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# Noncommutative geometry of angular momentum space $U(\mathfrak{su}(2))$

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We study the standard angular momentum algebra $[x_i, x_j] = \imath\lambda\epsilon_{ijk}x_k$ as a noncommutative manifold $\mathbb{R}^3_\lambda$. We show that there is a natural 4D differential calculus and obtain its cohomology and Hodge $*$ operator. We solve the spin 0 wave equation and some aspects of the Maxwell or electromagnetic theory including solutions for a uniform electric current density, and we find a natural Dirac operator $\partial\!\!\!/$. We embed $\mathbb{R}^3_\lambda$ inside a 4D noncommutative space–time which is the limit $q \to 1$ of $q$-Minkowski space and show that $\mathbb{R}^3_\lambda$ has a natural quantum isometry group given by the quantum double $\mathbb{C}(SU(2)) \rtimes\!\!\!\lhd U(\mathfrak{su}(2))$ which is a singular limit of the $q$-Lorentz group. We view $\mathbb{R}^3_\lambda$ as a collection of all fuzzy spheres taken together. We also analyze the semiclassical limit via minimum uncertainty states $|j,\theta,\phi\rangle$ approximating classical positions in polar coordinates. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1517395]

## I. INTRODUCTION

There has been much interest in recent years in the possibility that classical space or space–time itself (not only phase space) is in fact noncommutative and not a classical manifold. One simple model where

$$[x_i, t] = \imath\lambda x_i \tag{1}$$

has already been shown[3] to have physically measurable effects even if $\lambda \sim 10^{-44}$ s (the Planck time). So such a conjecture is not out of reach of experiment *even if* the noncommutativity is due to quantum gravity effects. Such noncommutativity of space–time, if verified, would amount to a new physical effect which could be called ‘‘cogravity’’ because it corresponds under non Abelian Fourier transform to curvature in momentum space.[17] We are usually familiar with this correspondence the other way around, i.e., on a curved space such as a sphere the canonical momenta (angular momentum) form a noncommutative algebra

$$[J_a, J_b] = \imath\epsilon_{abc}J_c, \quad a,b,c = 1,2,3, \tag{2}$$

where $\epsilon_{abc}$ denotes the totally antisymmetric tensor; if one believes in Born reciprocity, then one should also allow the theoretical possibility of a sphere in momentum space, which would correspond to the algebra

$$[x_a, x_b] = \imath\lambda\epsilon_{abc}x_c. \tag{3}$$

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This is the algebra $\mathbb{R}^3_\lambda$ which we study in this article from the point of view of the $x_i$ as coordinates of a noncommutative position space. We insert here a parameter $\lambda$ of length dimension. The physical relevance of this algebra hardly needs to be justified, but we note some specific applications in string theory and quantum gravity in the discussion below. There are also possible other contexts where a noncommutative space–time might be a good effective model, not necessarily connected with gravity and indeed this is an entirely independent (dual) effect.

Also from a mathematical point of view, the algebra (3) is a standard example of a formal deformation quantization, namely of the Kirillov–Kostant Poisson bracket on $\mathfrak{su}_2^*$ in the coadjoint orbit method.[12] We identify $\mathfrak{su}(2)^*$ as the vector space $\mathbb{R}^3$ with basis $J_a^*$, say, dual to the $J_a$. Then among the algebra of suitable (polynomial) functions $\mathbb{C}(\mathbb{R}^3)$ on it we identify the $J_a$ themselves with the "coordinate functions" $J_a(v)=v_a$ for any $v\in\mathfrak{su}(2)^*$ with component $v_a$ in the $J_a^*$ direction. These generate the whole coordinate algebra and their Poisson bracket is defined by

$$\{J_a,J_b\}(v)=v([J_a,J_b]),\quad \forall v\in\mathfrak{su}(2)^*.$$

Hence when viewed as functions on $\mathbb{R}^3$, the Lie algebra generators have a Poisson bracket given by the Lie bracket. Their standard "quantization" is evidently provided by (3) with deformation parameter $\lambda$.

Our goal in the present work is to use modern quantum group methods to take this further by developing the noncommutative differential geometry of this quantum space at the level of scaler fields, forms, and spinors, i.e., classical field theory. We will solve wave equations, etc., and generally show that physics is fully possible on $\mathbb{R}^3_\lambda$. Note that the earlier example (1) above was also of "dual Lie" type but there the Lie algebra was solvable whereas the $\mathfrak{su}(2)$ case that we address here is at the other extreme and very much harder to work with. We expect our methods to extend also to $U(g)$ for other simple $g$.

The article begins in Sec. II with some mathematical preliminaries on quantum group methods and noncommutative geometry. As a quantum group, $\mathbb{R}^3_\lambda\cong U(\mathfrak{su}(2))$ (the enveloping Hopf algebra) which means that at the end of the day all computations can be reduced to the level of $\mathfrak{su}(2)$ and Pauli matrices. One of the first things implied by quantum group theory is that $\mathbb{R}^3_\lambda$ has an isometry quantum group given by the Drinfeld quantum double $D(U(\mathfrak{su}(2)))$ and we describe this first, in Sec. III. A suitable Casimir of this induces a scalar wave operator $\Box$ and we also describe spherical harmonics $Y_l^m$ dictated by action of rotations. This theory could be called the "level 0" noncommutative geometry where we think of the space through its symmetries rather than its differential structure.

In Sec. IV we start the noncommutative differential geometry, introducing a natural differential calculus on $\mathbb{R}^3_\lambda$. The cotangent directions or basic forms are given literally by Pauli matrices plus an additional generator $\theta$:

$$\mathrm{d}x_a=\tfrac{1}{2}\sigma_a,\quad \theta=\sigma_0, \tag{4}$$

where $\sigma_0=\mathrm{id}$ (the identity matrix). There are also noncommutation relations between functions and one-forms:

$$x_a\mathrm{d}x_b=(\mathrm{d}x_b)x_a+\frac{\imath\lambda}{2}\epsilon_{abc}\mathrm{d}x_c+\frac{\lambda}{4}\delta_{ab}\theta,\quad x_a\theta=\theta x_a+\lambda\,\mathrm{d}x_a. \tag{5}$$

Some other calculi are mentioned in the Appendix for comparison, but in fact this four-dimensional one appears to be the most reasonable one. The extra $\theta$ direction turns out to generate the cohomology, i.e., is not d of anything in $\mathbb{R}^3_\lambda$. We interpret it as a local time direction in the same spirit as in a different model.[9]

In Sec. V we introduce a Hodge $*$ operator and solve the resulting wave equations for spin 0 and spin 1 (the Maxwell equations). We also find a natural Dirac operator for spin $\frac{1}{2}$. Among the solutions of interest are plane waves obeying

$$\Box e^{\imath k\cdot x}=-\frac{1}{\lambda^2}\left\{4\sin^2\left(\frac{\lambda|k|}{2}\right)+\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)^2\right\}e^{\imath k\cdot x},$$

for momentum $k\in\mathbb{R}^3$. Among spin 1 solutions is a uniform electric current density in some direction and magnetic field increasing with normal distance. This is computationally the easiest case; we expect that the theory should similarly allow more conventional decaying solutions. In Sec. VI we briefly consider quantum spheres $S^2_\lambda$ inside $\mathbb{R}^3_\lambda$ by setting $\Sigma_i x_i^2=1$. These are then the usual quantization of coadjoint orbits in $\mathfrak{su}(2)^*$ [as opposed to all of $\mathfrak{su}(2)^*$ as described above] and we show that they inherit a three-dimensional differential geometry. This case could be viewed as a slightly different approach to fuzzy spheres[14,5,11,20,21] that is more adapted to their classical limit $\lambda\to 0$. Fuzzy spheres also arise as world volume algebras in string theory,[2] hence it would be interesting to develop this point of contact further. In our case we obtain a 3D differential calculus on $S^2_\lambda$.

In Sec. VII we explain the origin of the $\theta$ direction as the remnant of the time direction d$t$ of a standard four-dimensional noncommutative space–time $\mathbb{R}^{1,3}_q$ in a certain scaling limit as $q\to 1$. In the $q\neq 1$ setting the theory is much more nonsingular and there is a full $q$-Lorentz symmetry already covered in the $q$-deformation literature.[6,16,15] On the other hand, as $q\to 1$ we obtain either usual commutative Minkowski space or we can make a scaling limit and obtain the algebra

$$[x_a,x_b]=\imath ct\,\epsilon_{abc}x_c,\quad [x_a,t]=0, \tag{6}$$

where the parameter $c$ has dimensions of velocity. Mathematically this is homogenized $\widetilde{\mathrm{U}(\mathfrak{su}(2))}$ and we see that it projects onto our above algebra (3) by sending $ct\to\lambda$. This algebra (6) is not itself a good noncommutative Minkowski space because the $q$-Lorentz group action becomes singular as $q\to 1$ and degenerates into an action of the above quantum double isometry group. On the other hand, it is the boundary point $q=1$ of a good and well-studied noncommutative Minkowski space.

