G}_0 \rightarrow \mathcal{E}$  sending

$x \mapsto \varphi_x(v_i)$ . Since  $\{\varphi_x(v_i)\}_{1, \dots, n}$  is a basis for any of the  $E_x$ 's, so for any section  $s \in \Gamma(\mathcal{E})$ , we can write  $s(x) = \sum_i a_i(x) s_i(x)$ , for some  $a_i(x) \in \mathbb{k}$  which depends only on  $s(x)$ <sup>(4)</sup>. Thus, the maps  $a_i$ 's define an  $M_{\mathbb{k}}(\mathcal{G}_0)$ -linear maps  $s_i^* : \Gamma(\mathcal{E}) \rightarrow M_{\mathbb{k}}(\mathcal{G}_0)$  by sending  $s \mapsto a_i$ . Now, one can easily check that  $\{s_i, s_i^*\}_i$  form a dual basis for the  $M_{\mathbb{k}}(\mathcal{G}_0)$ -module  $\Gamma(\mathcal{E})$ .

Evidently, after forgetting the  $\mathcal{G}$ -action,  $\Gamma$  establishes a  $\mathbb{k}$ -linear functor from the category of representations  $\mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$  to the category of modules  $\mathbf{proj}(M_{\mathbb{k}}(\mathcal{G}_0))$ . Specifically, we have the following well known result (see the last observation in [11, Example 1.24]).

PROPOSITION 1.10. *Let  $\mathcal{G}$  be a groupoid. Then the functor*

$$\begin{array}{ccc} \omega_{\mathcal{G}} : \mathbf{Rep}_{\mathbb{k}}(\mathcal{G}) & \longrightarrow & \mathbf{proj}(M_{\mathbb{k}}(\mathcal{G}_0)) \\ (\mathcal{E}, \varrho^{\mathcal{E}}) & \longrightarrow & \Gamma(\mathcal{E}) \end{array}$$

is a non trivial  $\mathbb{k}$ -linear monoidal right exact functor. In particular  $\omega_{\mathcal{G}}$  is a fiber functor and the pair  $(\mathbf{Rep}_{\mathbb{k}}(\mathcal{G}), \omega_{\mathcal{G}})$  is then a Tannakian  $\mathbb{k}$ -linear category.

*Proof.* We refer to the Appendix for the definitions and terminology employed in the statement. Notice that in our case there are various ways to deduce that  $\omega_{\mathcal{G}} := \omega$  is a monoidal functor. Here we give an elementary proof. First we clearly have that  $\omega(\mathcal{I}) \cong M_{\mathbb{k}}(\mathcal{G}_0)$ . Now, take two  $\mathcal{G}$ -representations  $(\mathcal{E}, \varrho^{\mathcal{E}})$  and  $(\mathcal{F}, \varrho^{\mathcal{F}})$ , and consider the following  $\mathbb{k}$ -linear map

$$\psi : \Gamma(\mathcal{E}) \otimes_{\mathbb{k}} \Gamma(\mathcal{F}) \longrightarrow \Gamma(\mathcal{E} \otimes \mathcal{F}), \left( p \otimes_{\mathbb{k}} q \longmapsto \left\{ x \mapsto p(x) \otimes_{\mathbb{k}} q(x) \right\} \right).$$

This map can be clearly extended to an  $M_{\mathbb{k}}(\mathcal{G}_0)$ -linear map on the tensor product  $\Gamma(\mathcal{E}) \otimes_{M_{\mathbb{k}}(\mathcal{G}_0)} \Gamma(\mathcal{F})$ . We denote also by  $\psi$  this extension. It is easily checked that  $\psi$  is injective. On the other hand, take an element  $s \in \Gamma(\mathcal{E} \otimes \mathcal{F})$ , so for every  $x \in \mathcal{G}_0$ , we can write  $s(x) = \sum_{i,j} a_{i,j}(x) s_i(x) \otimes_{\mathbb{k}} r_j(x)$ , where  $\{s_i, s_i^*\}$  and  $\{r_j, r_j^*\}$ , are respectively, the ('global') dual basis of  $\Gamma(\mathcal{E})$  and  $\Gamma(\mathcal{F})$  constructed as above. We clearly have that  $\psi\left(\sum_{i,j} s_i \otimes_{M_{\mathbb{k}}(\mathcal{G}_0)} a_{i,j} r_j\right) = s$  and then  $\psi$  is surjective. This establishes a natural isomorphism  $\omega(-) \otimes_{M_{\mathbb{k}}(\mathcal{G}_0)} \omega(-) \cong \omega(- \otimes -)$  obviously satisfying the necessary coherence conditions which convert  $\omega$  into a monoidal functor.

We need to check that  $\omega$  is right exact. So take an epimorphism  $\alpha : (\mathcal{E}, \varrho^{\mathcal{E}}) \rightarrow (\mathcal{F}, \varrho^{\mathcal{F}}) \rightarrow 0$  in  $\mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$ . Then  $\omega_{\alpha} : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})$  is defined by sending any section  $\Gamma(\mathcal{E}) \ni s \mapsto \omega_{\alpha}(s) \in \Gamma(\mathcal{F})$  to the section defined by  $\omega_{\alpha}(s)(x) = \alpha_x(s(x))$ , for every  $x \in \mathcal{G}_0$ . Given a section  $r \in \Gamma(\mathcal{F})$ , we know that, for every  $x \in \mathcal{G}_0$ , there are elements  $e^{(x)} \in E_x$  such that  $\alpha_x(e^{(x)}) = r(x)$ . Fix an element  $x \in \mathcal{G}_0$  and choose only one element say  $e_x$  of those  $e^{(x)} \in E_x$ . In this way, we have define a section  $s : \mathcal{G}_0 \rightarrow \mathcal{E}$  which sends  $x \mapsto e_x$  and satisfies  $\alpha_x(s(x)) = \alpha_x(e_x) = r(x)$ . That is,  $\omega_{\alpha}(s) = r$ , and so  $\omega(\alpha)$  is an epimorphism<sup>(5)</sup>. The particular statement follows by Lemma 1.5, using the fact that  $\mathbb{k} \cong \text{End}_{\mathbf{Rep}_{\mathbb{k}}(\mathcal{G})}(\varrho^{\mathcal{I}})$  and by knowing that  $\omega$  is a faithful functor.  $\square$

REMARK 1.11. For a topological groupoid  $\mathcal{G}$  with compact Hausdorff base space  $\mathcal{G}_0$  a variant of Proposition 1.10 could be formulated. Explicitly, take the category  $\mathbf{Rep}_{\mathbb{k}}^{\text{top}}(\mathcal{G})$  of  $\mathcal{G}$ -representations as was defined in Remark 1.2 with  $\mathbb{k}$  denotes the field of real or complex numbers. As it was mentioned before, this category is a  $\mathbb{k}$ -linear pseudo-abelian symmetric rigid monoidal category with  $\text{End}(\mathcal{I}) \cong \mathbb{k}$ . A functor  $\omega^{\text{top}} : \mathbf{Rep}_{\mathbb{k}}^{\text{top}}(\mathcal{G}) \rightarrow \text{Mod}_{C_{\mathbb{k}}(\mathcal{G}_0)}$  can be naturally defined here by taking continuous global sections and the ring of continuous functions  $C_{\mathbb{k}}(\mathcal{G}_0)$  as the base ring. Hence we can show thanks to [26, 25] that its image lands in the sub-category  $\mathbf{proj}(C_{\mathbb{k}}(\mathcal{G}_0))$  of finitely generated and projective  $C_{\mathbb{k}}(\mathcal{G}_0)$ -modules. To show that  $\omega^{\text{top}}$  is a monoidal functor, one perhaps could uses similar arguments as in the proof of Proposition 1.10, since we know in this case that a 'global' dual basis always exists; in the sense that for any vector bundle  $\mathcal{E}$  over  $\mathcal{G}_0$  (of locally constant rank) there are continuous sections  $s_1, \dots, s_n \in \Gamma(\mathcal{E})$  such that for any  $x \in \mathcal{G}_0$  there is a neighborhood  $U$  of  $x$  where the set  $\{s_1(y), \dots, s_n(y)\}$  is a  $\mathbb{k}$ -basis of  $E_y$ , for any  $y \in U$ .

On the other hand, since in general we know that the fiber functor on Tannakian  $\mathbb{k}$ -linear categories is in fact an exact functor, so it is left exact. We could then expect in the topological

<sup>(4)</sup> Notice here that  $a_i(x) = v_i^* \circ \varphi_x^*$ , where the  $v_i^*$ 's are the dual maps of the basis  $\{v_1, \dots, v_n\}$ .

<sup>(5)</sup> This step can be directly deduced by applying Urysohn's lemma to  $\mathcal{G}_0$  considering it as a normal space with discrete topology.



case that the functor  $\omega^{top}$  preserves the kernels of projectors<sup>(6)</sup>. As conclusion, a topological version of Proposition 1.10 could be formulated by saying that  $(\mathbf{Rep}_k^{top}(\mathcal{G}), \omega^{top})$  is a *pseudo-Tannakian  $k$ -linear category*.

Recall that a  $k$ -linear category is said to be *locally of finite type over  $k$* , if any object is of finite length and the  $k$ -vector space of morphisms between arbitrary two objects is finite dimensional.

**COROLLARY 1.12.** *Let  $\mathcal{G}$  be a groupoid and consider  $\mathbf{Rep}_k(\mathcal{G})$  its Tannakian  $k$ -linear category of representations. Then  $\mathbf{Rep}_k(\mathcal{G})$  is locally of finite type over  $k$  and the canonical map*

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Rep}_k(\mathcal{G})}(\varrho^{\mathcal{E}}, \varrho^{\mathcal{F}}) \otimes_k M_k(\mathcal{G}_0) &\longrightarrow \mathrm{Hom}_{M_k(\mathcal{G}_0)}(\Gamma(\mathcal{E}), \Gamma(\mathcal{F})) \\ \alpha \otimes_k a &\longmapsto \left[ s \longmapsto [x \mapsto \alpha_x(s(x))a(x)] \right] \end{aligned}$$

is injective.

*Proof.* This is a direct consequence of Proposition 1.10 and [6, Proposition 2.5].  $\square$

## 2. THE HOPF ALGEBROID OF REPRESENTATIVE FUNCTIONS.

In this section we introduce the Hopf algebroid of representative functions on a given discrete groupoid. In the discrete group case, we recover the construction of the commutative Hopf algebra of representative functions [1, 15], see also [9, 13, 23] for topological groups, Lie groups and algebraic linear groups with infinite base field.

By groupoid we mean a discrete groupoid, except where otherwise it is explicitly specified. The notation is that of Section 1. We fix a groupoid  $\mathcal{G} : \mathcal{G}_1 \rightleftarrows \mathcal{G}_0$  and denote by  $B := M_k(\mathcal{G}_0)$  its base  $k$ -algebra. As was mentioned before, we consider the total  $k$ -algebra  $M_k(\mathcal{G}_1)$  as an  $(B \otimes_k B)$ -algebra via the duals of both  $s$  and  $t$ , i.e.  $M_k(s)$  and  $M_k(t)$ .

By using Proposition 1.10, we can apply the procedure described in Appendix to construct the universal object  $\mathcal{R}_k(\mathcal{G}) := \mathcal{L}_k(\omega_{\mathcal{G}})$ , see [10] or precisely [6, Théorème 5.2], see also [12, §4].

Let us first fix some new notations. For a  $\mathcal{G}$ -representation  $(\mathcal{E}, \varrho^{\mathcal{E}})$  we denote  $\omega(\mathcal{E}, \varrho^{\mathcal{E}}) := \Gamma(\mathcal{E})$  its image by the fiber functor of Proposition 1.10. We will often use the isomorphisms  $\Gamma(\mathcal{E}^*) \cong \Gamma(\mathcal{E})^* = \mathrm{Hom}_B(\Gamma(\mathcal{E}), B)$ . By applying the description offered in equation (17) (see Appendix) to the fiber functor  $\omega_{\mathcal{G}} : \mathbf{Rep}_k(\mathcal{G}) \rightarrow \mathrm{proj}(M_k(\mathcal{G}_0))$ , we obtain

**COROLLARY AND DEFINITION 2.1.** Let  $\mathcal{G}$  a groupoid and  $B = M_k(\mathcal{G}_0)$  its base  $k$ -algebra. Then the universal  $B$ -bimodule  $\mathcal{R}_k(\mathcal{G})$  is the commutative  $(B \otimes_k B)$ -algebra

$$\mathcal{R}_k(\mathcal{G}) \cong \int^{\varrho^{\mathcal{E}} \in \mathbf{Rep}_k(\mathcal{G})} \Gamma(\mathcal{E})^* \otimes_k \Gamma(\mathcal{E}) \cong \frac{\bigoplus_{\varrho^{\mathcal{E}} \in \mathbf{Rep}_k(\mathcal{G})} \Gamma(\mathcal{E}^*) \otimes_{T_{\mathcal{E}}} \Gamma(\mathcal{E})}{\mathcal{I}_{\mathbf{Rep}_k(\mathcal{G})}} \cong \Sigma^\dagger \otimes_B \Sigma, \quad (7)$$

see the Appendix for the notation. The multiplication of  $\mathcal{R}_k(\mathcal{G})$  is defined using the tensor product in  $\mathbf{Rep}_k(\mathcal{G})$  and the unit is the image of  $B \otimes_k B$  by taking the identity representation  $(\mathcal{I}, \varrho^{\mathcal{I}})$ .

The algebra  $\mathcal{R}_k(\mathcal{G})$  is called *the algebra of representative functions* on the groupoid  $\mathcal{G}$ . The terminology will be soon justified. We will use the notation  $\overline{\varphi \otimes_{T_{\mathcal{E}}} p}$  to denote elements in  $\mathcal{R}_k(\mathcal{G})$  (here we are taking the second description in equation (7)). Such an element corresponds then to the  $\mathcal{G}$ -representation  $\varrho^{\mathcal{E}}$  and it is represented by the generic element  $\varphi \otimes_{T_{\mathcal{E}}} p \in \Gamma(\mathcal{E})^* \otimes_{T_{\mathcal{E}}} \Gamma(\mathcal{E})$ .

It is convenient to give explicitly the Hopf algebroid structure of  $(B, \mathcal{R}_k(\mathcal{G}))$ . The multiplication and the unit are defined as was mentioned above, using the tensor product of  $\mathcal{G}$ -representations and the unit uses the identity representation  $\varrho^{\mathcal{I}}$ .

Precisely, given two  $\mathcal{G}$ -representations  $\varrho^{\mathcal{E}}, \varrho^{\mathcal{F}}$  and two elements of the form  $\overline{\varphi \otimes_{T_{\mathcal{E}}} p}, \overline{\psi \otimes_{T_{\mathcal{F}}} q}$  in  $\mathcal{R}_k(\mathcal{G})$  with  $\varphi \in \Gamma(\mathcal{E})^*, p \in \Gamma(\mathcal{E})$  and  $\psi \in \Gamma(\mathcal{F})^*, q \in \Gamma(\mathcal{F})$ , we have

$$\overline{\varphi \otimes_{T_{\mathcal{E}}} p} \cdot \overline{\psi \otimes_{T_{\mathcal{F}}} q} = \overline{(\varphi \otimes_B \psi) \otimes_{T_{\mathcal{E} \otimes_{\mathcal{F}}}} (p \otimes_B q)},$$

where we have identified  $\Gamma(\mathcal{E}^*) \otimes_B \Gamma(\mathcal{F}^*)$  with  $\Gamma((\mathcal{E} \otimes_{\mathcal{F}})^*)$  via the canonical isomorphism which sends  $\varphi \otimes_B \psi$  to the section  $\mathcal{G}_0 \ni x \mapsto \varphi(x) \otimes_k \psi(x) \in E_x^* \otimes_k F_x^*$ . Of course this operation

<sup>(6)</sup> That are endomorphisms  $f$  such that  $f^2 = f$ .

is independent from the chosen representing elements in the equivalence classes of the second description in equation (7).

The reminder of structure maps are described as follows:

- *Source and target*:  $\mathfrak{s} : B \rightarrow \mathcal{R}_{\mathbb{k}}(\mathcal{G})$ ,  $\left(a \mapsto \overline{1_B \otimes_{\mathbb{T}_Z} a}\right)$ ,  $\mathfrak{t} : B \rightarrow \mathcal{R}_{\mathbb{k}}(\mathcal{G})$ ,  $\left(a \mapsto \overline{a \otimes_{\mathbb{T}_Z} 1_B}\right)$ .
- *Counit*:  $\varepsilon : \mathcal{R}_{\mathbb{k}}(\mathcal{G}) \rightarrow B$ ,  $\left(\varepsilon(\overline{\varphi \otimes_{\mathbb{T}_E} p}) : \mathcal{G}_0 \rightarrow \mathbb{k}, \left[x \mapsto \varphi(x)(p(x))\right]\right)$
- *Comultiplication*:  $\Delta : \mathcal{R}_{\mathbb{k}}(\mathcal{G}) \rightarrow \mathcal{R}_{\mathbb{k}}(\mathcal{G}) \otimes_B \mathcal{R}_{\mathbb{k}}(\mathcal{G})$ ,  $\left(\overline{\varphi \otimes_{\mathbb{T}_E} p} \mapsto \sum_i^n \overline{\varphi \otimes_{\mathbb{T}_E} s_i} \otimes_B \overline{s_i^* \otimes_{\mathbb{T}_E} p}\right)$  where  $\{s_i, s_i^*\}_i^n$  is the 'global' dual basis of  $\Gamma(\mathcal{E})_B$  and  $n$  is the rank of the bundle  $\mathcal{E}$ .
- *Antipode*:  $\mathcal{S} : \mathcal{R}_{\mathbb{k}}(\mathcal{G}) \rightarrow \mathcal{R}_{\mathbb{k}}(\mathcal{G})$ ,  $\left(\overline{\varphi \otimes_{\mathbb{T}_E} p} \mapsto \overline{\widetilde{p} \otimes_{\mathbb{T}_{E^*}} \varphi}\right)$ , where we have used the isomorphism  $\widetilde{(-)} : \mathcal{E} \cong (\mathcal{E}^*)^*$ .

The following proposition justifies the terminology used for  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$ .

PROPOSITION 2.2. *Let  $\mathcal{G}$  be a groupoid and  $B = M_{\mathbb{k}}(\mathcal{G}_0)$  its base  $\mathbb{k}$ -algebra. The following map*

$$\begin{array}{ccc} \mathcal{R}_{\mathbb{k}}(\mathcal{G}) & \xrightarrow{\zeta} & M_{\mathbb{k}}(\mathcal{G}_1) \\ \overline{\varphi \otimes_{\mathbb{T}_E} p} & \longmapsto & \left\{ g \mapsto \varphi(\mathfrak{t}(g)) \left( \varrho_g^{\mathcal{E}}(p(\mathfrak{s}(g))) \right) \right\} \end{array} \quad (8)$$

is an injective  $(B \otimes_{\mathbb{k}} B)$ -algebras map to the total algebra of  $\mathcal{G}$ . Moreover, we have

- (1)  $\iota^* \circ \zeta = \varepsilon$ .
- (2)  $\zeta \circ \mathcal{S}(\overline{\varphi \otimes_{\mathbb{T}_E} p})(g) = \zeta(\overline{\varphi \otimes_{\mathbb{T}_E} p})(g^{-1})$ , for every  $\overline{\varphi \otimes_{\mathbb{T}_E} p} \in \mathcal{R}_{\mathbb{k}}(\mathcal{G})$  and  $g \in \mathcal{G}_1$ .
- (3) For every pair of arrows  $f, g \in \mathcal{G}_1$  with  $\mathfrak{t}(f) = \mathfrak{s}(g)$  and every element  $F \in \mathcal{R}_{\mathbb{k}}(\mathcal{G})$ :

$$\zeta(F)(g \circ f) = \zeta(F_1)(g)\zeta(F_2)(f), \quad \text{where } \Delta(F) = F_1 \otimes_B F_2. \text{ (summation understood)}$$

*Proof.* An easy verification, using the definition of the two-sided ideal  $\mathcal{I}_{\mathcal{R}\mathbf{ep}_{\mathbb{k}}(\mathcal{G})}$  of equation (15) (see Appendix) and diagrams (2), shows that  $\zeta$  is a well defined map. Since the multiplication in  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$  is defined via the tensor product of representations and that of  $M_{\mathbb{k}}(\mathcal{G}_1)$  is defined componentwise, we easily check that  $\zeta$  is multiplicative and unital.

We need to check that  $\zeta$  is injective. To this end we define the following  $(B \otimes_{\mathbb{k}} B)$ -subalgebra of the total algebra  $M_{\mathbb{k}}(\mathcal{G}_1)$ . Take a  $\mathcal{G}$ -representation  $\varrho^{\mathcal{E}}$  with rank  $n$ , we denote by  $\mathcal{V}(\varrho^{\mathcal{E}})$  the  $(B \otimes_{\mathbb{k}} B)$ -sub-bimodule of  $M_{\mathbb{k}}(\mathcal{G}_1)$  generated by the set of functions  $\{a_{ij}\}_{1 \leq i, j \leq n}$  where for each  $g \in \mathcal{G}_1$  we have  $\varrho_g^{\mathcal{E}} = (a_{ij}^g)_{1 \leq i, j \leq n}$  the  $n$ -square matrix representing the  $\mathbb{k}$ -linear isomorphism  $\varrho_g^{\mathcal{E}}$ . That is, an element of the form  $(\lambda \otimes_{\mathbb{k}} \gamma) \cdot a_{ij} \in \mathcal{V}(\varrho^{\mathcal{E}})$  defines on  $\mathcal{G}_1$  the function  $g \mapsto \lambda(\mathfrak{t}(g))a_{ij}^g \gamma(\mathfrak{s}(g))$ . Now, we define the following  $(B \otimes_{\mathbb{k}} B)$ -algebra  $\mathcal{V}(\mathcal{G}) = \sum_{\varrho^{\mathcal{E}} \in \mathcal{R}\mathbf{ep}_{\mathbb{k}}(\mathcal{G})} \mathcal{V}(\varrho^{\mathcal{E}})$  whose multiplication is defined as in  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$  by using the tensor product of  $\mathcal{G}$ -representations and the unit is given by the identity representation  $\varrho^{\mathcal{I}}$ . An easy computation shows now that  $\zeta(\mathcal{R}_{\mathbb{k}}(\mathcal{G})) = \mathcal{V}(\mathcal{G})$ .

On the other hand, for any  $\mathcal{G}$ -representation  $\varrho^{\mathcal{E}}$ , we have a well defined morphism of  $(B \otimes_{\mathbb{k}} B)$ -modules

$$\xi_E : \Gamma(\mathcal{E})^* \otimes_{\mathbb{k}} \Gamma(\mathcal{E}) \longrightarrow \mathcal{V}(\mathcal{G}), \quad \left( \varphi \otimes_{\mathbb{k}} p \longmapsto \sum_i \varphi_i \otimes_{\mathbb{k}} p_j \cdot a_{ij} \right),$$

where, for every  $x \in \mathcal{G}_0$ , we expressed  $\varphi(x) = \sum_i \varphi_i(x)s_i(x)$  in  $E_x^*$  and  $p(x) = \sum_j p_j(x)s_j(x)$  in  $E_x$  by taking a 'global' dual basis  $\{s_i, s_i^*\}_i$  of the  $B$ -module  $\Gamma(\mathcal{E})$ . It turns out that the pair  $(\mathcal{V}(\mathcal{G}), \xi_-)$  is a dinatural transformation. Furthermore, by construction one can show that  $(\mathcal{V}(\mathcal{G}), \xi_-)$  is the coend of the functor  $\omega^*(-) \otimes_{\mathbb{k}} \omega(-)$ . Therefore, by the universal property we have an isomorphism<sup>(7)</sup>  $\mathcal{V}(\mathcal{G}) \cong \mathcal{R}_{\mathbb{k}}(\mathcal{G})$  via the map  $\zeta$ , and thus  $\zeta$  is injective.

The equality of item (1) in the last statement is clear. The second item follows by an easy computation using the formula of equation (4). The equality of item (3) is obtained as follows. Take  $g, f \in \mathcal{G}_1$  with  $\mathfrak{s}(g) = \mathfrak{t}(f)$ , it suffices to check the stated equality of elements of the form

<sup>(7)</sup> In fact an isomorphism of commutative Hopf  $B$ -algebroids, where  $\mathcal{V}(\mathcal{G})$  can be endowed with a natural structure of a Hopf algebroid.

$\overline{\varphi \otimes_{\mathbb{T}_\varepsilon} p}$  in  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$  given by some  $\mathcal{G}$ -representation  $\varrho^\varepsilon$  with rank  $n$  and a global dual basis  $\{s_i, s_i^*\}_i^n$  of sections. So, from one hand we have

$$\begin{aligned} \zeta(\overline{\varphi \otimes_{\mathbb{T}_\varepsilon} p})(gf) &= \varphi(\mathbf{t}(gf)) \left( \varrho_{gf}^\varepsilon(p(\mathbf{s}(gf))) \right) \\ &= \varphi(\mathbf{t}(g)) \left( \varrho_{gf}^\varepsilon(p(\mathbf{s}(f))) \right) \\ &\stackrel{(1)}{=} \varphi(\mathbf{t}(g)) \left( \varrho_g^\varepsilon(\varrho_f^\varepsilon(p(\mathbf{s}(f)))) \right) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \zeta\left(\left(\overline{\varphi \otimes_{\mathbb{T}_\varepsilon} p}\right)_1\right)(g) \zeta\left(\left(\overline{\varphi \otimes_{\mathbb{T}_\varepsilon} p}\right)_2\right)(f) &= \sum_i \zeta\left(\overline{\varphi \otimes_{\mathbb{T}_\varepsilon} s_i}\right)(g) \zeta\left(\overline{s_i^* \otimes_{\mathbb{T}_\varepsilon} p}\right)(f) \\ &= \sum_i \varphi(\mathbf{t}(g)) \left( \varrho_g^\varepsilon(s_i(\mathbf{s}(g))) \right) s_i^*(\mathbf{t}(f)) \left( \varrho_f^\varepsilon(p(\mathbf{s}(f))) \right) \\ &= \varphi(\mathbf{t}(g)) \left( \sum_i \varrho_g^\varepsilon(s_i(\mathbf{s}(g))) s_i^*(\mathbf{t}(f)) \left( \varrho_f^\varepsilon(p(\mathbf{s}(f))) \right) \right) \\ &= \varphi(\mathbf{t}(g)) \left( \varrho_g^\varepsilon \left( \sum_i s_i(\mathbf{s}(g)) s_i^*(\mathbf{t}(f)) \left( \varrho_f^\varepsilon(p(\mathbf{s}(f))) \right) \right) \right) \\ &= \varphi(\mathbf{t}(g)) \left( \varrho_g^\varepsilon \left( \varrho_f^\varepsilon(p(\mathbf{s}(f))) \right) \right) \end{aligned}$$

Therefore,  $\zeta(\overline{\varphi \otimes_{\mathbb{T}_\varepsilon} p})(gf) = \zeta\left(\left(\overline{\varphi \otimes_{\mathbb{T}_\varepsilon} p}\right)_1\right)(g) \zeta\left(\left(\overline{\varphi \otimes_{\mathbb{T}_\varepsilon} p}\right)_2\right)(f)$  which finishes the proof.  $\square$

EXAMPLE 2.3. If we start with a groupoid whose source is equal to its target, in other words, a disjoint union of groups (or group bundle)  $\cup_{x \in X} G_x$  parametrized by a non empty set  $X$ . Then the associated algebra of representative functions is in this case a commutative Hopf algebra over the base algebra  $B = M_{\mathbb{k}}(X)$  (i.e. a Hopf algebroid with same source and target). In this case if  $X$  is reduced to a point, which means that we are given a single group  $G$ . Then  $B = \mathbb{k}$  and by Example 1.3,  $\mathcal{R}_{\mathbb{k}}(G)$  is exactly the commutative Hopf  $\mathbb{k}$ -algebra of representative functions on  $G$ . That is, it coincides with the *finite dual*  $(\mathbb{k}G)^\circ$  of the group algebra  $\mathbb{k}G$ , see [1, 15].