The article concludes in Sec. VIII with a proposal for the interpretation which is needed before the noncommutative geometry can be compared with experiment. In addition to a normal ordering postulate [i.e., noncommutative $f(x)$ are compared with classical ones only when normal ordered] along the lines of Ref. 3, we also propose a simple quantum mechanical point of view inspired by Penrose's spin network theorem.[19] In our case we construct minimum uncertainty states $|j,\theta,\phi\rangle$ for each spin $j$ in which expectations $\langle f(x)\rangle$ behave approximately like classical functions in polar coordinates $r,\theta,\phi$ with $r=\lambda j$. In effect we view $\mathbb{R}^3_\lambda$ as a collection of fuzzy spheres for all spins $j$ taken together. There are some similarities also with the star product and coherent states discussed recently in Ref. 11.

Finally, whereas the above includes electromagnetic theory on $\mathbb{R}^3_\lambda$, we explain now that exactly this noncommutative space is needed for a geometric picture underlying the approach to $2+1$ quantum gravity of Refs. 4 and 22. When a Euclidean signature and vanishing cosmological constant are assumed, the gauge group of the classical gravitational action (as a Chern–Simons field theory) is the group ISO(3).[23] Considering the three-dimensional space as the direct product $\Sigma\times\mathbb{R}$, where $\Sigma$ is Riemann surface of genus $g$, one can find the space of solutions of the gravitational field in terms of the topology of $\Sigma$.[1,8] The simplest case is to consider $\Sigma$ as a sphere with a puncture, which represents the topological theory of one particle coupled to gravity. It is known that the quantum states of this kind of theory correspond to irreducible representations of the quantum double $D(\mathrm{U}(\mathfrak{su}(2)))$.[4] A more detailed explanation, based on representation theory, of how the quantum double is a deformation "quantization" of the Euclidean group in three dimensions can be found in Ref. 22. However, the direct geometrical role of the quantum double has been missing except as an 'approximate' isometry of $\mathbb{R}^3$. Our present results therefore provide a new point of view, namely of the quantum double symmetry as an *exact* symmetry but of the noncommutative space $\mathbb{R}^3_\lambda$ on which we should build a noncommutative Chern–Simons action, etc. This fits with the discussion above that noncommutative space–time could be used as a better effective description of corrections to geometry coming out of quantum gravity. Details of the required noncommutative Chern–Simons theory as well as gravity in the frame bundle approach of Ref. 18 will be presented in a sequel.

## II. MATHEMATICAL PRELIMINARIES

Here we outline some notions from quantum group theory into which our example fits. For Hopf algebras (i.e., quantum groups) we use the conventions of Ref. 15. It means an algebra $H$ equipped with a coproduct $\Delta:H\to H\otimes H$, counit $\epsilon:H\to\mathbb{C}$ and antipode $S:H\to H$. We will sometimes use the formal sum notation $\Delta(h)=\Sigma h_{(1)}\otimes h_{(2)}$, for any $h\in H$. The usual universal enveloping algebra algebra $\mathrm{U}(\mathfrak{su}(2))$ has a structure of cocommutative Hopf algebra generated by 1 and $J_a$, $a=1,2,3$ with relations (2) and

$$\Delta(J_a)=J_a\otimes 1+1\otimes J_a,\quad \epsilon(J_a)=0,\quad S(J_a)=-J_a. \tag{7}$$

We also recall that as for Abelian groups, for each Hopf algebra there is a dual one where the product of one is adjoint to the coproduct of the other. $\mathrm{U}(\mathfrak{su}(2))$ is dually paired with the commutative Hopf algebra $\mathbb{C}(\mathrm{SU}(2))$ generated by coordinate functions $t^i{}_j$, for $i,j=1,2$ on SU(2) satisfying the determinant relation $t^1{}_1t^2{}_2-t^1{}_2t^2{}_1=1$ and with

$$\Delta(t^i{}_j)=\sum_{k=1}^{2} t^i{}_k\otimes t^k{}_j,\quad \epsilon(t^i{}_j)=\delta^i{}_j,\quad St^i{}_j=t^{-1i}{}_j, \tag{8}$$

where inversion is as an algebra-valued matrix. The pairing between the algebras $\mathrm{U}(\mathfrak{su}(2))$ and $\mathbb{C}(\mathrm{SU}(2))$ is defined by

$$\langle \xi,f\rangle=\frac{d}{dt}f(e^{t\xi})\big|_{t=0},$$

where $\xi\in\mathfrak{su}(2)$ and $f\in\mathbb{C}(\mathrm{SU}(2))$ which results in particular in

$$\langle J_a,t^i{}_j\rangle=\tfrac{1}{2}\sigma_a{}^i{}_j, \tag{9}$$

where $\sigma_a{}^i{}_j$ are the $i$, $j$ entries of the Pauli matrices for $a=1,...,3$. We omit here a discussion of unitarity, but this is implicit and achieved by making the above into Hopf $*$-algebras (see Ref. 15 for further details).

We also need standard notions of actions and coactions. A left coaction of a Hopf algebra $H$ on a space $V$ means a map $V\to H\otimes V$ obeying axioms like those of an action but reversing all maps. So a coaction of $\mathbb{C}(\mathrm{SU}(2))$ essentially corresponds to an action of $\mathrm{U}(\mathfrak{su}(2))$ via the pairing. Examples are

$$\mathrm{Ad}_L(h)(g)=h\triangleright g=\sum h_{(1)}gS(h_{(2)}), \tag{10}$$

the left adjoint action $\mathrm{Ad}_L:H\otimes H\to H$. Its arrow-reversal is the left adjoint coaction $\mathrm{Ad}^L:H\to H\otimes H$,

$$\mathrm{Ad}^L(h)=\sum h_{(1)}S(h_{(3)})\otimes h_{(2)}. \tag{11}$$

There are also the regular action (given by the product), regular coaction (given by $\Delta:H\to H\otimes H$), and coadjoint actions and coregular actions of the dual, given via the pairing from the adjoint and regular coactions, etc.[15] We will need the left coadjoint action of $H$ on a dual quantum group $A$:

$$\mathrm{Ad}_L^*(h)(\phi)=h\triangleright\phi=\sum\ \phi_{(2)}\langle(S\phi_{(1)})\phi_{(3)},h\rangle,\quad\forall h\in H,\quad\phi\in A,\tag{12}$$

and the right coregular action of $A$ on $H$ which we will view as a left action of the opposite algebra $A^{\mathrm{op}}$:

$$\phi\triangleright h=\sum\ \langle\phi,h_{(1)}\rangle h_{(2)},\quad\forall h\in H,\quad\phi\in A.\tag{13}$$

Given a quantum group $H$ dual to a quantum group $A$, there is a quantum double written loosely as $D(H)$ and containing $H$, $A$ as sub-Hopf algebras. More precisely it is a double cross product $A^{\mathrm{op}}\bowtie H$ where there are cross relations given by mutual coadjoint actions.[15] Also, $D(H)$ is formally quasitriangular in the sense of a formal "universal R matrix" $\mathcal{R}$ with terms in $D(H)\otimes D(H)$. The detailed structure of $D(\mathrm{U}(\mathfrak{su}(2)))$ is covered in Sec. III and in this case is more simply a semidirect product $\mathbb{C}(\mathrm{SU}(2))\rtimes\mathrm{U}(\mathfrak{su}(2))$ by the coadjoint action.

We will also need the quantum double $D(H)$ when $H$ is some other quasitriangular quantum group such as $\mathrm{U}_q(\mathfrak{su}(2))$. This is a standard deformation of (2) and the coproduct, etc., with a parameter $q$. In this case there is a second "braided" or covariantized version of $A=\mathbb{C}_q(\mathrm{SU}(2))$ which we denote by $\mathrm{BSU}_q(2)$. Then

$$D(\mathrm{U}_q(\mathfrak{su}(2)))\cong\mathrm{BSU}_q(2)\rtimes\mathrm{U}_q(\mathfrak{su}(2)),\tag{14}$$

where the product is a semidirect one by the adjoint action of $\mathrm{U}_q(\mathfrak{su}(2))$ and the coproduct is also a semidirect one. We will use this nonstandard "bosonization" version of $D(H)$ when $H$ is quasitriangular. Also when $H$ is quasitriangular with $\mathcal{R}_{21}\mathcal{R}$ nondegenerate, there is a third "twisting" version of the quantum double:

$$D(\mathrm{U}_q(\mathfrak{su}(2)))\cong\mathrm{U}_q(\mathfrak{su}(2))\blacktriangleright\!\!\blacktriangleleft_{\mathcal{R}}\mathrm{U}_q(\mathfrak{su}(2)),\tag{15}$$

where the algebra is a tensor product and the coproduct is

$$\Delta(h\otimes g)=\mathcal{R}_{23}^{-1}\Delta_{H\otimes H}(h\otimes g)\mathcal{R}_{23}.$$

We will use both versions in Sec. VII. Note that both isomorphisms are formal but the right hand sides are well defined and we take them as definitions. Especially, the isomorphism (15) is highly singular as $q\to1$. In that limit the twisted version tends to U(so(1,3)) while the bosonization version tends to U(iso(3)).

Finally, we will need the notion of differential calculus on an algebra $H$. This is common to several approaches to noncommutative geometry including that of Connes.[7] A first order calculus means to specify $(\Omega^1,\mathrm{d})$, where $\Omega^1$ is an $H-H$-bimodule, $\mathrm{d}:H\to\Omega^1$ obeys the Leibniz rule,

$$\mathrm{d}(hg)=(\mathrm{d}h)g+h(\mathrm{d}g),\tag{16}$$

and $\Omega^1$ is spanned by elements of the form $(\mathrm{d}h)g$. A bimodule just means that one can multiply "one-forms" in $\Omega^1$ by "functions" in $H$ from the left or the right without caring about brackets.