A trivial case of the disjoint union of groups  $\cup_{x \in X} G_x$  is when each group of the  $G_x$ 's has only one element (the neutral element). Thus, for any non empty set  $X$  one can consider the groupoid  $\mathcal{U}(X)$  known as *the unit groupoid* of  $X$ <sup>(8)</sup>, where  $\mathcal{U}(X)_1 = \mathcal{U}(X)_0 = X$  and  $\mathbf{t} = \mathbf{s} = \iota = id_X$ . A finite dimensional  $\mathcal{U}(X)$ -representation is nothing but a set of the form  $X \times V$  where  $V$  is a finite dimensional  $\mathbb{k}$ -vector space. Under this description, the Hopf  $B$ -algebra of representative functions is given by following quotient of  $B$ -module

$$\mathcal{R}_{\mathbb{k}}(\mathcal{U}(X)) = \frac{\left( \bigoplus_{n \in \mathbb{N}} B^n \otimes_{M_n(\mathbb{k})} B^n \right)}{\left\langle u \otimes_{M_n(\mathbb{k})} (\lambda_{ij})v - u(\lambda_{ij}) \otimes_{M_m(\mathbb{k})} v \right\rangle_{u \in B^n, v \in B^m, (\lambda_{ij}) \in M_{n,m}(\mathbb{k})}}.$$

Another less trivial groupoid, is the *groupoid of pairs*  $\mathcal{V}(X)$  (or *fine groupoid* in the terminology of [5] which is a particular case of *principal groupoids* [21]). Here  $\mathcal{V}(X)_1 = X \times X$ ,  $\mathcal{V}(X)_0 = X$  with  $\mathbf{s} = pr_1$ ,  $\mathbf{t} = pr_2$  (the first and second projections) and  $\iota = \delta$  (the diagonal map). The composition is understood. In this case a finite dimensional  $\mathcal{V}(X)$ -representation is a bundle  $\cup_{x \in X} E_x$  with a 'type fiber' an  $n$ -dimensional  $\mathbb{k}$ -vector space  $V$ , together with a family of invertible  $n$ -square matrices  $\{(a_{ij}^{(x,y)})_{i,j}\}_{x,y \in X}$  in  $M_n(\mathbb{k})$  satisfying

$$\left( a_{ij}^{(x,y)} \right)_{i,j} \left( a_{ij}^{(y,z)} \right)_{i,j} = \left( a_{ij}^{(y,z)} \right)_{i,j}, \quad \left( a_{ij}^{(x,x)} \right)_{i,j} = I_n, \quad \text{for every } x, y, z \in X.$$

The description of the commutative Hopf algebroid  $\mathcal{R}_{\mathbb{k}}(\mathcal{V}(X))$  is approximately similar to that of  $\mathcal{R}_{\mathbb{k}}(\mathcal{U}(X))$ . Thus, instate of taking the  $n$ -square matrix algebra in the above nominator of

<sup>(8)</sup> The terminology is that of [8].

$\mathcal{R}_{\mathbb{k}}(\mathcal{U}(X))$ , we take the following  $\mathbb{k}$ -subalgebra of  $M_n(B)$  consisting of matrices which are 'locally' conjugated by the  $(a_{i,j})_{i,j}$ 's. That is, matrices  $(\lambda_{ij})_{i,j} \in M_n(B)$  such that

$$\left(\lambda_{ij}^y\right)_{i,j} \left(a_{ij}^{(x,y)}\right)_{i,j} = \left(a_{ij}^{(x,y)}\right)_{i,j} \left(\lambda_{ij}^x\right)_{i,j}, \quad \text{for every } x, y \in X.$$

It is noteworthy to mention, that a more precise and complete description of both Hopf algebroids, by means of generators and relations, is far from being obvious even under strong hypothesis of finiteness on the set  $X$ .

EXAMPLE 2.4. Let  $H$  be a group with identity element  $e$ , and  $X$  any set. Consider the following groupoid

$$\mathcal{G} : X \times H \times X \begin{array}{c} \xrightarrow{pr_1} \\ \xleftarrow{pr_3} \\ \xrightarrow{\quad} \end{array} X,$$

with  $s = pr_1$  and  $t = pr_3$ , the first and third projections, and where  $\iota(x) = (x, e, x)$ , for every  $x \in X$ . The composition and the inverse maps are given by

$$(x, g, y) \cdot (y, h, z) = (x, gh, z), \quad (x, g, y)^{-1} = (y, g^{-1}, x).$$

Let  $\mathcal{R}_{\mathbb{k}}(H)$  be the Hopf  $\mathbb{k}$ -algebra of representative functions on  $H$  which we consider as a Hopf algebroid with source equal target. Consider then its extended Hopf algebroid  $(M_{\mathbb{k}}(X), M_{\mathbb{k}}(X) \otimes_{\mathbb{k}} \mathcal{R}_{\mathbb{k}}(H) \otimes_{\mathbb{k}} M_{\mathbb{k}}(X))$ . One can easily check then that the image of  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$  in the total algebra  $M_{\mathbb{k}}(X \times H \times X)$  by the map  $\zeta$  of Proposition 2.2 coincides with the image of the canonical map

$$M_{\mathbb{k}}(X) \otimes_{\mathbb{k}} \mathcal{R}_{\mathbb{k}}(H) \otimes_{\mathbb{k}} M_{\mathbb{k}}(X) \hookrightarrow M_{\mathbb{k}}(X \times H \times X).$$

Therefore, there is an isomorphism of  $M_{\mathbb{k}}(X)$ -bimodules  $\mathcal{R}_{\mathbb{k}}(\mathcal{G}) \cong M_{\mathbb{k}}(X) \otimes_{\mathbb{k}} \mathcal{R}_{\mathbb{k}}(H) \otimes_{\mathbb{k}} M_{\mathbb{k}}(X)$  which by the universal property of  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$  is an isomorphism of Hopf  $M_{\mathbb{k}}(X)$ -algebroids. Furthermore, the canonical morphism of Hopf algebroids

$$(\mathbb{k}, \mathcal{R}_{\mathbb{k}}(H)) \longrightarrow (M_{\mathbb{k}}(X), M_{\mathbb{k}}(X) \otimes_{\mathbb{k}} \mathcal{R}_{\mathbb{k}}(H) \otimes_{\mathbb{k}} M_{\mathbb{k}}(X))$$

coincides, up to this isomorphism, with the morphism  $\mathcal{R}(pr_2) : (\mathbb{k}, \mathcal{R}_{\mathbb{k}}(H)) \rightarrow (M_{\mathbb{k}}(X), \mathcal{R}_{\mathbb{k}}(\mathcal{G}))$  of Hopf algebroids which corresponds to the obvious morphism of groupoids  $pr_2 : \mathcal{G} \rightarrow H$ .

Now we come back to the general situation. So let  $\mathcal{G}$  be a groupoid with base algebra  $B = M_{\mathbb{k}}(\mathcal{G}_0)$ . By applying Deligne's Theorem [6, Théorème 5.2] together with [6, Théorème 7.1] to Corollary 1.12, we know that  $(B, \mathcal{R}_{\mathbb{k}}(\mathcal{G}))$  is a *transitive commutative Hopf algebroid*, and so it is *geometrically transitive* by [6, Théorème 8.2], see also Lemma A.1 in the Appendix. More consequences of this property are stated in Example 3.4(2).

PROPOSITION 2.5. *Let  $\mathcal{G}$  be a groupoid with  $B$  its base  $\mathbb{k}$ -algebra. Then the algebra of representative functions  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$  is a geometrically transitive Hopf  $B$ -algebroid. Equivalently,  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$  is a projective and faithfully flat  $(B \otimes_{\mathbb{k}} B)$ -module.*

*Proof.* This is a direct consequence of [6, Théorèmes 5.2 et 7.1] (see also Lemma A.1).  $\square$

REMARK 2.6. Let us consider now a topological groupoid  $\mathcal{G}$  with compact Hausdorff base space  $\mathcal{G}_0$ . Take its category of representations  $\mathbf{Rep}_{\mathbb{k}}^{top}(\mathcal{G})$  with the "fiber" functor  $\omega^{top} : \mathbf{Rep}_{\mathbb{k}}^{top}(\mathcal{G}) \rightarrow \mathbf{proj}(C_{\mathbb{k}}(\mathcal{G}_0))$  as in Remark 1.11 ( $\mathbb{k}$  is the field of real or complex numbers). The construction of the  $(C_{\mathbb{k}}(\mathcal{G}_0) \otimes_{\mathbb{k}} C_{\mathbb{k}}(\mathcal{G}_0))$ -algebra of *continuous representative functions* is also possible in this context. In fact, associated to the pair  $(\mathbf{Rep}_{\mathbb{k}}^{top}(\mathcal{G}), \omega^{top})$  we can construct, in similar way as in equation (17) (see Appendix below), a commutative Hopf  $C_{\mathbb{k}}(\mathcal{G}_0)$ -algebroid  $\mathcal{R}_{\mathbb{k}}^{top}(\mathcal{G})$  which of course in this case is no longer geometrically transitive. The image of  $\mathcal{R}_{\mathbb{k}}^{top}(\mathcal{G})$  by the algebra map  $\zeta$  of Proposition 2.2, lands now in the total algebra  $C_{\mathbb{k}}(\mathcal{G}_1)$  of all continuous functions from  $\mathcal{G}_1$  to  $\mathbb{k}$ . The fact that  $\mathbf{Rep}_{\mathbb{k}}^{top}(\mathcal{G})$  is no longer abelian prevents us to apply Deligne-Bruguères's machinery to well understand the geometric nature of the commutative Hopf algebroid  $\mathcal{R}_{\mathbb{k}}^{top}(\mathcal{G})$ . This is why perhaps this context suggests then to study pseudo-Tannakian  $\mathbb{k}$ -linear categories (see Remark 1.11) in relation with commutative  $C^*$ -algebras.

## 3. DUALITY BETWEEN DISCRETE GROUPOIDS AND HOPF ALGEBROIDS.

This section contains our main result, namely, Theorem 3.13. We show that there is a duality between the category of discrete groupoids and the category of geometrically transitive Hopf algebroids. By a duality we mean here a kind of an adjunction between contravariant functors. Specifically, let  $\mathcal{A}$  and  $\mathcal{B}$  two additive categories. We say that there is a duality between  $\mathcal{A}$  and  $\mathcal{B}$ , if there exists a pair  $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$  of contravariant additive functors together with natural transformations  $\theta : id_{\mathcal{A}} \rightarrow GF$ ,  $\eta : id_{\mathcal{B}} \rightarrow FG$  such that  $F\theta \circ \eta_F = F$  and  $G\eta \circ \theta_G = G$ .

**3.1. The representative functions functor  $\mathcal{R}_{\mathbb{k}} : \text{Grpd}^{op} \rightarrow \text{CHAlgd}_{\mathbb{k}}$ .** We denote by  $\text{Grpd}$  the category of discrete groupoids and by  $\text{CHAlgd}_{\mathbb{k}}$  the category of commutative Hopf algebroids with ground field  $\mathbb{k}$ , that is, all the involved commutative rings are  $\mathbb{k}$ -algebras.

Let  $\phi : \mathcal{G} \rightarrow \mathcal{H}$  be a morphism of groupoids. To distinguish between the fiber functors of  $\mathcal{G}$  and  $\mathcal{H}$ , we use the following notations  $\omega^{\mathcal{G}} : \mathbf{Rep}_{\mathbb{k}}(\mathcal{G}) \rightarrow \text{proj}(\text{M}_{\mathbb{k}}(\mathcal{G}_0))$  and  $\omega^{\mathcal{H}} : \mathbf{Rep}_{\mathbb{k}}(\mathcal{H}) \rightarrow \text{proj}(\text{M}_{\mathbb{k}}(\mathcal{H}_0))$ . We consider  $\text{M}_{\mathbb{k}}(\mathcal{G}_0)$  as an  $\text{M}_{\mathbb{k}}(\mathcal{H}_0)$ -module via the extension  $\text{M}_{\mathbb{k}}(\phi_0) : \text{M}_{\mathbb{k}}(\mathcal{H}_0) \rightarrow \text{M}_{\mathbb{k}}(\mathcal{G}_0)$ . Our aim here is to show that representative functions is a functorial construction.

**LEMMA 3.1.** *Let  $\phi : \mathcal{G} \rightarrow \mathcal{H}$  be a morphism of groupoids and consider the restriction functor  $\mathcal{R}(\phi)$  as defined in Lemma 1.6. Then the following diagram of functors*

$$\begin{array}{ccc} \mathbf{Rep}_{\mathbb{k}}(\mathcal{H}) & \xrightarrow{\mathcal{R}(\phi)} & \mathbf{Rep}_{\mathbb{k}}(\mathcal{G}) \\ \omega^{\mathcal{H}} \downarrow & & \downarrow \omega^{\mathcal{G}} \\ \text{proj}(\text{M}_{\mathbb{k}}(\mathcal{H}_0)) & \xrightarrow{\text{M}_{\mathbb{k}}(\phi_0)^*} & \text{proj}(\text{M}_{\mathbb{k}}(\mathcal{G}_0)) \end{array}$$

is strictly commutative.