When we have a Hopf algebra $H$, a differential calculus can be asked to be "bicovariant,"[25] which means that there are left and right coactions of $H$ in $\Omega^1$ (a bicomodule) which are themselves bimodule homomorphisms, and d intertwines the coactions with the regular coactions of $H$ on itself. Given a bicovariant calculus one can find invariant forms

$$\omega(h)=\sum\ (\mathrm{d}h_{(1)})Sh_{(2)}\tag{17}$$

for any $h\in H$. The span of such invariant forms is a space $\Lambda^1$ and all of $\Omega^1$ can be reconstructed from them via

$$\mathrm{d}h=\sum \omega(h_{(1)})h_{(2)}. \tag{18}$$

As a result, the construction of a differential structure on a quantum group rests on that of $\Lambda^1$, with $\Omega^1=\Lambda^1.H$. They in turn can be constructed in the form

$$\Lambda^1=\ker\ \epsilon/\mathcal{I},$$

where $\mathcal{I}\subset\ker\ \epsilon$ is some left ideal in $H$ that is $\mathrm{Ad}^L$-stable.[25] We will use this method in Sec. IV to introduce a reasonable calculus on $\mathrm{U}(\mathfrak{su}(2))$. Some general remarks (but not our calculus, which seems to be new) appeared in Ref. 18.

Any bicovariant calculus has a "minimal" extension to an entire exterior algebra.[25] One uses the universal R-matrix of the quantum double to define a braiding operator on $\Lambda^1\otimes\Lambda^1$ and uses it to "antisymmetrize" the formal algebra generated by the invariant forms. These and elements of $H$ define $\Omega$ in each degree. In our case of $\mathrm{U}(\mathfrak{su}(2))$, because it is cocommutative, the braiding is the usual flip. Hence we have the usual anticommutation relations among invariant forms. We also extend $\mathrm{d}:\Omega^k\to\Omega^{k+1}$ as a (right-handed) super-derivation by

$$\mathrm{d}(\omega\wedge\eta)=\omega\wedge\mathrm{d}\eta+(-1)^{\deg\eta}\mathrm{d}\omega\wedge\eta.$$

A differential calculus is said to be inner if the exterior differentiation in $\Omega^1$ (and hence in all degrees) is given by the (graded) commutator with an invariant one-form $\theta\in\Lambda^1$, that is,

$$\mathrm{d}\omega=\omega\wedge\theta-(-1)^{\deg\omega}\theta\wedge\omega.$$

Almost all noncommutative geometries that one encounters are inner, which is the fundamental reason that they are in many ways better behaved than the classical case.

## III. THE QUANTUM DOUBLE AS EXACT ISOMETRIES OF $\mathbb{R}^3_\lambda$

In this section we first of all recall the structure of the quantum double $D(\mathrm{U}(\mathfrak{su}(2)))$ in the context of Hopf algebra theory. We will then explain its canonical action on a second copy $\mathbb{R}^3_\lambda\cong\mathrm{U}(\mathfrak{su}(2))$ arising from the general Hopf algebra theory, thereby presenting it explicitly as an exact quantum symmetry group of that. Here $x_a=\lambda J_a$ is the isomorphism valid for $\lambda\neq 0$. By an exact quantum symmetry we mean that the quantum group acts on $\mathbb{R}^3_\lambda$ with the product of $\mathbb{R}^3_\lambda$ an intertwiner (i.e. the algebra is covariant).

Because $\mathrm{U}(\mathfrak{su}(2))$ is cocommutative, its quantum double $D(\mathrm{U}(\mathfrak{su}(2)))$ is a usual crossed product[15]

$$D(\mathrm{U}(\mathfrak{su}(2)))=\mathbb{C}(\mathrm{SU}(2))_{\mathrm{Ad}^*_L}\rtimes\mathrm{U}(\mathfrak{su}(2)),$$

where the action is induced by the adjoint action [it is the coadjoint action on $\mathbb{C}(\mathrm{SU}(2))$]. This crossed product is isomorphic as a vector space with $\mathbb{C}(\mathrm{SU}(2))\otimes\mathrm{U}(\mathfrak{su}(2))$ but with algebra structure given by

$$(a\otimes h)(b\otimes g)=\sum a\mathrm{Ad}^*_{L\,h_{(1)}}(b)\otimes h_{(2)}g,$$

for $a,b\in\mathbb{C}(\mathrm{SU}(2))$ and $h,g\in\mathrm{U}(\mathfrak{su}(2))$. In terms of the generators, the left coadjoint action (12) takes the form

$$\mathrm{Ad}^*_{L\,J_a}(t^i{}_j)=\sum t^k{}_l\langle J_a,S(t^i{}_k)t^l{}_j\rangle=\frac{1}{2}(t^i{}_k\sigma_a{}^k{}_j-\sigma_a{}^j{}_k t^k{}_j). \tag{19}$$

As a result we find that $D(\mathrm{U}(\mathfrak{su}(2)))$ is generated by $\mathrm{U}(\mathfrak{su}(2))$ and $\mathbb{C}(\mathrm{SU}(2))$ with cross relations

$$[J_a, t^i{}_j] = \tfrac{1}{2}(t^i{}_k \sigma_a{}^k{}_j - \sigma_a{}^j{}_k t^k{}_j). \tag{20}$$

Meanwhile the coproducts are the same as those of $U(\mathfrak{su}(2))$ and $\mathbb{C}(SU(2))$.

Next, a general feature of any quantum double is a canonical or "Schrödinger" representation, where $U(\mathfrak{su}(2)) \subset D(U(\mathfrak{su}(2)))$ acts on $U(\mathfrak{su}(2))$ by the left adjoint action (10) and $\mathbb{C}(SU(2)) \subset D(U(\mathfrak{su}(2)))$ acts by the coregular one (13), see Ref. 15. We denote the acted-upon copy by $\mathbb{R}^3_\lambda$. Then $J_a$ simply act by

$$J_a \triangleright f(x) = \lambda^{-1} \sum x_{a(1)} f(x) S(x_{a(2)}) = \lambda^{-1}[x_a, f(x)], \quad \forall f(x) \in \mathbb{R}^3_\lambda, \tag{21}$$

e.g.,

$$J_a \triangleright x_b = \imath \epsilon_{abc} x_c,$$

while the co-regular action reads

$$t^i{}_j \triangleright f(x) = \langle t^i{}_j, f(x)_{(1)} \rangle f(x)_{(2)}, \quad \text{e.g.,} \quad t^i{}_j \triangleright x_a = \frac{\lambda}{2} \sigma_a{}^i{}_j 1 + \delta^i{}_j x_a.$$

The general expression is given by a shuffle product (see Sec. IV). *With this action*, $\mathbb{R}^3_\lambda$ *turns into a left* $D(U(\mathfrak{su}(2)))$*-covariant algebra*.

In order to analyze the classical limit of this action, let us consider the role of the numerical parameter $\lambda$ used to define the algebra $\mathbb{R}^3_\lambda$. Considering the relations (3), we have already explained that $\mathbb{R}^3_\lambda$ becomes the usual algebra of functions on $\mathbb{R}^3$ as $\lambda \to 0$. The same parameter $\lambda$ can be introduced into the quantum double by means of a redefinition of the generators of $\mathbb{C}(SU(2))$ to

$$M^i{}_j = \frac{1}{\lambda}(t^i{}_j - \delta^i{}_j), \tag{22}$$

so that $t^i{}_j = \delta^i{}_j + \lambda M^i{}_j$. We stress that we are dealing with the same Hopf Algebra $D(U(\mathfrak{su}(2)))$, but written in terms of new generators, it is only a change of variables. The homomorphism property of $\Delta$ gives

$$\Delta M^i{}_j = \sum_{k=1}^{2} (\delta^i{}_k \otimes M^k{}_j + M^i{}_k \otimes \delta^k{}_j + \lambda M^i{}_k \otimes M^k{}_j),$$

while the condition on the determinant, $t^1{}_1 t^2{}_2 - t^1{}_2 t^2{}_1 = 1$, implies that

$$\mathrm{Tr}(M) = M^1{}_1 + M^2{}_2 = -\lambda \det(M).$$

This means that in the limit $\lambda \to 0$, the elements $M^i{}_j$ have to obey $M^1{}_1 = -M^2{}_2$ and $\mathbb{C}(SU(2))$ becomes the commutative Hopf algebra $U(\mathbb{R}^3)$. To make this explicit, we can define the momentum generators

$$P_1 = -\imath(M^1{}_2 + M^2{}_1), \quad P_2 = M^1{}_2 - M^2{}_1, \quad P_3 = -\imath(M^1{}_1 - M^2{}_2), \tag{23}$$

or

$$P_a = -\imath \sigma_a{}^i{}_j M^j{}_i, \quad a = 1,2,3 \tag{24}$$

(sum over $i,j$). The inverse of this relationship is

$$M^i{}_j = \frac{\imath}{2} \sigma_a{}^i{}_j P_a + \delta^i{}_j P_0, \quad P_0 = \frac{1}{2}\mathrm{Tr}(M) = -\frac{1}{\lambda}\left(1 - \sqrt{1 - \frac{\lambda^2}{4} P^2}\right). \tag{25}$$