*Proof.* Let  $\varrho^{\mathcal{F}}$  be an  $\mathcal{H}$ -representation. We know that the underlying bundle of  $\mathcal{R}(\phi)(\varrho^{\mathcal{F}})$  is the induced (pull-back) bundle  $\phi_0^*(\mathcal{F})$ . Therefore, its 'global' sections are given by the tensor product module  $\Gamma(\phi_0^*(\mathcal{F})) = \Gamma(\mathcal{F}) \otimes_{\text{M}_{\mathbb{k}}(\mathcal{H}_0)} \text{M}_{\mathbb{k}}(\mathcal{G}_0)$ . The same happens to morphisms between  $\mathcal{H}$ -representations. That is, we have  $\omega^{\mathcal{H}} \circ \mathcal{R}(\phi) = (- \otimes_{\text{M}_{\mathbb{k}}(\mathcal{H}_0)} \text{M}_{\mathbb{k}}(\mathcal{G}_0)) \circ \omega^{\mathcal{G}}$  and the state diagram is commutative.  $\square$

**PROPOSITION 3.2.** *The assignment  $\mathcal{R}_{\mathbb{k}} : \text{Grpd} \rightarrow \text{CHAlgd}_{\mathbb{k}}$  which sends any discrete groupoid  $\mathcal{G}$  to its Hopf algebroid  $(\text{M}_{\mathbb{k}}(\mathcal{G}_0), \mathcal{R}_{\mathbb{k}}(\mathcal{G}))$  of representative functions, is a well defined contravariant functor with image in the full subcategory of geometrically transitive Hopf algebroids. Furthermore, for any morphism of groupoids  $\phi : \mathcal{G} \rightarrow \mathcal{H}$ , we have the following commutative diagram of algebras*

$$\begin{array}{ccc} \mathcal{R}_{\mathbb{k}}(\mathcal{H}) & \xrightarrow{\mathcal{R}_{\mathbb{k}}(\phi)} & \mathcal{R}_{\mathbb{k}}(\mathcal{G}) \\ \zeta_{\mathcal{H}} \downarrow & & \downarrow \zeta_{\mathcal{G}} \\ \text{M}_{\mathbb{k}}(\mathcal{H}_1) & \xrightarrow{\text{M}_{\mathbb{k}}(\phi_1)} & \text{M}_{\mathbb{k}}(\mathcal{G}_1) \end{array}$$

where  $\zeta$  is the algebra map of Proposition 2.2.

*Proof.* The first part of this proof is in fact a sketch of the proof of the general statement given in Lemma A.1, see Appendix.

For simplicity we denote here the functor  $\mathcal{R}(\phi)$  by  $(-)\phi$ . So let  $\varrho^{\mathcal{F}}$  be any  $\mathcal{H}$ -representation, using Lemma 3.1, we can show that there is a morphism of  $\text{M}_{\mathbb{k}}(\mathcal{H}_0)$ -bimodules:

$$\begin{array}{ccc} \omega^{\mathcal{H}}(\varrho^{\mathcal{F}})^* \otimes_{\mathbb{k}} \omega^{\mathcal{H}}(\varrho^{\mathcal{F}}) & \xrightarrow{\quad} & \omega^{\mathcal{G}}(\varrho^{\mathcal{F}\phi})^* \otimes_{\mathbb{k}} \omega^{\mathcal{G}}(\varrho^{\mathcal{F}\phi}) \\ & \searrow \theta_{\mathcal{F}} \text{ (dashed)} & \downarrow \\ & & \mathcal{R}_{\mathbb{k}}(\mathcal{G}) \end{array}$$

where the horizontal arrow is clear and the vertical arrow is one of the canonical morphisms of  $\text{M}_{\mathbb{k}}(\mathcal{G}_0)$ -bimodules defining the universal object  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$ .

It turns out that the pair  $(\mathcal{R}_k(\mathcal{G}), \theta_{\mathcal{F}})_{\mathcal{G}^{\mathcal{F}} \in \mathcal{R}\text{ep}_k(\mathcal{H})}$  is a dinatural transformation for the bifunctor  $\omega^{\mathcal{H}}(-)^* \otimes_k \omega^{\mathcal{H}}(-)$  in  $M_k(\mathcal{H}_0)$ -bimodules. Therefore, by the universal property of  $\mathcal{R}_k(\mathcal{H})$ , there is a morphism of  $M_k(\mathcal{H}_0)$ -bimodules  $\mathcal{R}_k(\phi) : \mathcal{R}_k(\mathcal{H}) \rightarrow \mathcal{R}_k(\mathcal{G})$  which in fact is a morphism of Hopf algebroids. The remainder axioms on  $\mathcal{R}_k$  to be a functor, are easily checked. The commutativity of the stated diagram is directly obtained from the basic properties of the restriction functor  $\mathcal{R}$  given in Lemma 1.6.  $\square$

Before given examples concerning Hopf algebroids of representative functions, let us first recall the notion of *induced groupoid* which we will use in the sequel.

EXAMPLE 3.3. Let  $\mathcal{H}$  be a groupoid and  $u : P \rightarrow \mathcal{H}_0$  any map. Consider the following set

$$P_1 := P \times_u \times_s \mathcal{H}_1 \times_u P = \left\{ (p, g, q) \in P \times \mathcal{H}_1 \times P \mid u(p) = s(g), u(q) = t(g) \right\}.$$

Clearly  $(P_1, P)$  has a structure of groupoid<sup>(9)</sup> and  $(pr_2, u) : (P_1, P) \rightarrow (\mathcal{H}_1, \mathcal{H}_0)$  is a morphism of groupoids, where  $pr_2 : P_1 \rightarrow \mathcal{H}_1$  is the second projection. This is the *induced groupoid* by  $u$  and denoted by  $\mathcal{H}_u$ . For instance, if we fix a point  $x \in \mathcal{H}_0$ , we can then take  $P = \{x\}$  and show that the groupoid  $(\{x\}_1, \{x\})$  is in fact a group which coincides with the isotropy group of  $\mathcal{H}$  at  $x$ .

Furthermore, any morphism  $\phi : \mathcal{G} \rightarrow \mathcal{H}$  of groupoids can be lifted to the induced groupoid by  $\phi_0$ . That is, we have a morphism of groupoids  $\psi : \mathcal{G} \rightarrow \mathcal{G}_{\phi_0}$ , where  $\psi_0 = \phi_0$  and  $\psi_1 : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times_s \mathcal{H}_1 \times_{\phi_0} \mathcal{G}_0$  sends  $g \mapsto (s(g), \phi_1(g), t(g))$ .

EXAMPLES 3.4. Since groups are in a canonical way related with groupoids, it is natural to try to see, using the functor  $\mathcal{R}_k$ , how this relation behave at the level of Hopf algebras and Hopf algebroids (here Hopf algebra are consider as Hopf algebroid with source equal target).

- (1) Let  $\mathcal{G}$  be a groupoid and consider the (full) sub-groupoid  $\mathcal{G}^i \hookrightarrow \mathcal{G}$  consisting of arrows  $\mathcal{G}^i_1 = \{g \in \mathcal{G}_1 \mid s(g) = t(g)\}$  and objects  $\mathcal{G}^i_0 = \mathcal{G}_0$  (i.e., the *isotropy groupoid of  $\mathcal{G}$* ). Then we have a surjective map  $\mathcal{R}_k(\mathcal{G}) \twoheadrightarrow \mathcal{R}_k(\mathcal{G}^i)$  of Hopf  $M_k(\mathcal{G}_0)$ -algebroids, where  $\mathcal{R}_k(\mathcal{G})$  is the quotient Hopf  $M_k(\mathcal{G}_0)$ -algebra of  $\mathcal{R}_k(\mathcal{G})$  by the ideal generated by the set  $\{b \in M_k(\mathcal{G}_0) \mid s(b) - t(b)\}$ .
- (2) Let  $\mathcal{G}$  be a groupoid and fix a point  $x \in \mathcal{G}_0$ . Consider the isotropy group at  $x$ , that is, the group  $G_x = \{g \in \mathcal{G}_1 \mid s(g) = t(g) = x\}$ . This point gives a rise to the  $\mathbb{k}$ -algebra map  $ev_x : M_k(\mathcal{G}_0) \rightarrow \mathbb{k}$  sending  $a \mapsto a(x)$ . We denote by  $\mathbb{k}_x$  the  $M_k(\mathcal{G}_0)$ -algebra  $\mathbb{k}$  via the extension  $ev_x$ . In this way the monomorphism of groupoids  $\times = (i_x, x) : (G_x, \{x\}) \hookrightarrow \mathcal{G}$ , where we denote  $x : \{x\} \hookrightarrow \mathcal{G}_0$ , leads to a morphism of Hopf algebroid

$$(ev_x, \mathcal{R}_k(\times)) : (M_k(\mathcal{G}_0), \mathcal{R}_k(\mathcal{G})) \longrightarrow (\mathbb{k}_x, \mathcal{R}_k(G_x)). \quad (9)$$

This Hopf  $\mathbb{k}$ -algebra is called *the isotropy Hopf algebra* at the point  $ev_x$  (which in fact corresponds to the isotropy group at  $x$ ).

On the other hand, it is easily checked that the base extension Hopf algebroid  $(\mathbb{k}_x, \mathbb{k}_x \otimes_{M_k(\mathcal{G}_0)} \mathcal{R}_k(\mathcal{G}) \otimes_{M_k(\mathcal{G}_0)} \mathbb{k}_x)$  coincides with the Hopf algebra  $(\mathbb{k}_x, \mathcal{R}_k(\mathcal{G}_x))$  of the induced groupoid  $\mathcal{G}_x$  by  $x$ . Therefore, it is isomorphic to the isotropy Hopf  $\mathbb{k}$ -algebra  $\mathcal{R}_k(G_x)$ , since  $\mathcal{G}_x$  and  $G_x$  are isomorphic as groups, see Example 3.3.

The geometrically transitive property of the Hopf algebroid  $(M_k(\mathcal{G}_0), \mathcal{R}_k(\mathcal{G}))$  could be then interpreted by saying that, for any point  $x \in \mathcal{G}_0$ , the extended map of Hopf  $\mathbb{k}$ -algebras

$$\mathbb{k}_x \otimes_{M_k(\mathcal{G}_0)} \mathcal{R}_k(\mathcal{G}) \otimes_{M_k(\mathcal{G}_0)} \mathbb{k}_x \longrightarrow \mathcal{R}_k(G_x) \quad (10)$$

is an isomorphism and that the category of  $\mathcal{R}_k(\mathcal{G})$ -comodules is equivalent (as symmetric monoidal category) to the category of  $\mathcal{R}_k(G_x)$ -comodules, as was expound in [10].

- (3) Let  $X$  be a non empty set and consider its unit groupoid  $\mathcal{U}(X)$  together with its groupoid of pairs  $\mathcal{V}(X)$ , see Example 2.3. The diagonal map clearly gives a monomorphism of groupoids  $\mathcal{U}(X) \hookrightarrow \mathcal{V}(X)$ . Then we have a morphism of Hopf  $M_k(X)$ -algebroids  $\mathcal{R}_k(\mathcal{V}(X)) \rightarrow \mathcal{R}_k(\mathcal{U}(X))$  which in fact can be constructed elementarywise using the description offered in Example 2.3. Although,  $\mathcal{R}_k(\mathcal{V}(X))$  and  $\mathcal{R}_k(\mathcal{U}(X))$  are not isomorphic part of their structures behave similar in the sense that they have the same class of isotropy Hopf algebras.

<sup>(9)</sup> If  $\mathcal{H}$  is a groupoid with only one object, that is, a group, and  $P$  is any set then one recovers the groupoid defined in Example 2.4.

- (4) Consider an action groupoid  $\mathcal{G}$  with group  $G$  acting on non empty set  $X$ . As we have seen in Example 1.7, the projection  $G \times X \rightarrow G$  is a morphism of groupoids which then leads to a morphism of Hopf algebroids  $(\mathbb{k}, \mathcal{R}_{\mathbb{k}}(G)) \rightarrow (\mathbb{M}_{\mathbb{k}}(X), \mathcal{R}_{\mathbb{k}}(G \times X))$ . At this level of generality, one can not expect to have an isomorphism  $\mathcal{R}_{\mathbb{k}}(G \times X) \cong \mathbb{M}_{\mathbb{k}}(X) \otimes_{\mathbb{k}} \mathcal{R}_{\mathbb{k}}(G)$  of Hopf  $\mathbb{M}_{\mathbb{k}}(X)$ -algebroid, since in general there is no way to convert  $\mathbb{M}_{\mathbb{k}}(X)$  into  $\mathcal{R}_{\mathbb{k}}(G)$ -comodule algebra. This happens perhaps only when one is restricted to a special class of representative functions and assuming a more rich structure on both  $G$  and  $X$ , see the forthcoming examples.