The other square root is also allowed, but then $P_0$ is not $\mathcal{O}(\lambda)$, i.e., this is not the "patch" of $\mathbb{C}(\mathrm{SU}(2))$ that concerns us. Note also that there are unitarity conditions that we do not explicitly discuss (if we put them in then the $P_a$ are Hermitian). In these terms we have

$$\Delta P_a = P_a \otimes 1 + 1 \otimes P_a + \mathcal{O}(\lambda),$$

so that we have the usual additive coproduct in the $\lambda \to 0$ limit. Meanwhile, the left coadjoint action (19) and the resulting cross relations in the double become

$$\mathrm{Ad}^*_{L\,J_a}(P_b) = \imath \epsilon_{abc} P_c, \quad [J_a, P_b] = \imath \epsilon_{abc} P_c,$$

i.e., $D(\mathrm{U}(\mathfrak{su}(2)))$ in the limit $\lambda \to 0$ with these generators becomes the usual U(iso(3)). This part is essentially known.[4,22]

Moreover, our action of these scaled generators on $\mathbb{R}^3_\lambda$ is

$$M^i{}_j \triangleright f(x) = \partial^i{}_j(f(x)), \quad \text{e.g.,} \quad M^i{}_j \triangleright x_a = \langle J_a, t^i{}_j \rangle 1 = \tfrac{1}{2} \sigma_a{}^i{}_j 1, \tag{26}$$

where the operators $\partial^i{}_j$ are the same as those in the next section. We can also write the action of $P_a$ as partial derivatives defined there [in (35)] by

$$P_a \triangleright f(x) = -\imath \partial^a f(x), \quad P_0 \triangleright f(x) = \frac{1}{c} \partial^0 f(x),$$

where the constant $c$ is put in order to make the equations have the same form as the classical ones, interpreting roughly the zero-direction as a "time" direction. This relation will become clearer in Sec. VII.

In the limit $\lambda \to 0$, the action of $J_a$ becomes usual rotations in three-dimensional Euclidean space while the action of $P_a$ becomes the action of translation operators of the algebra $\mathrm{U}(\mathbb{R}^3)$, so we indeed recover the classical action of U(iso(3)) on $\mathbb{R}^3$. In three-dimensional gravity, considering the dimension of the gravitational constant $G_3$ and the speed of light to be equal to 1, we have that $\lambda$ must be proportional to the Planck constant.[22]

Next, there are several applications of the action of the double based on the above point of view. First and foremost, we could look for a wave operator from a Fourier transform point of view as in Ref. 3 (we give a different point of view later). Namely we look for a Casimir of $D(\mathrm{U}(\mathfrak{su}(2)))$ lying in momentum space $\mathbb{C}(\mathrm{SU}(2))$, and define the wave operator as its action. The possible such Casimirs are the $\mathrm{U}(\mathfrak{su}(2))$-invariant functions, which means basically the class functions on SU(2). In our case this just means any function of the trace function $\tau = t^1{}_1 + t^2{}_2$. The one suggested by the noncommutative geometry in the next sections is

$$\mathcal{E} \equiv -P^2 - \frac{4}{\lambda^2}\left(1 - \sqrt{1 - \frac{\lambda^2}{4} P^2}\right)^2 = \frac{4}{\lambda^2}(\tau - 2) \tag{27}$$

and its action on $\mathbb{R}^3_\lambda$ is then the wave operator $\square$ on degree zero in Sec. V, but with metric $-4$ in the time direction. Note that $S\tau = \tau$ for $\mathbb{C}(\mathrm{SU}(2))$ so any such wave operator is invariant under group inversion, which appears as the antipode $SP_a = -P_a$.

A different question we can also ask is about the noncommutative analogs of spherical harmonics as functions in $\mathbb{R}^3_\lambda$ in the sense of irreducible representations $Y_l{}^m$ under the above action (21) of the rotation group. We find the (unnormalized) lowest ones for $l \in \mathbb{Z}_+$ and $m = -l, -l + 1, \ldots, l$ as

$$Y_0{}^0 = 1,$$

$$Y_1{}^{\pm 1} = \mp \frac{1}{\sqrt{2}}(x_1 \pm \imath x_2), \quad Y_1{}^0 = x_3,$$

$$Y_2{}^{\pm 2}=(x_1\pm \imath x_2)^2,\quad Y_2{}^{\pm 1}=\mp((x_1\pm \imath x_2)x_3+x_3(x_1\pm \imath x_2)),$$

$$Y_2{}^{0}=\frac{1}{\sqrt{6}}(4x_3^2-(x_1+\imath x_2)(x_1-\imath x_2)-(x_1-\imath x_2)(x_1+\imath x_2)).$$

Let us note that such spherical harmonics can have many applications beyond their usual role in physics. For example, they classify the possible noncommutative differential calculi on the classical coordinate algebra $\mathbb{C}(\mathrm{SU}(2))$ which is dual to the space we study here.

## IV. THE FOUR-DIMENSIONAL CALCULUS ON $\mathbb{R}^3_\lambda$

The purpose of this section is to construct a bicovariant calculus on the algebra $\mathbb{R}^3_\lambda$ following the steps outlined in Sec. II, the calculus we obtain being that on the algebra $U(\mathfrak{su}(2))$ on setting $\lambda=1$. We write $\mathbb{R}^3_\lambda$ as generated by $x_+$, $x_-$ and $h$, say, and with the Hopf algebra structure given explicitly in terms of the generators as

$$[h,x_\pm]=\pm 2\lambda x_\pm;\quad [x_+,x_-]=\lambda h, \tag{28}$$

and the additive coproduct as before. The particular form of the coproduct, the relations and (17) show that $\mathrm{d}\xi=\omega(\xi)$ for all $\xi\in\mathfrak{su}(2)$. Because of the cocommutativity, all ideals in $\mathbb{R}^3_\lambda$ are invariant under adjoint coactions (11) so that first order differential calculi $\Omega^1$ on $\mathbb{R}^3_\lambda$ are classified simply by the ideals $\mathcal{I}\subset\ker\epsilon$. In order to construct an ideal of $\ker\epsilon$, consider a two-dimensional representation $\rho:\mathbb{R}^3_\lambda\to\mathrm{End}\,\mathbb{C}^2$, which in the basis $\{e_1,e_2\}$ of $\mathbb{C}^2$ is given by

$$\rho(x_+)e_1=0,\quad \rho(x_+)e_2=\lambda e_1,$$

$$\rho(x_-)e_1=\lambda e_2,\quad \rho(x_-)e_2=0,$$

$$\rho(h)e_1=\lambda e_1,\quad \rho(h)e_2=-\lambda e_2.$$

The representation $\rho$ is a surjective map onto $M_2(\mathbb{C})$, even when restricted to $\ker\epsilon$. The kernel of $\rho|_{\ker\epsilon}$ is a two-sided ideal in $\ker\epsilon$. Then we have

$$M_2(\mathbb{C})\equiv\ker\ \epsilon/\ker\ \rho. \tag{29}$$

This isomorphism allows us to identify the basic one-forms with $2\times 2$ matrices, $\{e_{ij}\}$, for $i,j=1,2$, where $e_{ij}$ is the matrix with 1 in the $(i,j)$ entry and 0 otherwise. Then the first order differential calculus is

$$\Omega^1(\mathbb{R}^3_\lambda)=M_2(\mathbb{C})\otimes\mathbb{R}^3_\lambda.$$

The exterior derivative operator is

$$\mathrm{d}f(x)=\lambda^{-1}\sum\ \rho(f(x)_{(1)}-\epsilon(f(x)_{(1)})1)f(x)_{(2)}=e_{ij}\partial^i{}_j(f),$$

where the last equality is a definition of the partial derivatives $\partial^i{}_j:\mathbb{R}^3_\lambda\to\mathbb{R}^3_\lambda$. In particular, we have

$$\mathrm{d}\xi=\lambda^{-1}\rho(\xi),\quad \forall\xi\in\mathfrak{su}(2),$$

which, along with id, span the whole space $\mathrm{M}_2(\mathbb{C})$ of invariant one-forms. For a general monomial $\xi_1\cdots\xi_n$, the expression of the derivative is

$$\mathrm{d}(\xi_1\cdots\xi_n)=\lambda^{-1}\sum_{k=1}^{n}\sum_{\sigma\in S_{(n,k)}}\rho(\xi_{\sigma(1)}\cdots\xi_{\sigma(k)})\xi_{\sigma(k+1)}\cdots\xi_{\sigma(n)},$$

where $\sigma$ is a permutation of 1,...,$n$, such that $\sigma(1)<\cdots<\sigma(k)$ and $\sigma(k+1)<\cdots<\sigma(n)$. This kind of permutation is called a $(n,k)$-shuffle. And finally, for a (formal power series) grouplike element $g$ (where $\Delta g=g\otimes g$), the derivative is

$$\mathrm{d}g=\lambda^{-1}(\rho(g)-\theta)g.$$

On our basis we have

$$\mathrm{d}x_+=e_{12},\quad \mathrm{d}x_-=e_{21},\quad \mathrm{d}h=e_{11}-e_{22},\quad \theta=e_{11}+e_{22}.$$