EXAMPLE 3.5. Assume that  $\mathcal{G}$  is an algebraic affine action  $\mathbb{k}$ -groupoid with  $\mathbb{k}$  an infinite field. Precisely, consider  $\mathcal{G}_1 = G \times X$ , where  $G$  is an algebraic affine  $\mathbb{k}$ -group acting on an affine algebraic  $\mathbb{k}$ -set  $X = \mathcal{G}_0$  and assume that the action is a morphism of algebraic  $\mathbb{k}$ -sets. Then the algebra  $\mathcal{P}(G \times X)$  of polynomial representative functions on  $G \times X$  can be endowed within a Hopf  $\mathcal{P}(X)$ -algebroid structure which splits into a tensor product  $\mathcal{P}(X) \otimes_{\mathbb{k}} \mathcal{P}(G)$  (see [15] for basic definitions and notations). This is of course an example of a topological action groupoid  $\mathcal{G}$  where we have a chain of homomorphisms of Hopf algebroids

$$(\mathcal{P}(X), \mathcal{P}(G \times X)) \subseteq (\mathbb{C}_{\mathbb{k}}(\mathcal{G}_0), \mathcal{R}_{\mathbb{k}}^{top}(\mathcal{G})) \subseteq (\mathbb{M}_{\mathbb{k}}(\mathcal{G}_0), \mathcal{R}_{\mathbb{k}}(\mathcal{G})).$$

In this situation there is no a clearer way to assert that  $\mathcal{P}(G \times X)$  is geometrically transitive Hopf algebroid. However, if we assume that the  $G$ -action is free and transitive, then  $X \rightarrow X/G$  is a principal  $G$ -bundle and we have an isomorphism  $\mathcal{P}(X) \otimes_{\mathbb{k}} \mathcal{P}(X) \rightarrow \mathcal{P}(G \times X)$  of Hopf algebroids, From which we deduce that  $\mathcal{P}(G \times X)$  is geometrically transitive, since so is  $\mathcal{P}(X) \otimes_{\mathbb{k}} \mathcal{P}(X)$ .

On the other hand, the geometric transitive property of the Hopf algebroid  $\mathcal{P}(G \times X)$  could be related as follows to the Equivariant Serre Problem [19]. Assume that  $X$  is the affine  $n$ -space and that  $\mathcal{P}(G \times X)$  is geometrically transitive. Then any object in the category  $\text{comod}_{\mathcal{P}(G \times X)}$  is finitely generated and projective  $\mathcal{P}(X)$ -module and thus free by Quillen-Suslin's theorem. In particular, this means that any algebraic  $G$ -equivariant bundle is trivial.

There is also another situation where the Equivariant Serre Problem for general affine algebraic  $\mathbb{k}$ -set  $X$ , have a positive answer. Specifically, since we know that there is a Hopf algebroid morphism  $(\eta_0, \eta_1) : (\mathbb{k}, \mathcal{P}(G)) \rightarrow (\mathcal{P}(X), \mathcal{P}(G \times X))$ , we can associate to it the induction functor  $\eta^* : \text{Comod}_{\mathcal{P}(G)} \rightarrow \text{Comod}_{\mathcal{P}(G \times X)}$  which have the ad-induction functor  $\eta_*$  as right adjoint. So if we assume that the counit of this adjunction is a natural isomorphism, then any object  $M$  in the category  $\text{comod}_{\mathcal{P}(G \times X)}$  is isomorphic to  $\eta_*(M) \otimes_{\mathbb{k}} \mathcal{P}(X) \cong M$ , and thus it is a free  $\mathcal{P}(X)$ -module, which in particular means that any algebraic  $G$ -equivariant bundle is trivial.

EXAMPLE 3.6. Let  $\mathcal{G}$  be a transitive groupoid, that is, the map  $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$  is surjective. Fix an object  $x \in \mathcal{G}_0$  and choose a family of arrows  $\{\tau_y\}_{y \in \mathcal{G}_0} \subseteq \mathcal{G}_1$  where each  $\tau_y \in t^{-1}(\{x\})$  and  $s(\tau_y) = y$ , for  $y \neq x$  and  $\tau_x = \iota(x)$ , for  $y = x$ . It is well known (see for instance [5]), that there is a (non canonical) isomorphism of groupoids

$$\phi^x : (\mathcal{G}_1, \mathcal{G}_0) \xrightarrow{\cong} (\mathcal{G}_0 \times G_x \times \mathcal{G}_0, \mathcal{G}_0) \quad \left( g \mapsto (s(g), \tau_{t(g)} g \tau_{s(g)}^{-1}, t(g)), id_{\mathcal{G}_0} \right)$$

where the right-hand side groupoid is the one defined as in Example 2.4. By applying the functor  $\mathcal{R}_{\mathbb{k}}$  and the isomorphism established in Example 2.4, we obtain then a chain of isomorphism of Hopf algebroids

$$(\mathbb{M}_{\mathbb{k}}(\mathcal{G}_0), \mathcal{R}_{\mathbb{k}}(\mathcal{G})) \cong (\mathbb{M}_{\mathbb{k}}(\mathcal{G}_0), \mathcal{R}_{\mathbb{k}}(\mathcal{G}_0 \times G_x \times \mathcal{G}_0)) \cong (\mathbb{M}_{\mathbb{k}}(\mathcal{G}_0), \mathbb{M}_{\mathbb{k}}(\mathcal{G}_0) \otimes_{\mathbb{k}} \mathcal{R}_{\mathbb{k}}(G_x) \otimes_{\mathbb{k}} \mathbb{M}_{\mathbb{k}}(\mathcal{G}_0))$$

whose composition turns to be the extended Hopf  $\mathbb{M}_{\mathbb{k}}(\mathcal{G}_0)$ -algebroids morphism of the Hopf algebra isomorphism given in (10).

On the other hand, as was shown in Example 3.4(2), for a general groupoid  $\mathcal{G}$  the geometrically transitive property of its Hopf algebroid  $(\mathbb{M}_{\mathbb{k}}(\mathcal{G}_0), \mathcal{R}_{\mathbb{k}}(\mathcal{G}))$  says that, up to isomorphisms of Hopf algebras, we only have one type of isotropy Hopf algebras. However, there is no way to give an explicit description of such isomorphisms. In difference when  $\mathcal{G}$  is transitive, these isomorphisms are precisely given by conjugating the isotropy groups of  $\mathcal{G}$ , before applying the functor  $\mathcal{R}_{\mathbb{k}}$ .

**3.2. The character functor**  $\mathcal{X}_{\mathbb{k}} : \text{CHAlgd}_{\mathbb{k}}^{\circ p} \rightarrow \text{Grpd}$ . In this section we recall the definition of the character functor. All Hopf algebroids are considered over the ground field  $\mathbb{k}$ . We are implicitly assuming that the base  $\mathbb{k}$ -algebra  $R$  of any Hopf algebroid have the property that  $R(\mathbb{k}) \neq \emptyset$ .

Let  $(R, \mathcal{H})$  be a Hopf algebroid. As in the case of Hopf algebra over fields, it is well known that we can define the *characters groupoid* of  $(R, \mathcal{H})$  as the groupoid

$$\mathcal{X}_{\mathbb{k}}(R, \mathcal{H}) : \mathcal{H}(\mathbb{k}) \begin{array}{c} \xrightarrow{s^*} \\ \xleftarrow{c^*} \\ \xrightarrow{t^*} \\ \xleftarrow{c^*} \end{array} R(\mathbb{k}),$$

by dualizing the source, target and the counit of  $(R, \mathcal{H})$ . The rest of the axioms defining the underlying category of this groupoid are easily verified, whence we have identified  $(\mathcal{H} \otimes_R \mathcal{H})(\mathbb{k})$  with  $\mathcal{H}(\mathbb{k})_{s^*} \times_{t^*} \mathcal{H}(\mathbb{k})$  using  $\mathcal{H}$  as an  $R$ -bimodule with  $t$  acting on the left and  $s$  on the right. The inverse map is given by  $\mathcal{S}^* : \mathcal{H}(\mathbb{k}) \rightarrow \mathcal{H}(\mathbb{k})$ , where  $\mathcal{S} : \mathcal{H} \rightarrow \mathcal{H}$  is the antipode of  $(R, \mathcal{H})$ .

Now, let  $\alpha = (\alpha_0, \alpha_1) : (R, \mathcal{H}) \rightarrow (S, \mathcal{K})$  be a morphism of Hopf algebroids. Then clearly we obtain a morphism between character groupoids:

$$\begin{array}{ccc} \mathcal{X}_{\mathbb{k}}(S, \mathcal{K}) : & & \mathcal{K}(\mathbb{k}) \begin{array}{c} \xrightarrow{s^*} \\ \xleftarrow{c^*} \\ \xrightarrow{t^*} \\ \xleftarrow{c^*} \end{array} S(\mathbb{k}) \\ \downarrow \mathcal{X}_{\mathbb{k}}(\alpha) & & \downarrow \alpha_1^* \quad \quad \quad \downarrow \alpha_0^* \\ \mathcal{X}_{\mathbb{k}}(R, \mathcal{H}) : & & \mathcal{H}(\mathbb{k}) \begin{array}{c} \xrightarrow{s^*} \\ \xleftarrow{c^*} \\ \xrightarrow{t^*} \\ \xleftarrow{c^*} \end{array} R(\mathbb{k}). \end{array}$$

LEMMA 3.7. *The characters groupoid*

$$\mathcal{X}_{\mathbb{k}} : \text{CHAlgd}_{\mathbb{k}} \longrightarrow \text{Grpd}$$

*establishes a well defined contravariant functor.*

*Proof.* Straightforward. □

**3.3. The unit and counit of the duality.** The restriction of the characters groupoid functor to the full subcategory of geometrically transitive Hopf algebroids  $\text{GTCHAlgd}_{\mathbb{k}}$  will be also denoted by  $\mathcal{X}_{\mathbb{k}}$ . Let  $\mathcal{G}$  be an object in  $\text{Grpd}$  and consider the Hopf algebroid  $(M_{\mathbb{k}}(\mathcal{G}_0), \mathcal{R}_{\mathbb{k}}(\mathcal{G}))$  of representative functions on  $\mathcal{G}$  which by Proposition 2.5 we know that it is an object in  $\text{GTCHAlgd}_{\mathbb{k}}$ .

Let us define the following two maps  $\Theta_{\mathcal{G}_0} = ev : \mathcal{G}_0 \rightarrow M_{\mathbb{k}}(\mathcal{G}_0)(\mathbb{k})$  by evaluating on each point of  $\mathcal{G}_0$  and  $\Theta_{\mathcal{G}_1} : \mathcal{G}_1 \rightarrow \mathcal{R}_{\mathbb{k}}(\mathcal{G})(\mathbb{k})$  by applying the map  $\zeta$  of Proposition 2.2. Explicitly, using the notation of Proposition 2.2,  $\Theta_{\mathcal{G}_1}$  is defined by sending

$$\mathcal{G}_1 \ni g \longmapsto \left[ \overline{\varphi \otimes_{T_{\mathcal{E}}} p} \longmapsto \varphi(t(g)) \left( \varrho_g^{\mathcal{E}}(p(s(g))) \right) \right] \in \mathcal{R}_{\mathbb{k}}(\mathcal{G})(\mathbb{k}).$$

LEMMA 3.8. *Keep the above notations. We have*

- (i) *For each groupoid  $\mathcal{G}$ , the pair  $(\Theta_{\mathcal{G}_0}, \Theta_{\mathcal{G}_1}) : \mathcal{G} \rightarrow \mathcal{X}_{\mathbb{k}} \circ \mathcal{R}_{\mathbb{k}}(\mathcal{G})$  is a morphism of groupoids.*
- (ii)  $\Theta_- : id_{\text{Grpd}} \rightarrow \mathcal{X}_{\mathbb{k}} \circ \mathcal{R}_{\mathbb{k}}$  *is a natural transformation.*

*Proof.* (i). The compatibility of  $\Theta_{\mathcal{G}}$  with both sources and targets of the two groupoids is easily checked. To verify the compatibility of  $\Theta_{\mathcal{G}}$  with the identity and the inverse maps, we use the first two items stated in Proposition 2.2. Let us then check that  $\Theta_{\mathcal{G}}$  is compatible with the compositions. Take  $g, f \in \mathcal{G}_1$  with  $s(g) = t(f)$ , and fix an element of the form  $\overline{\varphi \otimes_{T_{\mathcal{E}}} p}$  in  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$  given by the  $\mathcal{G}$ -representation  $\varrho^{\mathcal{E}}$  with rank  $n$  and a global dual basis  $\{s_i, s_i^*\}_i^n$  of the sections of its underlying bundle. Since we know that

$$\Theta_{\mathcal{G}_1}(h)(\overline{\varphi \otimes_{T_{\mathcal{E}}} p}) = \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(h), \quad \forall h \in \mathcal{G}_1,$$

we have, by applying the equality of item (3) in Proposition 2.2, that

$$\begin{aligned} \Theta_{\mathcal{G}_1}(gf)(\overline{\varphi \otimes_{T_{\mathcal{E}}} p}) &= \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(gf) \\ &= \zeta\left(\overline{(\varphi \otimes_{T_{\mathcal{E}}} p)_1}\right)(g) \zeta\left(\overline{(\varphi \otimes_{T_{\mathcal{E}}} p)_2}\right)(f) \\ &= \sum_i \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} s_i})(g) \zeta(\overline{s_i^* \otimes_{T_{\mathcal{E}}} p})(f) \\ &= \sum_i \Theta_{\mathcal{G}_1}(g)(\overline{\varphi \otimes_{T_{\mathcal{E}}} s_i}) \Theta_{\mathcal{G}_1}(f)(\overline{s_i^* \otimes_{T_{\mathcal{E}}} p}) \end{aligned}$$



$$= (\Theta_{\mathcal{G}_1}(g)\Theta_{\mathcal{G}_1}(f)) \left( \overline{\varphi \otimes_{\mathbb{T}_\varepsilon} p} \right)$$

where the last equality follows from the definition of the functor  $\chi_{\mathbb{k}}$  and the comultiplication of  $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$ . This shows that  $\Theta_{\mathcal{G}_1}(g)\Theta_{\mathcal{G}_1}(f) = \Theta_{\mathcal{G}_1}(gf)$  and finishes the proof of item (i).

(ii). Let  $\phi : \mathcal{G} \rightarrow \mathcal{H}$  be a morphism of groupoids. We need then to check the following two equalities:

$$\Theta_{\mathcal{H}_0} \circ \phi_0 = (\mathcal{X}_{\mathbb{k}} \circ \mathcal{R}_{\mathbb{k}}(\phi))_0 \circ \Theta_{\mathcal{G}_0}, \quad (\mathcal{X}_{\mathbb{k}} \circ \mathcal{R}_{\mathbb{k}}(\phi))_1 \circ \Theta_{\mathcal{G}_1} = \Theta_{\mathcal{H}_1} \circ \phi_1.$$

The first equality is easily deduced from the definition of the involved maps. The second one uses the commutative diagram of Proposition 3.2.  $\square$

Consider now a geometrically transitive Hopf algebroid  $(R, \mathcal{H})$  and denote by  $\text{ev} : R \rightarrow M_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0) = M_{\mathbb{k}}(R(\mathbb{k}))$  the evaluation algebra map. We know from [6, Théorème 7.1, or Proposition 5.8] that  $\mathcal{H}$  can be reconstructed from its category  $\text{comod}_{\mathcal{H}}$  of  $\mathcal{H}$ -comodules which are finitely generated as  $R$ -modules. On the other hand, by [10, Corollaire 3.9, item (b) in page 114], this category consist in fact of those comodule which are finitely generated and projective of constant rank over  $R$ , since we are assuming that  $R(\mathbb{k}) \neq \emptyset$ . Taking this observation into account, next we want to construct a functor from  $\text{comod}_{\mathcal{H}}$  to the category of representations of the characters groupoid of  $(R, \mathcal{H})$ . Let us denote by  $\mathcal{U}_{\mathcal{H}} : \text{comod}_{\mathcal{H}} \rightarrow \text{proj}(R)$  the forgetfull functor.

LEMMA 3.9. *Let  $(R, \mathcal{H})$  be a geometrically transitive Hopf algebroid. Then there is a functor  $\mathcal{F} : \text{comod}_{\mathcal{H}} \rightarrow \mathbf{Rep}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H}))$  which turns commutative the following diagram*

$$\begin{array}{ccc} \text{comod}_{\mathcal{H}} & \xrightarrow{\mathcal{F}} & \mathbf{Rep}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})) \\ \mathcal{U}_{\mathcal{H}} \downarrow & & \downarrow \omega \\ \text{proj}(R) & \xrightarrow{\text{ev}^*} & \text{proj}(M_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0)) \end{array}$$

*Proof.* Let  $P \in \text{comod}_{\mathcal{H}}$  with  $\mathcal{H}$ -coaction  $P \ni p \mapsto p_0 \otimes_R p_1 \in P \otimes_R \mathcal{H}$  (summation understood). Consider  $P_x = P \otimes_R \mathbb{k}_x$ , for any algebra map  $x \in R(\mathbb{k})$ . Each of the  $P_x$ 's is a finite dimensional vector  $\mathbb{k}$ -space with dimension the rank of  $P$ . We thus obtain a vector bundle  $\mathcal{E}(P) = \cup_{x \in R(\mathbb{k})} P_x$  over  $R(\mathbb{k}) = \mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0$ . Now we want to endow this bundle within an action of the groupoid  $\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})$ . Take an algebra map  $g \in \mathcal{H}(\mathbb{k})$ , we define the following map

$$\varrho_g^{\mathcal{E}(P)} : P \otimes_R \mathbb{k}_{s^*(g)} \longrightarrow P \otimes_R \mathbb{k}_{t^*(g)}, \quad \left( p \otimes_R k \longmapsto p_0 \otimes_R g(p_1)k \right) \quad (11)$$

where  $s, t$  are the source and the target of  $(R, \mathcal{H})$ . Since the identity arrow of any  $x \in R(\mathbb{k})$  is given by  $\varepsilon^*(x)$ , we have

$$\varrho_{\varepsilon^*(x)}^{\mathcal{E}(P)}(p \otimes_R k) = p_0 \otimes_R \varepsilon^*(x)(p_1)k = p_0 \otimes_R x(\varepsilon(p_1))k = p_0 \varepsilon(p_1) \otimes_R k = p \otimes_R k.$$

Hence  $\varrho_{\varepsilon^*(x)}^{\mathcal{E}(P)} = id_{P_x}$ , for every  $x \in R(\mathbb{k})$ . On the other hand, if we take two arrows  $g, f \in \mathcal{H}(\mathbb{k})$  with  $t^*(f) = s^*(g)$ , then we have the following equalities<sup>(10)</sup>

$$\varrho_g^{\mathcal{E}(P)} \circ \varrho_f^{\mathcal{E}(P)}(p \otimes_R k) = \varrho_g^{\mathcal{E}(P)}(p_0 \otimes_R f(p_1)k) = p_0 \otimes_R g(p_1)f(p_2)k = p_0 \otimes_R gf(p_1)k = \varrho_{gf}^{\mathcal{E}(P)}(p \otimes_R k),$$

which means that  $\varrho_g^{\mathcal{E}(P)} \circ \varrho_f^{\mathcal{E}(P)} = \varrho_{gf}^{\mathcal{E}(P)}$ . We have then show that  $\varrho^{\mathcal{E}(P)}$  satisfies the cocycle condition, and also it is an isomorphism at each  $g \in \mathcal{H}(\mathbb{k})$ . Therefore,  $\varrho^{\mathcal{E}(P)}$  is an  $\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})$ -representation. This gives the definition of the functor  $\mathcal{F}$  on objects.

Now it is easily seen, using the definition of the action given in (11), that any morphism  $P \rightarrow Q$  in  $\text{comod}_{\mathcal{H}}$  gives a rise to a morphism  $\mathcal{E}(P) \rightarrow \mathcal{E}(Q)$  between the associated  $\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})$ -representations, and the functor  $\mathcal{F}$  is now established. Lastly, the stated diagram is commutative since we know that, for every comodule  $P \in \text{comod}_{\mathcal{H}}$ , the module of global section of  $\mathcal{E}(P)$  can be identified, as  $M_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0)$ -module, with the tensor product  $P \otimes_R M_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0)$ .  $\square$

<sup>(10)</sup> This is essentially the proof presented by diagrams in [16, p.1299]

PROPOSITION 3.10. *Let  $(R, \mathcal{H})$  be a geometrically transitive Hopf algebroid. Then there is a morphism of Hopf algebroids  $\Omega_{(R, \mathcal{H})} : (R, \mathcal{H}) \rightarrow (\mathbb{M}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0), \mathcal{R}_{\mathbb{k}} \circ \mathcal{X}_{\mathbb{k}}(R, \mathcal{H}))$ . Furthermore,  $\Omega_- : id_{\text{GTCHAL}_{\text{gd}_{\mathbb{k}}}} \rightarrow \mathcal{R}_{\mathbb{k}} \circ \mathcal{X}_{\mathbb{k}}$  is a natural transformation.*

*Proof.* Since  $(R, \mathcal{H})$  is geometrically transitive we know from [6, Théorème 7.1] that the pair  $(\text{comod}_{\mathcal{H}}, \mathcal{U}_{\mathcal{H}})$  is a Tannakian  $\mathbb{k}$ -linear category and that the canonical morphism of Hopf algebroids  $(R, \mathcal{L}_{\mathbb{k}}(\mathcal{U}_{\mathcal{H}})) \rightarrow (R, \mathcal{H})$  is an isomorphism, where  $\mathcal{L}_{\mathbb{k}}(\mathcal{U}_{\mathcal{H}})$  is the universal object attached to  $(\text{comod}_{\mathcal{H}}, \mathcal{U}_{\mathcal{H}})$ , see the Appendix. Using the functor  $\mathcal{F}_{\mathcal{H}}$  constructed in Lemma 3.9, we know by Lemma A.1 that there is a morphism

$$\mathcal{L}_{\mathbb{k}}(\mathcal{F}_{\mathcal{H}}) : (R, \mathcal{L}_{\mathbb{k}}(\mathcal{U}_{\mathcal{H}})) \rightarrow (\mathbb{M}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0), \mathcal{R}_{\mathbb{k}} \mathcal{X}_{\mathbb{k}}(R, \mathcal{H}))$$

of Hopf algebroids with base change map  $\mathbf{ev} : R \rightarrow \mathbb{M}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0) = \mathbb{M}_{\mathbb{k}}(R(\mathbb{k}))$  the evaluation algebra map. Therefore, we have a composition of morphism

$$\Omega_{(R, \mathcal{H})} : (R, \mathcal{H}) \cong (R, \mathcal{L}_{\mathbb{k}}(\mathcal{U}_{\mathcal{H}})) \rightarrow (\mathbb{M}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0), \mathcal{R}_{\mathbb{k}} \mathcal{X}_{\mathbb{k}}(R, \mathcal{H})) \quad (12)$$

of Hopf algebroids as it was stated.

To show that  $\Omega_-$  is natural, one need to check the following two equalities

$$(\mathcal{R}_{\mathbb{k}} \circ \mathcal{X}_{\mathbb{k}}(\alpha))_0 \circ \Omega_{(R, \mathcal{H})_0} = \Omega_{(S, \mathcal{K})_0} \circ \alpha_0, \quad \Omega_{(S, \mathcal{K})_1} \circ \alpha_1 = (\mathcal{R}_{\mathbb{k}} \circ \mathcal{X}_{\mathbb{k}}(\alpha))_1 \circ \Omega_{(R, \mathcal{H})_1}$$

for any morphism  $\alpha = (\alpha_0, \alpha_1) : (R, \mathcal{H}) \rightarrow (S, \mathcal{K})$  between geometrically transitive Hopf algebroids. The first equality is obviously obtained from the definitions. While the proof of the second is given as follows. Since  $\alpha$  is a morphism of Hopf algebroids, we have the following commutative diagram

$$\begin{array}{ccccc}
 & & \text{comod}_{\mathcal{K}} & \xrightarrow{\mathcal{F}_{\mathcal{K}}} & \mathbf{Rep}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(S, \mathcal{K})) \\
 & \nearrow \text{---} \otimes_R S & & \nearrow \mathcal{R}(\mathcal{X}_{\mathbb{k}}(\alpha)) & \downarrow \omega_{\mathcal{X}_{\mathbb{k}}(S, \mathcal{K})} \\
 \text{comod}_{\mathcal{H}} & \xrightarrow{\mathcal{F}_{\mathcal{H}}} & \mathbf{Rep}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})) & & \downarrow \mathcal{U}_{\mathcal{K}} \\
 \downarrow \mathcal{U}_{\mathcal{H}} & & \downarrow \omega_{\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})} & & \downarrow \omega_{\mathcal{X}_{\mathbb{k}}(S, \mathcal{K})} \\
 \text{proj}(R) & \xrightarrow{\mathbf{ev}^*} & \text{proj}(S) & \xrightarrow{\mathbf{ev}^*} & \text{proj}(\mathbb{M}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(S, \mathcal{K})_0)) \\
 & \nearrow \alpha_0^* & \nearrow \mathbb{M}_{\mathbb{k}}(\alpha_0(\mathbb{k}))^* & & \\
 & & \text{proj}(\mathbb{M}_{\mathbb{k}}(\mathcal{X}_{\mathbb{k}}(R, \mathcal{H})_0)) & & 
 \end{array}$$

which implies the equality  $\mathcal{F}_{\mathcal{K}} \circ (- \otimes_R S) = \mathcal{R}(\mathcal{X}_{\mathbb{k}}(\alpha)) \circ \mathcal{F}_{\mathcal{H}}$  in the category  $\text{Tanna}_{\mathbb{k}}$  (see the Appendix), where  $\mathcal{F}_-$  are the functors defined in Lemma 3.9. Now applying the functor  $\mathcal{L}_{\mathbb{k}}$  to this equality gives the desired equation.  $\square$

**3.4. The main Theorem.** Now we dispose of all ingredients to state our main theorem. We will show that the contravariant functors  $\mathcal{X}_{\mathbb{k}}$  and  $\mathcal{R}_{\mathbb{k}}$  establish a duality between geometrically transitive Hopf algebroids and discrete groupoids.

Let  $\mathcal{G}$  be a groupoid and  $B = \mathbb{M}_{\mathbb{k}}(\mathcal{G}_0)$  its base  $\mathbb{k}$ -algebra. There are various evaluating maps which will be used. So a distinguishing notation is helpful. These are

$$ev : \mathcal{G}_0 \rightarrow B(\mathbb{k}), \quad \mathbf{ev} : B \rightarrow \mathbb{M}_{\mathbb{k}}(B(\mathbb{k})), \quad \mathbf{ev} : \mathbb{M}_{\mathbb{k}}(B(\mathbb{k})) \rightarrow \mathbb{M}_{\mathbb{k}}(\mathcal{G}_0) = B$$

where the last algebra map uses the first one and we have  $\mathbf{ev} \circ \mathbf{ev} = id_B$ .

LEMMA 3.11. *Consider the natural transformations  $\Theta$  and  $\Omega$ , respectively, of Lemma 3.8(ii) and Proposition 3.10. Then for any groupoid  $\mathcal{G}$  with base  $\mathbb{k}$ -algebra  $B$ , we have*

$$\mathcal{R}_{\mathbb{k}}(\Theta_{\mathcal{G}}) \circ \Omega_{(B, \mathcal{R}_{\mathbb{k}}(\mathcal{G}))} = id_{(B, \mathcal{R}_{\mathbb{k}}(\mathcal{G}))}.$$

*Proof.* We know that  $(B, \mathcal{R}_{\mathbb{k}}(\mathcal{G}))$  is geometrically transitive. In particular, we have a monoidal equivalence of categories  $\text{comod}_{\mathcal{R}_{\mathbb{k}}(\mathcal{G})} \cong \mathbf{Rep}_{\mathbb{k}}(\mathcal{G})$  (in fact an isomorphism in the category  $\text{Tanna}_{\mathbb{k}}$ ).