The compatibility conditions of this definition of the derivative with the Leibniz rule is due to the following commutation relations between the generators of the algebra and the basic one-forms:

$$\begin{aligned}
x_\pm \mathrm{d}x_\pm &= (\mathrm{d}x_\pm)x_\pm,\\
x_\pm \mathrm{d}x_\mp &= (\mathrm{d}x_\mp)x_\pm+\frac{\lambda}{2}(\theta\pm \mathrm{d}h),\\
x_\pm \mathrm{d}h &= (\mathrm{d}h)x_\pm \mp\lambda\, \mathrm{d}x_\pm,\\
h\mathrm{d}x_\pm &= (\mathrm{d}x_\pm)h\pm\lambda\, \mathrm{d}x_\pm,\\
h\mathrm{d}h &= (\mathrm{d}h)h+\lambda\theta,\\
x_\pm\theta &= \theta x_\pm+\lambda\, \mathrm{d}x_\pm,\\
h\theta &= \theta h+\lambda\, \mathrm{d}h.
\end{aligned}\tag{30}$$

From these commutation relations, we can see that this calculus is inner, that is, the derivatives of any element of the algebra can be basically obtained by the commutator with the one-form $\theta$. In the classical limit, this calculus turns out to be the commutative calculus on usual three-dimensional Euclidean space. The explicit expression for the derivative of a general monomial $x_-^a h^b x_+^c$ is given by

$$\begin{aligned}
\mathrm{d}(x_-^a h^b x_+^c)=&\ \mathrm{d}h\left(\sum_{i=0}^{[(b-1)/2]}\binom{b}{2i+1}\lambda^{2i}x_-^a h^{b-2i-1}x_+^c\right)+\theta\left(\sum_{i=1}^{[b/2]}\binom{b}{2i}\lambda^{2i-1}x_-^a h^{b-2i}x_+^c\right)\\
&+\mathrm{d}x_+\left(\sum_{i=0}^{b}\binom{b}{i}\lambda^{i}c x_-^a h^{b-i}x_+^{c-1}\right)+\mathrm{d}x_-\left(\sum_{i=0}^{b}\binom{b}{i}\lambda^{i}a x_-^{a-1} h^{b-i}x_+^{c}\right)+\frac{1}{2}(\theta-\mathrm{d}h)\\
&\times\left(\sum_{i=0}^{b}\binom{b}{i}\lambda^{i+1}ac x_-^{a-1} h^{b-i}x_+^{c-1}\right),
\end{aligned}\tag{31}$$

where the symbol $[z]$ denotes the greatest integer less than $z$ and only terms with $\geqslant 0$ powers of the generators are included. Note that this expression becomes in the limit $\lambda\to 0$ the usual expression for the derivative of a monomial in three commuting coordinates.

In terms of the generators $x_a$, $a=1,2,3$, which are related to the previous generators by

$$x_1=\frac{1}{2}(x_++x_-),\quad x_2=\frac{\imath}{2}(x_--x_+),\quad x_3=\frac{1}{2}h,$$

we have

$$\mathrm{d}x_a=\tfrac{1}{2}\sigma_a,\quad \theta=\sigma_0,\tag{32}$$

i.e., the Pauli matrices are nothing other than three of our basic one-forms, and together with $\sigma_0 = \mathrm{id}$ form a basis of the invariant one-forms. The commutation relations (30) have a simple expression:

$$x_a \mathrm{d}x_b = (\mathrm{d}x_b) x_a + \frac{\imath\lambda}{2} \epsilon_{abc} \mathrm{d}x_c + \frac{\lambda}{4} \delta_{ab} \theta,$$
$$x_a \theta = \theta x_a + \lambda \mathrm{d}x_a . \tag{33}$$

In this basis the partial derivatives defined by

$$\mathrm{d}f(x) = (\mathrm{d}x_a) \partial^a f(x) + \theta \frac{1}{c} \partial^0 f(x) \tag{34}$$

are related to the previous ones by

$$\partial^i{}_j = \frac{1}{2} \sigma_a{}^i{}_j \partial^a + \frac{1}{c} \sigma_0{}^i{}_j \partial^0 \tag{35}$$

as in (25). The exterior derivative of a general monomial $x_1^a x_2^b x_3^c$ is quite complicated to write down explicitly, but we find it as

$$\begin{aligned}
\mathrm{d}(x_1^a x_2^b x_3^c) = {} & \sum_{i=0}^{[a/2]} \sum_{j=0}^{[b/2]} \sum_{k=0}^{[c/2]} \theta \frac{\lambda^{2(i+j+k)-1}}{2^{2(i+j+k)}} \binom{a}{2i} \binom{b}{2j} \binom{c}{2k} x_1^{a-2i} x_2^{b-2j} x_3^{c-2k} \\
& + \sum_{i=0}^{[a/2]} \sum_{j=0}^{[b/2]} \sum_{k=0}^{[(c-1)/2]} \mathrm{d}x_3 \frac{\lambda^{2(i+j+k)}}{2^{2(i+j+k)}} \binom{a}{2i} \binom{b}{2j} \binom{c}{2k+1} x_1^{a-2i} x_2^{b-2j} x_3^{c-2k-1} \\
& + \sum_{i=0}^{[a/2]} \sum_{j=0}^{[(b-1)/2]} \sum_{k=0}^{[c/2]} \mathrm{d}x_2 \frac{\lambda^{2(i+j+k)}}{2^{2(i+j+k)}} \binom{a}{2i} \binom{b}{2j+1} \binom{c}{2k} x_1^{a-2i} x_2^{b-2j-1} x_3^{c-2k} \\
& + \sum_{i=0}^{[a/2]} \sum_{j=0}^{[(b-1)/2]} \sum_{k=0}^{[(c-1)/2]} \imath \mathrm{d}x_1 \frac{\lambda^{2(i+j+k)+1}}{2^{2(i+j+k)+1}} \binom{a}{2i} \binom{b}{2j+1} \\
& \times \binom{c}{2k+1} x_1^{a-2i} x_2^{b-2j-1} x_3^{c-2k-1} + \sum_{i=0}^{[(a-1)/2]} \sum_{j=0}^{[b/2]} \sum_{k=0}^{[c/2]} \mathrm{d}x_1 \frac{\lambda^{2(i+j+k)}}{2^{2(i+j+k)}} \binom{a}{2i+1} \binom{b}{2j} \\
& \times \binom{c}{2k} x_1^{a-2i-1} x_2^{b-2j} x_3^{c-2k} - \sum_{i=0}^{[(a-1)/2]} \sum_{j=0}^{[b/2]} \sum_{k=0}^{[(c-1)/2]} \imath \mathrm{d}x_2 \frac{\lambda^{2(i+j+k)+1}}{2^{2(i+j+k)+1}} \binom{a}{2i+1} \binom{b}{2j} \\
& \times \binom{c}{2k+1} x_1^{a-2i-1} x_2^{b-2j} x_3^{c-2k-1} + \sum_{i=0}^{[(a-1)/2]} \sum_{j=0}^{[(b-1)/2]} \sum_{k=0}^{[c/2]} \imath \mathrm{d}x_3 \frac{\lambda^{2(i+j+k)+1}}{2^{2(i+j+k)+1}} \binom{a}{2i+1} \\
& \times \binom{b}{2j+1} \binom{c}{2k} x_1^{a-2i-1} x_2^{b-2j-1} x_3^{c-2k} \\
& + \sum_{i=0}^{[(a-1)/2]} \sum_{j=0}^{[(b-1)/2]} \sum_{k=0}^{[(c-1)/2]} \theta \frac{\lambda^{2(i+j+k)+2}}{2^{2(i+j+k)+3}} \binom{a}{2i+1} \binom{b}{2j+1} \\
& \times \binom{c}{2k+1} x_1^{a-2i-1} x_2^{b-2j-1} x_3^{c-2k-1} - \frac{\theta}{\lambda} x_1^a x_2^b x_3^c .
\end{aligned} \tag{36}$$

In both cases the expression for the derivatives of plane waves is very simple. In terms of generators $x_a$, the derivative of the plane wave $e^{\imath \Sigma_a k^a x_a} = e^{ik\cdot x}$ is given by

$$\mathrm{d}e^{\imath k\cdot x} = \left\{ \frac{\theta}{\lambda}\left( \cos\left(\frac{\lambda|k|}{2}\right) - 1 \right) + \frac{2\imath \sin(\lambda|k|/2)}{\lambda|k|} k\cdot \mathrm{d}x \right\} e^{\imath k\cdot x}. \tag{37}$$

One can see that the limit $\lambda \to 0$ gives the correct formula for the derivative of plane waves, that is,

$$\lim_{\lambda\to 0} \mathrm{d}e^{\imath k\cdot x} = \left( \sum_{a=1}^{3} \imath k_a \mathrm{d}\overline{x_a} \right) e^{\imath k\cdot \overline{x}} = \imath k\cdot(\mathrm{d}\overline{x}) e^{\imath k\cdot \overline{x}},$$

where at $\lambda = 0$ on the right hand side we have the classical coordinates and the classical one-forms in usual three-dimensional commutative calculus. In terms of the generators $x_\pm$, $h$, the plane wave $e^{\imath(k_+x_+ + k_-x_- + k_0 h)} = e^{\imath k\cdot x}$ is given by

$$\mathrm{d}e^{\imath k\cdot x} = \left\{ \frac{\theta}{\lambda}(\cos(\lambda\sqrt{k_0^2 + k_+k_-}) - 1) + \frac{\imath(k_+\mathrm{d}x_+ + k_-\mathrm{d}x_- + k_0\mathrm{d}h)}{\lambda\sqrt{k_0^2 + k_+k_-}} \sin(\lambda\sqrt{k_0^2 + k_+k_-}) \right\} e^{\imath k\cdot x}. \tag{38}$$

This calculus is four-dimensional, in the sense that one has four basic one-forms, but these dimensions are entangled in a nontrivial way. For example, note that they satisfy the relation

$$\epsilon_{abc} x_a (\mathrm{d}x_b) x_c = 0.$$

We can see that in the classical limit $\lambda \to 0$, the calculus turns out to be commutative and the extra dimension, namely the one-dimensional subspace generated by the one-form $\theta$, decouples totally from the calculus generated by the other three one-forms. The relation between this extra dimension and quantization can also be perceived by considering the derivative of the Casimir operator

$$C = \sum_{a=1}^{3} (x_a)^2,$$

which implies

$$\mathrm{d}C = 2\sum_{a=1}^{3} (\mathrm{d}x_a)x_a + \frac{3\lambda}{4}\theta.$$

The coefficient of the term in $\theta$ is exactly the eigenvalue of the Casimir in the spin $\frac{1}{2}$ representation, the same used to construct the differential calculus, and also vanishes when $\lambda \to 0$. We shall see later that this extra dimension can also be seen as a remnant of the time coordinate in the $q$-Minkowski space $\mathbb{R}_q^{1,3}$ when the limit $q \to 1$ is taken. A semi-classical analysis on this calculus can also be made in order to recover an interpretation of time in the three-dimensional noncommutative space.

We can also construct the full exterior algebra $\Omega^{\cdot}(\mathbb{R}_\lambda^3) = \oplus_{n=0}^{\infty} \Omega^n(\mathbb{R}_\lambda^3)$. In our case the general braiding[25] becomes the trivial flip homomorphism because the right invariant basic one-forms are also left invariant. Hence our basic one-forms in $M_2(\mathbb{C})$ are totally anticommutative and their usual antisymmetric wedge product generates the usual exterior algebra on the vector space $M_2(\mathbb{C})$. The full $\Omega^{\cdot}(\mathbb{R}_\lambda^3)$ is generated by these and elements of $\mathbb{R}_\lambda^3$ with the relations (30). The exterior differentiation in $\Omega^{\cdot}(\mathbb{R}_\lambda^3)$ is given by the graded commutator with the basic one-form $\theta$, that is,

$$\mathrm{d}\omega = \omega\wedge\theta - (-1)^{\deg\omega}\theta\wedge\omega.$$

In particular, the basic one-forms $M_2(\mathbb{C})$ are all closed, among which $\theta$ is not exact. The cohomologies of this calculus were also calculated giving the following results:

**Theorem 4.1:** *The noncommutative de Rham cohomology of* $\mathbb{R}^3_\lambda$ *is*

$$H^0=\mathbb{C}.1, \quad H^1=\mathbb{C}.\theta, \quad H^2=H^3=H^4=\{0\}.$$

*Proof:* This is by direct (and rather long) computation of the closed forms and the exact ones in each degree using the explicit formula (31) on general monomials. To give an example of the procedure, we will do it in some detail for the case of one-forms of the particular type

$$\omega=\alpha(\mathrm{d}x_+)x_-^a h^b x_+^c+\beta(\mathrm{d}x_-)x_-^m h^n x_+^p+\gamma(\mathrm{d}h)x_-^r h^s x_+^t+\delta\theta x_-^u h^v x_+^w,$$

and impose $\mathrm{d}\omega=0$. We start analyzing the simplest cases, and then going to more complex ones.

Taking $\beta=\gamma=\delta=0$, then

$$\omega=\alpha(\mathrm{d}x_+)x_-^a h^b x_+^c.$$

The term in $dx_-\wedge dx_+$ leads to the conclusion that $c=0$. Similarly, the term in $\mathrm{d}h\wedge\mathrm{d}x_+$ leads to $b=0$ so that

$$\omega=\alpha(\mathrm{d}x_+)x_+^c=\mathrm{d}\left(\frac{1}{c+1}x_+^{c+1}\right),$$

which is an exact form, hence belonging to the null cohomology class. The cases $\alpha=\gamma=\delta=0$ and $\alpha=\beta=\delta=0$ also lead to exact forms. The case $\alpha=\beta=\gamma=0$ leads to the one-form

$$\omega=\delta\theta x_-^u h^v x_+^w.$$

The vanishing of the term in $\mathrm{d}x_+\wedge\theta$ implies that $w=0$, the term in $\mathrm{d}x_-\wedge\theta$ vanishes if and only if $u=0$ and the term in $\mathrm{d}h\wedge\theta$ has its vanishing subject to the condition $v=0$. Hence we have only the closed, nonexact form $\theta$ from this case.

Let us now analyze the case with two nonzero terms:

$$\omega=\alpha(\mathrm{d}x_+)x_-^a h^b x_+^c+\beta(\mathrm{d}x_-)x_-^m h^n x_+^p.$$

The vanishing condition in the term on $\mathrm{d}x_-\wedge\mathrm{d}x_+$ reads

$$\alpha\sum_{i=0}^{b}\binom{b}{i}\lambda^i a x_-^{a-1}h^{b-i}x_+^c=\beta\sum_{i=0}^{n}\binom{n}{i}\lambda^i p x_-^m h^{n-i}x_+^{p-1}.$$

Then we conclude that $b=n$, $a-1=m$, $c=p-1$ and $\alpha a=\beta(c+1)$. The vanishing of the term in $\mathrm{d}h\wedge\mathrm{d}x_+$ reads

$$\sum_{i=0}^{[(b-1)/2]}\binom{b}{2i+1}\lambda^{2i}x_-^a h^{b-2i-1}x_+^c=\frac{1}{2}\sum_{i=0}^{b}\binom{b}{i}\lambda^{i+1}acx_-^{a-1}h^{b-i}x_+^{c-1}.$$

The terms in odd powers of $\lambda$ vanish if and only if $ac=0$. Then the left hand side vanishes if and only if $b=0$. The case $a=0$ implies that $\beta=0$, which reduces to the previous case already mentioned. For the case $c=0$ we have $\beta=\alpha a$ so that

$$\omega=\alpha((\mathrm{d}x_+)x_-^a+a(\mathrm{d}x_-)x_-^{a-1}x_+).$$

It is easy to see that $\omega$ is closed if and only if $a=1$. But

$$(\mathrm{d}x_+)x_- + (\mathrm{d}x_-)x_+ = \mathrm{d}\left(x_-x_+ + \frac{\lambda}{2}h\right) - \frac{\lambda}{2}\theta,$$

which is a form homologous to $\theta$. It is a long, but straightforward, calculation to prove that all the other cases of closed one-forms of the type above rely on these cases. The general case is still more complicated.

The proof that all higher cohomologies are trivial is also an exhaustive analysis of all the possible cases and inductions on powers of $h$, as exemplified here for the four-forms: It is clear that all four-forms

$$\omega = \mathrm{d}x_- \wedge \mathrm{d}h \wedge \mathrm{d}x_+ \wedge \theta x_-^m h^n x_+^p$$

are closed. We use induction on $n$ to prove that there exists a three-form $\eta$ such that $\omega = d\eta$. For $n=0$, we have

$$\mathrm{d}x_- \wedge \mathrm{d}h \wedge \mathrm{d}x_+ \wedge \theta x_-^m x_+^p = \mathrm{d}\left(-\frac{1}{m+1}\mathrm{d}h \wedge \mathrm{d}x_+ \wedge \theta x_-^{m+1} x_+^p\right).$$

Suppose that there exist three-forms $\eta_k$, for $0 \leqslant k < n$, such that

$$\mathrm{d}x_- \wedge \mathrm{d}h \wedge \mathrm{d}x_+ \wedge \theta x_-^m h^k x_+^p = \mathrm{d}\eta_k .$$

Then

$$\begin{aligned}\mathrm{d}x_- \wedge \mathrm{d}h \wedge \mathrm{d}x_+ \wedge \theta x_-^m h^n x_+^p &= \mathrm{d}\left(-\frac{1}{m+1}\mathrm{d}h \wedge \mathrm{d}x_+ \wedge \theta x_-^{m+1} h^n x_+^p\right) \\ &\quad - \mathrm{d}x_- \wedge \mathrm{d}h \wedge \mathrm{d}x_+ \wedge \theta \sum_{i=1}^{n} \binom{n}{i} \lambda^i x_-^m h^{n-i} x_+^p \\ &= \mathrm{d}\left(-\frac{1}{m+1}\mathrm{d}h \wedge \mathrm{d}x_+ \wedge \theta x_-^{m+1} h^n x_+^p - \sum_{i=1}^{n} \binom{n}{i} \lambda^i \eta_{n-i}\right).\end{aligned}$$

Hence all four-forms are exact. The same procedure is used to show the triviality of the other cohomologies. ◇

For $\mathbb{R}^3_\lambda$ we should expect the cohomology to be trivial, since this corresponds to Stokes theorem and many other aspects taken for granted in physics. We find almost this except for the generator $\theta$ which generates the calculus and which has no three-dimensional classical analogs. We will see in Sec. VII that $\theta$ is a remnant of a time direction even though from the point of view of $\mathbb{R}^3_\lambda$ there is no time coordinate. The cohomology result says exactly that $\theta$ is an allowed direction but not d of anything.