In one direction the functor  $\mathbf{Rep}_k(\mathcal{G}) \rightarrow \mathbf{comod}_{\mathcal{R}_k(\mathcal{G})}$  takes any  $\mathcal{G}$ -representation  $\varrho^\mathcal{E}$ , to its global sections module  $\Gamma(\mathcal{E})$  with the following  $\mathcal{R}_k(\mathcal{G})$ -comodule structure

$$\Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E}) \otimes_B \mathcal{R}_k(\mathcal{G}), \quad \left( s \longmapsto \sum_i s_i \otimes_B \overline{(s_i^* \otimes_{T_\mathcal{E}} s)} \right)$$

where  $\{s_i, s_i^*\}$  is the dual basis of the module  $\Gamma(\mathcal{E})$ . In this way, we compose this functor with the functor  $\mathcal{F}$  of Lemma 3.9 to obtain a new functor  $\mathcal{F}_{\mathcal{R}_k(\mathcal{G})} : \mathbf{Rep}_k(\mathcal{G}) \rightarrow \mathbf{Rep}_k(\mathcal{X}_k(B, \mathcal{R}_k(\mathcal{G})))$ . We then arrive to the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathbf{Rep}_k(\mathcal{X}_k(B, \mathcal{R}_k(\mathcal{G}))) & & \\
 & \nearrow \mathcal{F}_{\mathcal{R}_k(\mathcal{G})} & \downarrow & \searrow \mathcal{R}(\Theta_\mathcal{G}) & \\
 \mathbf{Rep}_k(\mathcal{G}) & \xrightarrow{\cong} & \mathbf{Rep}_k(\mathcal{G}) & & \mathbf{Rep}_k(\mathcal{G}) \\
 \downarrow \omega_\mathcal{G} & & \downarrow \omega_{\mathcal{X}_k(B, \mathcal{R}_k(\mathcal{G}))} & & \downarrow \omega_\mathcal{G} \\
 & \nearrow \mathbf{ev}^* & \text{proj}(M_k(B(\mathbb{k}))) & \searrow \mathbf{ev}^* & \\
 \text{proj}(B) & \xrightarrow{\cong} & \text{proj}(B) & \xrightarrow{\cong} & \text{proj}(B)
 \end{array}$$

whose proof is based on the fact that for any  $\mathcal{G}$ -representation  $\varrho^\mathcal{E}$  the fibers of the underlying bundle of the  $\mathcal{G}$ -representation  $\mathcal{R}(\Theta_\mathcal{G}) \circ \mathcal{F}_{\mathcal{R}_k(\mathcal{G})}(\varrho^\mathcal{E})$  are identified with those of  $\mathcal{E}$ , as well as their  $\mathcal{G}$ -actions. Now the stated equality follows by applying the functor  $\mathcal{L}_k$  (see Appendix) to the above diagram.  $\square$

LEMMA 3.12. Consider the natural transformations  $\Theta$  and  $\Omega$ , respectively, of Lemma 3.8(ii) and Proposition 3.10. Then for any geometrically transitive Hopf algebroid  $(R, \mathcal{H})$ , we have

$$\mathcal{X}_k(\Omega_{(R, \mathcal{H})}) \circ \Theta_{\mathcal{X}_k(R, \mathcal{H})} = id_{\mathcal{X}_k(R, \mathcal{H})}.$$

*Proof.* We need to check the following two equalities

$$\mathcal{X}_k(\Omega_{(R, \mathcal{H})})_0 \circ \Theta_{\mathcal{X}_k(R, \mathcal{H})_0} = id_{R(\mathbb{k})} \quad \text{and} \quad \mathcal{X}_k(\Omega_{(R, \mathcal{H})})_1 \circ \Theta_{\mathcal{X}_k(R, \mathcal{H})_1} = id_{\mathcal{H}(\mathbb{k})}, \quad (13)$$

The right hand term of the first one is just the composition of the following two maps

$$\begin{array}{ccc}
 R(\mathbb{k}) & \longrightarrow & M_k(R(\mathbb{k}))(\mathbb{k}) & \longrightarrow & R(\mathbb{k}) \\
 x \longmapsto & & \left[ \beta \mapsto \beta(x) \right] & & \delta \longmapsto \left[ r \mapsto \delta(ev_r) \right]
 \end{array}$$

where  $ev : R \rightarrow M_k(R(\mathbb{k}))$  sends  $r \mapsto [x \mapsto x(r)]$ , which is the identity of  $R(\mathbb{k})$ .

Now we want to check the second equality in (13). To this end, we first identify  $(R, \mathcal{H})$  with the Hopf algebroid  $(R, \mathcal{L}_k(\mathbf{comod}_{\mathcal{H}}))$ . Under this identification the coaction of  $Q \in \mathbf{comod}_{\mathcal{H}}$  is given by  $q \mapsto q_0 \otimes_R q_1 = q_i \otimes_R \overline{(q_i^* \otimes_{T_Q} q)}$  for some fixed dual basis  $\{q_i, q_i^*\}$  of  $Q_R$ . On the other hand the elements of  $\mathcal{H}$  are now considered as sum of generic elements of the form  $\overline{\psi \otimes_{T_Q} q} \in \mathcal{H}$  for some  $\mathcal{H}$ -comodule  $Q \in \mathbf{comod}_{\mathcal{H}}$  with  $\psi \in Q^*$ ,  $q \in Q$ . In this way the left hand-side term of the second equality in equation (13) is explicitly given by the map

$$\mathcal{H}(\mathbb{k}) \longrightarrow \mathcal{H}(\mathbb{k}), \quad \left( g \longmapsto \left[ \overline{\psi \otimes_{T_Q} q} \mapsto (\psi \otimes_R 1)(\mathbf{t}^*(g)) \varrho_g^{\mathcal{E}(Q)}((q \otimes_R 1)(\mathbf{s}^*(g))) \right] \right) \quad (14)$$

where the bundle  $\mathcal{E}(Q)$  is  $\cup_{x \in R(\mathbb{k})} Q \otimes_R \mathbb{k}_x$  and its action  $\varrho_g^{\mathcal{E}(Q)}$  is as in equation (11). Here  $\mathbf{s}$  and  $\mathbf{t}$  are, respectively, the source and the target of  $(R, \mathcal{H})$ . Computing the resulting value in (14), we find that

$$\begin{aligned}
 (\psi \otimes_R 1)(\mathbf{t}^*(g)) \varrho_g^{\mathcal{E}(Q)}((q \otimes_R 1)(\mathbf{s}^*(g))) &= (\psi \otimes_R 1)(\mathbf{t}^*(g))(q_0 \otimes_R g(q_1)) \\
 &= ((g \circ \mathbf{t} \circ \psi) \otimes_R 1)(q_0 \otimes_R g(q_1))
 \end{aligned}$$

$$\begin{aligned}
&= g(\mathfrak{t}(\psi(q_0))g(q_1)) \\
&= g\left(\mathfrak{t}(\psi(q_0))q_1\right) \\
&= g\left(\mathfrak{t}(\psi(q_i)q_i^* \otimes_R q)\right) \\
&= g\left(\overline{\mathfrak{t}(\psi(q_i)q_i^*) \otimes_R q}\right) \\
&= g\left(\overline{\psi \otimes_R q}\right),
\end{aligned}$$

which shows that the map of equation (14) is the identity and finishes the proof.  $\square$

Our main result is the following theorem.

**THEOREM 3.13.** *Let  $\mathbb{k}$  be a field. Then the contravariant functors of  $\mathbb{k}$ -characters groupoid and  $\mathbb{k}$ -valued representative functions*

$$\mathcal{X}_{\mathbb{k}} : \text{GTCHAlgd}_{\mathbb{k}} \xleftrightarrow{\quad} \text{Grpd} : \mathcal{R}_{\mathbb{k}}$$

*establish a duality between the category of geometrically transitive commutative Hopf algebroïds and the category of discrete groupoids. That is, there is a natural isomorphism:*

$$\text{Hom}_{\text{GTCHAlgd}_{\mathbb{k}}}\left((R, \mathcal{H}); (\mathbb{M}_{\mathbb{k}}(\mathcal{G}_0), \mathcal{R}_{\mathbb{k}}(\mathcal{G}))\right) \cong \text{Hom}_{\text{Grpd}}\left(\mathcal{G}; \mathcal{X}_{\mathbb{k}}(R, \mathcal{H})\right)$$

for every  $(R, \mathcal{H}) \in \text{GTCHAlgd}_{\mathbb{k}}$  and  $\mathcal{G} \in \text{Grpd}$ .

*Proof.* This is a direct consequence of Lemmas 3.11 and 3.12.  $\square$

**REMARK 3.14.** In analogy with the anti-equivalence between compact topological groups and commutative Hopf algebras with (positive) integral<sup>(11)</sup> and dense characters group, which was proved in [14, Theorem 3.5, page 30], see also [1, Theorem 3.4.3]. It is the belief of the author that a similar duality as in Theorem 3.13 by using the functor  $\mathcal{R}_{\mathbb{k}}^{\text{top}}$ , could be performed. In this direction, we expect that it is possible to construct an anti-equivalence of categories between the category of compact topological groupoids and a certain full subcategory of commutative Hopf algebroïds. Parallel to the results of [2], such an anti-equivalence could be taught as a Tannaka-Krein duality for compact topological groupoids.

Of course, similar to the classical case of compact topological groups, here there are surely two essential difficulties which one is required to overcome. The first one is after endowing a given compact topological groupoid  $\mathcal{G}$  within a (left) Haar system, we are required to check how this can be reflected in its Hopf algebroïd  $\mathcal{R}_{\mathbb{k}}^{\text{top}}(\mathcal{G})$ . This perhaps is reflected in terms of a system of integrals in the isotropy Hopf algebras. The second difficulty, is to prove a version of Peter-Weyl's theorem in this context, namely, that  $\mathcal{R}_{\mathbb{k}}^{\text{top}}(\mathcal{G})$  is dense in the total algebra  $C_{\mathbb{k}}(\mathcal{G}_1)$ .

#### APPENDIX A. BASIC TANNAKIAN $\mathbb{k}$ -LINEAR CATEGORIES AND THE FUNCTOR $\mathcal{L}_{\mathbb{k}}$ .

We present here a brief account on Tannakian  $\mathbb{k}$ -linear<sup>(12)</sup> categories. What is essential for our use is the construction of the functor  $\mathcal{L}_{\mathbb{k}}$  from the category of all Tannakian  $\mathbb{k}$ -linear categories to the category of geometrically transitive Hopf algebroïds with ground field  $\mathbb{k}$ .

For more details on Tannakian  $\mathbb{k}$ -linear categories, we refer to [10, 11, 6], see also [12]. Recall from [10, §2] (see also [6, §2] for a weaker definition) that a symmetric monoidal rigid (or autonomous)  $\mathbb{k}$ -linear (essentially small) category  $(\mathcal{T}, \otimes, \mathbb{I})$ <sup>(13)</sup>, is said to be a  $\mathbb{k}$ -tensorial category "est une catégorie tensorielle sur  $\mathbb{k}$ ", if its underlying category  $\mathcal{T}$  is abelian and the canonical algebra map  $\mathbb{k} \rightarrow \text{End}_{\mathcal{T}}(\mathbb{I})$  is an isomorphism. Let  $(\mathcal{T}, \otimes, \mathbb{I})$  be a  $\mathbb{k}$ -tensorial category, and  $R$  a commutative  $\mathbb{k}$ -algebra. Following [10, 1.9] (see also [6, Definition p.5826]), a *fiber functor of  $\mathcal{T}$  over  $R$*  is a monoidal  $\mathbb{k}$ -linear faithful and right exact functor from  $\mathcal{T}$  to the category  $\text{proj}(R)$  of finitely generated and projective  $R$ -modules. Such a functor  $\omega : \mathcal{T} \rightarrow \text{proj}(R)$  satisfies  $\omega(\mathbb{I}) \cong R$

<sup>(11)</sup> which corresponds to the Haar measure on the group.

<sup>(12)</sup>  $\mathbb{k}$ -linear means enriched in  $\mathbb{k}$ -vector spaces.

<sup>(13)</sup> We are implicitly assuming that  $\otimes$  is a  $\mathbb{k}$ -bilinear bi-functor.

and a natural isomorphism  $\omega(- \otimes -) \cong \omega(-) \otimes_R \omega(-)$  compatible with both associativity and symmetry of  $\otimes$ . A *Tannakian  $\mathbb{k}$ -linear category* [10, 2.8], is then a  $\mathbb{k}$ -tensorial category with a fiber functor  $\omega : \mathcal{T} \rightarrow \text{proj}(R)$  (here in fact we are restricting the definition [10, 2.8] to affine schemes).

Notice here that  $\omega$  is not trivial, since we are assuming that  $R(\mathbb{k}) \neq \emptyset$ . In particular, we have by [6, Proposition 2.5] that  $\mathcal{T}$  is *locally of finite type over  $\mathbb{k}$*  (see the definition before [6, Proposition 2.5]). We denote the situation of a given Tannakian  $\mathbb{k}$ -linear category by  $(\mathcal{T}, \omega)_R$ . Tannakian categories are objects of the category  $\text{Tanna}_{\mathbb{k}}$  where a morphism  $(\mathcal{T}, \omega)_R \rightarrow (\mathcal{P}, \gamma)_S$  between two Tannakian categories consists of  $\mathbb{k}$ -algebra map  $\theta : R \rightarrow S$  and a functor  $f : \mathcal{T} \rightarrow \mathcal{P}$  such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{f} & \mathcal{P} \\ \omega \downarrow & & \downarrow \gamma \\ \text{proj}(R) & \xrightarrow{\theta^*} & \text{proj}(S). \end{array}$$

To each object in  $(\mathcal{T}, \omega)_R \in \text{Tanna}_{\mathbb{k}}$  one can associate a universal object denoted  $\mathcal{L}_{\mathbb{k}}(\omega)$  which turns to be a commutative Hopf  $R$ -algebroid. In the notation of [10], this is  $\mathbf{Aut}^{\otimes}(\omega)$  the set of all monoidal natural isomorphisms of  $\omega$ . In categorical terms, the universality of  $\mathcal{L}_{\mathbb{k}}(\omega)$  can be expressed at least in two different (rather equivalent) ways. The first one says that  $\mathcal{L}_{\mathbb{k}}(\omega)$  is the  $(R \otimes_{\mathbb{k}} R)$ -bimodule which solves the following universal problem in  $R$ -bimodules between natural transformations and  $R$ -bilinear maps:

$$\begin{aligned} \text{Nat}\left(\omega, - \otimes_R \omega\right) &\cong \text{Hom}_{R-R}\left(\mathcal{L}_{\mathbb{k}}(\omega), -\right), \\ \text{Nat}\left(\omega \otimes_R \omega, - \otimes_R (\omega \otimes_R \omega)\right) &\cong \text{Hom}_{R-R}\left(\mathcal{L}_{\mathbb{k}}(\omega) \otimes_{R^e} \mathcal{L}_{\mathbb{k}}(\omega), -\right), \end{aligned}$$

where  $R^e = R \otimes_{\mathbb{k}} R$  is the enveloping algebra of  $R$ .