## V. HODGE $*$-OPERATOR AND ELECTROMAGNETIC THEORY

The above geometry also admits a metric structure. It is known that any nondegenerate bilinear form $\eta \in \Lambda^1 \otimes \Lambda^1$ defines an invariant metric on the Hopf algebra $H$.[18] For the case of $\mathbb{R}^3_\lambda$ we can define the metric

$$\eta = \mathrm{d}x_1 \otimes \mathrm{d}x_1 + \mathrm{d}x_2 \otimes \mathrm{d}x_2 + \mathrm{d}x_3 \otimes \mathrm{d}x_3 + \mu\,\theta \otimes \theta \tag{39}$$

for a parameter $\mu$. This bilinear form is nondegenerate, invariant by left and right coactions and symmetric in the sense that $\wedge(\eta) = 0$. With this metric structure, it is possible to define a Hodge $*$-operator and then explore the properties of the Laplacian and find some physical consequences. Our picture is similar to Ref. 9 where the manifold is similarly three-dimensional but there is an extra time direction $\theta$ in the local cotangent space.

The Hodge $*$-operator on an $n$-dimensional calculus (for which the top form is of order $n$), over a Hopf algebra $H$ with metric $\eta$ is a map $*:\Omega^k \to \Omega^{n-k}$ given by the expression

$$*(\omega_{i_1}\cdots\omega_{i_k})=\frac{1}{(n-k)!}\epsilon_{i_1\ldots i_k i_{k+1}\ldots i_n}\eta^{i_{k+1}j_1}\cdots\eta^{i_n j_{n\cdots k}}\omega_{j_1}\cdots\omega_{j_{n-k}}.$$

In the case of the algebra $\mathbb{R}^3_\lambda$, we have a four-dimensional calculus with $\omega_1=\mathrm{d}x_1$, $\omega_2=\mathrm{d}x_2$, $\omega_3=\mathrm{d}x_3$, $\omega_4=\theta$. The components of the metric inverse, as we can see from (39), are $\eta^{11}=\eta^{22}=\eta^{33}=1$, and $\eta^{44}=1/\mu$. The arbitrary factor $\mu$ in the metric can be set by imposing conditions on the map $*^2$. Then we have two possible choices for the constant $\mu$: The first is $\mu=1$ making a four-dimensional Euclidean geometry; then for a $k$-form $\omega$ we have the constraint $**(\omega)=(-1)^{k(4-k)}\omega$. The second possibility is $\mu=-1$; then the metric is Minkowskian and the constraint on a $k$-form $\omega$ is $**(\omega)=(-1)^{1+k(4-k)}\omega$. In what follows, we will be using the Minkowskian convention on the grounds that this geometry on $\mathbb{R}^3_\lambda$ is a remnant of a noncommutative geometry on a $q$-deformed version of the Minkowski space, as we shall explain in Sec. VII. The expressions for the Hodge $*$-operator are summarized as follows:

$$\begin{aligned}
*1&=-\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\theta,\\
*\mathrm{d}x_1&=-\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\theta,\\
*\mathrm{d}x_2&=\mathrm{d}x_1\wedge\mathrm{d}x_3\wedge\theta,\\
*\mathrm{d}x_3&=-\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\theta,\\
*\theta&=-\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3,\\
*(\mathrm{d}x_1\wedge\mathrm{d}x_2)&=-\mathrm{d}x_3\wedge\theta,\\
*(\mathrm{d}x_1\wedge\mathrm{d}x_3)&=\mathrm{d}x_2\wedge\theta,\\
*(\mathrm{d}x_1\wedge\theta)&=\mathrm{d}x_2\wedge\mathrm{d}x_3,\\
*(\mathrm{d}x_2\wedge\mathrm{d}x_3)&=-\mathrm{d}x_1\wedge\theta,\\
*(\mathrm{d}x_2\wedge\theta)&=-\mathrm{d}x_1\wedge\mathrm{d}x_3,\\
*(\mathrm{d}x_3\wedge\theta)&=\mathrm{d}x_1\wedge\mathrm{d}x_2,\\
*(\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3)&=-\theta,\\
*(\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\theta)&=-\mathrm{d}x_3,\\
*(\mathrm{d}x_1\wedge\mathrm{d}x_3\wedge\theta)&=\mathrm{d}x_2,\\
*(\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\theta)&=-\mathrm{d}x_1,\\
*(\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\theta)&=1.
\end{aligned}\tag{40}$$

Given the Hodge $*$-operator, one can write, for example, the coderivative $\delta=*\mathrm{d}*$ and the Laplacian operator $\Delta=\delta\mathrm{d}+\mathrm{d}\delta$. Note that the Laplacian maps to forms of the same degree. We prefer to work actually with the ''Maxwell-type'' wave operator

$$\Box=\delta\mathrm{d}=*\mathrm{d}*\mathrm{d},\tag{41}$$

which is just the same in degree 0 and the same in degree 1 if we work in a gauge where $\delta=0$. In the rest of this section, we are going to describe some features of the electromagnetic theory arising in this noncommutative context. The electromagnetic theory is the analysis of solutions $A\in\Omega^1(\mathbb{R}^3_\lambda)$ of the equation $\Box A=\mathrm{J}$ where J is a one-form which can be interpreted as a ''physical'' source. We demonstrate the theory on two natural choices of sources, namely an electrostatic and a magnetic one. We start with spin 0 and we limit ourselves to algebraic plus plane wave solutions.

### A. Spin 0 modes

The wave operator on $\Omega^0(\mathbb{R}^3_\lambda)=\mathbb{R}^3_\lambda$ is computed from the definitions above as

$$\Box=*\mathrm{d}*\mathrm{d}=(\partial^a)^2-\frac{1}{c^2}(\partial^0)^2,$$

where the partials are defined by (34). The algebraic massless modes ker $\Box$ are given by

(i) polynomials of degree one: $f(x)=\alpha+\beta_a x_a$,
(ii) linear combinations of polynomials of the type $f(x)=x_a^2-x_b^2$,
(iii) linear combinations of quadratic monomials of the type, $f(x)=\alpha_{ab}x_a x_b$, with $a\neq b$, and
(iv) The three particular combinations $f(x)=x_1x_2x_3-(\imath\lambda/4)\,x_a^2$, for $a=1,2,3$.

General eigenfunctions of $\Box$ in degree 0 are the plane waves; the expression for their derivatives can be seen in (37). Hence

$$\Box e^{\imath k\cdot x}=-\frac{1}{\lambda^2}\left\{4\sin^2\left(\frac{\lambda|k|}{2}\right)+\left(\cos\left(\frac{\lambda|k|}{2}\right)-1\right)^2\right\}e^{\imath k\cdot x}.$$

It is easy to see that this eigenvalue goes in the limit $\lambda\rightarrow 1$ to the usual eigenvalue of the Laplacian in three-dimensional commutative space acting on plane waves.

### B. Spin 1 electromagnetic modes

On $\Omega^1(\mathbb{R}^3_\lambda)$, the Maxwell operator $\Box_1=*\mathrm{d}*\mathrm{d}$ can likewise be computed explicitly. If we write $A=(\mathrm{d}x_a)A^a+\theta A^0$ for functions $A^\mu$, then

$$F=\mathrm{d}A=\mathrm{d}x_a\wedge\mathrm{d}x_b\partial^b A^a+\mathrm{d}x_a\wedge\theta\frac{1}{c}\partial^0A^a+\theta\wedge\mathrm{d}x_a\partial^a A^0.$$

If we break this up into electric and magnetic parts in the usual way, then

$$B_a=\epsilon_{abc}\partial^b A^c,\quad E_a=\frac{1}{c}\partial^0 A^a-\partial^a A^0.$$

These computations have just the same form as for usual space–time. The algebraic zero modes ker $\Box_1$ are given by

(i) forms of the type $A=\mathrm{d}x_a(\alpha+\beta_a x_a+\gamma_a x_a^2)$ with curvature

$$F=\frac{\lambda}{4}\gamma_a\mathrm{d}x_a\wedge\theta,$$

(ii) forms of the type $A=\beta_{ab}(\mathrm{d}x_a)x_b$, with $a\neq b$ and curvature

$$F=\beta_{ab}\mathrm{d}x_a\wedge\mathrm{d}x_b,$$

(iii) forms of the type $A=\theta f$ with $f\in\ker\Box$. The curvatures for the latter three $f(x)$ shown above are

$$F=-2(\mathrm{d}x_a\wedge\theta x_a-\mathrm{d}x_b\wedge\theta x_b),$$

$$F=\alpha_{ab}\left(\theta\wedge(\mathrm{d}x_a)x_b+\theta\wedge(\mathrm{d}x_b)x_a+\frac{\imath\lambda}{2}\epsilon_{abc}\theta\wedge\mathrm{d}x_c\right)$$

$$F=-\mathrm{d}x_1\wedge\theta\left(x_2x_3+\frac{\imath\lambda}{2}x_1\right)-\mathrm{d}x_2\wedge\theta\left(x_1x_3-\frac{\imath\lambda}{2}x_2\right)-\mathrm{d}x_3\wedge\theta\left(x_1x_2+\frac{\imath\lambda}{2}x_3\right)-\frac{\imath\lambda}{2}\mathrm{d}x_a\wedge\theta x_a\,.$$

These are ''self-propagating'' electromagnetic modes or solutions of the sourceless Maxwell equations for a one-form or ''gauge potential'' $A$.