The second one says that  $\mathcal{L}_{\mathbb{k}}(\omega)$  is the  $R$ -coring<sup>(14)</sup> which represents the following functor

$$R\text{-Corings} \longrightarrow \text{Sets}, \quad (\mathfrak{C} \rightarrow \{\text{functorial right } \mathfrak{C}\text{-coactions on } \omega\})$$

form the category of  $R$ -corings to sets, see [6, Proposition 4.2] for more details.

To our need a concert description of  $\mathcal{L}_{\mathbb{k}}(\omega)$  is primordial. As it is known by specialist such a description is not trivial and rather quite technical. Next, we propose three descriptions of  $\mathcal{L}_{\mathbb{k}}(\omega)$  two of them use the so called reconstruction process which was developed in [12].

Let us first fix some notations. For sake of simplicity, the vector spaces of morphisms in  $\mathcal{T}$  will be denoted by

$$T_{X,Y} := \text{Hom}_{\mathcal{T}}(X, Y), \quad T_X := \text{End}_{\mathcal{T}}(\varrho^X).$$

In this way we can perform the direct sum of  $R$ -bimodules

$$\bigoplus_{X \in \mathcal{T}} \omega(X)^* \otimes_{T_X} \omega(X), \text{ and } \mathcal{I}_{\mathcal{T}} = \left\langle \varphi \otimes_{T_Y} \alpha s - \varphi \alpha \otimes_{T_X} s \right\rangle_{\varphi \in \omega(Y)^*, s \in \omega(X), \alpha \in T_{X,Y}}, \quad (15)$$

its  $(R \otimes_{\mathbb{k}} R)$ -submodule generated by this set and where a certain canonical pairings were used.

On the other hand we set

$$\mathcal{B} := \bigoplus_{X,Y \in \mathcal{T}} T_{X,Y} = \bigoplus_{X,Y \in \mathcal{T}} \text{Hom}_{\mathcal{T}}(X, Y)$$

the Gabriel's ring<sup>(15)</sup> associated to the essentially small  $\mathbb{k}$ -linear category  $\mathcal{T}$ , together with the following unital bimodules:

$$\Sigma := \bigoplus_{X \in \mathcal{T}} \omega(X), \text{ and } \Sigma^{\dagger} := \bigoplus_{X \in \mathcal{T}} \omega(X)^*, \quad (16)$$

where  $\Sigma$  is an unital  $(\mathcal{B}, R)$ -bimodule while  $\Sigma^{\dagger}$  is an unital  $(R, \mathcal{B})$ -bimodule.

<sup>(14)</sup> What is called here an  $R$ -coring is called "k-cogébroïde de base  $R$ " in [10, 6].

<sup>(15)</sup> This is a ring with local units, namely, with enough orthogonal idempotents the identities arrows of  $\mathcal{T}$ .

In summary, the universal  $R$ -bimodule  $\mathcal{L}_k(\omega)$  which solves the above problems, is given by

$$\mathcal{L}_k(\omega) \cong \int^{X \in \mathcal{T}} \omega(X)^* \otimes_k \omega(X) \cong \frac{\bigoplus_{X \in \mathcal{T}} \omega(X^*) \otimes_{\mathcal{T}_X} \omega(X)}{\mathcal{I}_{\mathcal{T}}} \cong \Sigma^\dagger \otimes_{\mathcal{B}} \Sigma. \quad (17)$$

In this way we arrive to the following lemma which is extremely useful in this context.

LEMMA A.1. *The universal object of equation (17) establishes a well defined covariant functor*

$$\mathcal{L}_k : \mathbf{Tanna}_k \longrightarrow \mathbf{GTCHAlgd}_k$$

*to the category of geometrically transitive and commutative Hopf algebroids.*

*Proof.* This is a consequence of [6, Théorèmes 5.2, 7.1 et 8.2].  $\square$

As a matter of indication we recall that the functor  $\mathcal{L}_k$  is elementwise defined as follows. We use the second description of equation (17), that is, an element in  $\mathcal{L}_k(\omega)$  is a sum of generic elements of the form  $\overline{\varphi \otimes_{\mathcal{T}_X} x}$  which is the equivalence class of the element  $\varphi \otimes_{\mathcal{T}_X} x \in \omega(X)^* \otimes_{\mathcal{T}_X} \omega(X)$ . Consider a morphism  $(\theta, f) : (\mathcal{T}, \omega)_R \rightarrow (\mathcal{P}, \gamma)_S$  in  $\mathbf{Tanna}_k$ , then the corresponding morphism of Hopf algebroids is given by

$$(\theta, \mathcal{L}_k(f)) : (R, \mathcal{L}_k(\omega)) \longrightarrow (S, \mathcal{L}_k(\gamma)), \quad \left( \left( r, \overline{\varphi \otimes_{\mathcal{T}_X} x} \right) \mapsto \left( \theta(r), \overline{(\varphi \otimes_R 1) \otimes_{\mathcal{T}_f(X)} (x \otimes_R 1)} \right) \right)$$

where  $\varphi \otimes_R 1 \in \gamma(f(X)) = \omega(X)^* \otimes_R S$  is defined by sending  $u \otimes_R s \mapsto \varphi(u)s$ .

REMARK A.2. As one can realize the category  $\mathbf{Tanna}_k$  is in fact a 2-category with 0-cells are segments, 1-cells are squares and 2-cells are cubs. It is also possible to endow  $\mathbf{GTCHAlgd}_k$  within a structure of bicategory in such a way that  $\mathcal{L}_k$  becomes a (strict) homomorphism of bicategories. At this level, the search of a 2-adjoint of  $\mathcal{L}_k$  could be of extremely interest in this framework.

On the other hand, the construction of the functor  $\mathcal{L}_k$  can be performed in the more general case of non necessarily abelian  $\mathbb{k}$ -linear categories. Precisely, we can consider the category whose objects are three-tuples of the form  $(\mathcal{A}, \omega)_A$  where  $\mathcal{A}$  is a  $\mathbb{k}$ -linear symmetric and rigid monoidal (essentially small) category,  $\omega : \mathcal{A} \rightarrow \mathbf{proj}(A)$  is a monoidal faithful functor, and  $A$  is a commutative  $\mathbb{k}$ -algebra. The morphisms in this category are as above.

In this way, we can construct exactly as in equation (17) a functor  $\overline{\mathcal{L}}_k$  from this category to the category of commutative Hopf algebroids. For example, the algebra of continuous representative functions  $\mathcal{R}_k^{top}(\mathcal{G})$  of a topological groupoid  $\mathcal{G}$  with compact Hausdorff base space, described in Remark 2.6, is the image of the object  $(\mathbf{Rep}_k^{top}(\mathcal{G}), \omega^{top})_{C_k(\mathcal{G}_0)}$  by this functor  $\overline{\mathcal{L}}_k$ . In general, some of the properties of  $\overline{\mathcal{L}}_k(\omega)$ , the image of an object  $(\mathcal{A}, \omega)_A$ , can be deduced from that of the unital bimodule  ${}_{\mathcal{B}}\Sigma_A$  described in equation (16). For instance, one can show by using the third description of equation (7), that  $\overline{\mathcal{L}}_k(\omega)$  is projective as an  $(A \otimes_k A)$ -module, if the unital right  $(\mathcal{B} \otimes_k A)$ -module<sup>(17)</sup>  $\Sigma$  is projective.

**Acknowledgements.** I would like to thank Alain Bruguières for carefully reading the first version of the draft. I would like to thank also Fabio Gavarini and Niels Kowalzig for helpful discussions about this subject, for encourage me to write this note and for inviting me to visit the Dipartimento di Matematica, Università di Roma "Tor Vergata".

## REFERENCES

- [1] E. Abe, *Hopf algebras*. Cambridge University Press. Cambridge 1980.
- [2] M. Amini, *Tannaka-Krein duality for compact groupoids I. Representation theory*. Adv. Math. **214** (2007), 78-91.
- [3] J. Bernstein and V. Lunts, *Equivariant sheaves and functors.*, Lecture Notes in Mathematics, **1578**. Springer-Verlag, Berlin, 1994.
- [4] L. Breen, *Tannakian categories*. In Motives, Proc. Symp. Pure Math. **55**(1), (1994), 337-376.
- [5] R. Brown, *From groups to groupoids: A brief survey*. Bull. London Math. Soc. **19**(2), (1987), 113-134.
- [6] A. Bruguières, *Théorie Tannakienne non commutative*, Commun. in Algebra **22** (1994), 5817-5860.

<sup>(16)</sup> This is the coend object of the functor  $\omega(-)^* \otimes_k \omega(-)$  in  $R$ -bimodules, see [18].

<sup>(17)</sup> Here  $\mathcal{B} \otimes_k A$  is considered as a ring with enough orthogonal idempotents, namely, the set  $\{id_X \otimes_k 1_A\}_{X \in \mathcal{A}}$ , where  $\mathcal{B}$  is the Gabriel ring of  $\mathcal{A}$ .

- [7] R. Bos, *Continuous representations of groupoids*. Houston J. Math. **37**(3) (2011), 807-844.
- [8] P. Cartier, *Groupoïdes de Lie et leurs Algèbroïdes*. Séminaire Bourbaki 60<sup>e</sup> année, 2007-2008, n<sup>o</sup> 987, 165-196.
- [9] C. Chevalley, *Theory of Lie groups I*. Princeton University Press. 1946.
- [10] P. Deligne, *Catégories tannakiennes*. In *The Grothendieck Festschrift* (P. Cartier et al., eds), Progr. math., 87, vol. II, Birkhäuser, Boston, MA. 1990, pp. 111-195.
- [11] P. Deligne and J. S. Milne, *Tannakian Categories, in Hodge Cycles, Motives, and Shimura Varieties*. Lecture Notes in Mathematics **900**, 1982, pp. 101-228.
- [12] L. El Kaoutit and J. Gómez-Torrecillas, *Infinite comatrix corings*. Inter. Math. Res. Notices, **39** (2004), 2017-2037.
- [13] G. P. Hochschild and G. D. Mostow, *Representations and representative functions of Lie groups*. Ann. Math. **66** (3) (1957), 495-542.
- [14] G. P. Hochschild, *Structure of Lie groups*. Holden Day. 1965.
- [15] G. P. Hochschild, *Basic Theory of Algebraic Groups and Lie Algebras*. Graduate Texts in Mathematics, **75**. Springer-Verlag Berlin Heidelberg, 1981.
- [16] M. Hovey, *Morita theory for Hopf algebroids and presheaves of groupoids*. Amer. J. Math. **124**(6) (2002), 1289-1318.
- [17] M. Karoubi, *K-Theory, an introduction*. Grundlehren der mathematischen Wissenschaften, **226**, Springer-Verlag Berlin, 1978.
- [18] S. Mac Lane, *Categories for the working mathematician*. Graduate Texts in Mathematics, Vol. **5**. Springer-Verlag, New York-Berlin, 1971.
- [19] M. Masuda, T. Petrie, *Stably trivial equivariant algebraic vector bundles*. J. Amer. Math. Soc. **8** (3), (1995), 687-714.
- [20] D. C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*. Pure and Applied Mathematics Series, Academic Press, San Diego, 1986.
- [21] J. Renault, *A groupoid approach to  $C^*$ -algebras*. Lecture Notes in Mathematics **793**, Springer Verlag, 1980.
- [22] J. Renault, *Représentations des produits croisés d'algèbres de groupoïdes*. J. Operator theory, **18**, (1987), 67-97.
- [23] G. Segal, *The representation-ring of a compact Lie group*. Publ. Math. I.H.É.S. **34** (1968), 113-128.
- [24] G. Segal, *Equivariant K-theory*. Publ. Math. I.H.É.S. **34** (1968), 129-151.
- [25] J. P. Serre, *Modules projectifs et espace fibrés à fibre vectorielle*. Séminaire Dubreil-Pisot **11**(2), (1957-58), exposé n<sup>o</sup> 23, 1-18.
- [26] R. G. Swan, *Vector bundles and projective modules*. Trans. Amer. Math. Soc. **105**, (1962), 264-277.
- [27] A. Weinstein, *Groupoids: Unifying internal and external symmetry. A tour through some examples*. Notices of the AMS **43**(7), (1996), 744-752.
- [28] J. J. Westman, *Locally trivial  $C^r$ -groupoids and their representations*. Pacific J. Math. **20**, (1967), 339-349.

UNIVERSIDAD DE GRANADA, DEPARTAMENTO DE ÁLGEBRA. CAMPUS DE CEUTA. FACULTAD DE EDUCACIÓN Y HUMANIDADES. CORTADURA DEL VALLE S/N. E-51001 CEUTA, SPAIN

E-mail address: [kaoutit@ugr.es](mailto:kaoutit@ugr.es)

URL: <http://www.ugr.es/~kaoutit>