### C. Electrostatic solution

Here we take a uniform source in the ''purely time'' direction $\mathrm{J}=\theta$. In this case the solution of the gauge potential is

$$A=\tfrac{1}{6}\theta C,$$

where $C=\Sigma_a x_a^2$ is the Casimir operator. The curvature operator, which in this case can be interpreted as an electric field, is given by

$$F=\mathrm{d}A=\tfrac{1}{3}(\theta\wedge(\mathrm{d}x_1)x_1+\theta\wedge(\mathrm{d}x_2)x_2+\theta\wedge(\mathrm{d}x_3)x_3).$$

If $\theta$ is viewed as a time direction, then this curvature is a radial electric field. It has field strength increasing with the radius, which is a kind of solution exhibiting a confinement behavior. This is the correct physical solution for a uniform electric charge density throughout all space provided this is understood with the correct boundary conditions; if one builds the uniform charge density by a series of concentric shells about the origin, then, at radius $r$, all shells of greater radius produce no electric field and all shells of smaller radius total a charge proportional to $r^3$ and hence a radial electric field of strength proportional to $r$.

### D. Magnetic solution

Here we take a uniform electric current density along a direction vector $k\in\mathbb{R}^3$, i.e., $\mathrm{J}=k\cdot\mathrm{d}x=\Sigma_a k^a\mathrm{d}x_a$. In this case, the gauge potential can be written as

$$A=\frac{1}{4}\left\{\left(\sum_{a=1}^{3}k_a\mathrm{d}x_a\right)C+\frac{\theta}{2}\left(\sum_{a=1}^{3}k_a x_a\right)-\sum_{a=1}^{3}k_a(\mathrm{d}x_a)x_a^2\right\}.$$

The field strength is

$$F=\mathrm{d}A=\tfrac{1}{2}\{\mathrm{d}x_1\wedge\mathrm{d}x_2(k_1x_2-k_2x_1)+\mathrm{d}x_1\wedge\mathrm{d}x_3(k_1x_3-k_3x_1)+\mathrm{d}x_2\wedge\mathrm{d}x_3(k_2x_3-k_3x_2)\}. \quad (42)$$

If we decompose the curvature in the usual way, then this is a magnetic field in a direction $k\times x$ (the vector cross product). This is a ''confining'' (in the sense of increasing with normal distance) version of the field due to a uniform current density in direction $k$, taken with cylindrical boundary conditions at infinity.

We have considered for the electromagnetic solutions only uniform sources $J$; we can clearly put in a functional dependence for the coefficients of the source to similarly obtain other solutions of both the electric and magnetic types. Solutions more similar to the usual decaying ones, however, will not be polynomial (one would need the inverse of $\Sigma_a x_a^2$) and are therefore well outside our present scope; even at a formal level the problem of computing $\mathrm{d}(\Sigma_a x_a^2)^{-1}$ in a closed form appears to be formidable. On the other hand, these matters could probably be addressed by completing to $C^*$-algebras and using the functional calculus for such algebras.

### E. Spin $\frac{1}{2}$ equation

For completeness, let us mention here also a natural spin $\frac{1}{2}$ wave operator, namely the Dirac operator. We consider the simplest (Weyl) spinors as two components $\psi^i \in \mathbb{R}^3_\lambda$. In view of the fact that the partial derivatives $\partial^i{}_j$ already form a matrix, and following the similar phenomenon as for quantum groups,[18] we are led to define

$$(\partial\!\!\!/ \psi)^i = \partial^i{}_j \psi^j. \tag{43}$$

According to (35) this could also be written as

$$\partial\!\!\!/ = \frac{1}{2}\sigma_a \partial^a + \frac{1}{c}\partial^0,$$

where the second term is suggested by the geometry over and above what we might also guess. This term is optional in the same way as $(\partial^0)^2$ in $\Box$ is not forced by covariance, and is $\mathcal{O}(\lambda)$ for bounded spatial derivatives.

Here $\partial\!\!\!/$ is covariant under the quantum double action in Sec. III as follows (the same applies without the $\partial^0$ term). The action of $J_a$ on $\mathbb{R}^3_\lambda$ is that of orbital angular momentum and we have checked already that $\Box$ on degree 0 is covariant. For spin $\frac{1}{2}$ the total spin should be

$$S_a = \tfrac{1}{2}\sigma_a + J_a, \tag{44}$$

and we check that this commutes with $\partial\!\!\!/$:

$$\begin{aligned}(S_a \partial\!\!\!/ \psi)^i &= \tfrac{1}{2}\sigma_a{}^i{}_j \partial^j{}_k \psi^k + J_a M^i{}_j \triangleright \psi^j = \tfrac{1}{2}\sigma_a{}^i{}_j M^j{}_k \triangleright \psi^k + [J_a, M^i{}_j] \triangleright \psi^j + M^i{}_j J_a \triangleright \psi^j \\ &= \tfrac{1}{2} M^i{}_j \triangleright \sigma_a{}^j{}_k \psi^k + M^i{}_j J_a \triangleright \psi^j = M^i{}_j \triangleright (S_a \psi)^j = (\partial\!\!\!/ S_a \psi)^i,\end{aligned}$$

where we used the relations (20) (those with $M^i{}_j$ have the same form) and the action (26). The operator $\partial\!\!\!/$ is clearly also translation invariant under $\mathbb{C}(\mathrm{SU}(2))$ since the $\partial^i{}_j$ mutually commute. The operators $\sigma_a$ and $\partial^i{}_j$ also commute since one acts on the spinor indices and the other on $\mathbb{R}^3_\lambda$, so $S_a$ in place of $J_a$ still gives a representation of $D(\mathrm{U}(\mathfrak{su}(2)))$ on spinors, under which $\partial\!\!\!/$ is covariant.

### F. Yang–Mills U(1) fields

Finally, also for completeness, we mention that there is a different U(1) theory which behaves more like Yang–Mills. Namely instead of $F = \mathrm{d}A$ as in the Maxwell theory, we define $F = \mathrm{d}A + A \wedge A$ for a one-form $A$. This transforms by conjugation as $A \mapsto gAg^{-1} + g\mathrm{d}g^{-1}$ and is a nonlinear version of the above, where $g \in \mathbb{R}^3_\lambda$ is any invertible element, e.g., a plane wave. In this context one would expect to be able to solve for zero-curvature, i.e., $A$ such that $F(A) = 0$ and thereby demonstrate the Bohm–Aharanov effect, etc. This is part of the nonlinear theory, however, and beyond our present scope.

## VI. DIFFERENTIAL CALCULUS ON THE QUANTUM SPHERE

In this section we briefly analyze what happens if we try to set the ''length'' function given by the Casimir $C$ of $\mathbb{R}^3_\lambda$ to a fixed number, i.e., a sphere. We take this at unit radius, i.e., we define $S^2_\lambda$ as the algebra $\mathbb{R}^3_\lambda$ with the additional relation

$$C \equiv \sum_{a=1}^{3} (x_a)^2 = 1. \tag{45}$$

This immediately gives a "quantization condition" for the constant $\lambda$ if the algebra is to have an irreducible representation, namely $\lambda = 1/\sqrt{j(j+1)}$ for some $j \in \frac{1}{2}\mathbb{Z}_+$. The image of $S^2_\lambda$ in such a spin $j$ representation is a $(2j+1)\times(2j+1)$-matrix algebra which can be identified with the class of noncommutative spaces known as "fuzzy spheres."[14,2,5,20,21] In these works one does elements of noncommutative differential geometry directly on matrix algebras motivated by thinking about them as a projection of $\mathrm{U}(\mathfrak{su}(2))$ in the spin $j$ representation, and the greater the spin $j \to \infty$, the greater the resemblance with a classical sphere. The role of this in our case is played by $\lambda \to 0$ according to the above formula. On the other hand, note that we are working directly on $S^2_\lambda$ and are not required to look in one or any irreducible representation, i.e., this is a slightly more geometrical approach to "fuzzy spheres" where we deform the conventional geometry of $S^2$ by a parameter $\lambda$ and do not work with matrix algebras.

Specifically, when we make the constraint (45), the four-dimensional calculus given by relations (33) is reduced to a three-dimensional calculus on the sphere because

$$\mathrm{d}C = \sum_{a=1}^{3} 2(\mathrm{d}x_a)x_a + \frac{3\lambda}{4}\theta = 0,$$

which means that $\theta$ can be written as an expression on $\mathrm{d}x_a$. The remaining relations are given by

$$\begin{aligned} x_a \mathrm{d}x_b &= (\mathrm{d}x_b)x_a + \frac{i}{2}\lambda \epsilon_{abc}\mathrm{d}x_c - \frac{2}{3}\delta_{ab}\sum_{d=1}^{3}(\mathrm{d}x_d)x_d, \\ \lambda^2 \mathrm{d}x_a &= \frac{4i}{3}\lambda \epsilon_{abc}(\mathrm{d}x_b)x_c - \frac{16}{9}\sum_{d=1}^{3}(\mathrm{d}x_d)x_d x_a. \end{aligned} \tag{46}$$

In the limit $\lambda \to 0$ we recover the ordinary two-dimensional calculus on the sphere, given in terms of the classical variables $\overline{x_a} = \lim_{\lambda\to 0} x_a$. This can be seen by the relation

$$\sum_{a=1}^{3} (\mathrm{d}\overline{x_a})\overline{x_a} = 0,$$

allowing us to write one of the three one-forms in terms of the other two. For example, in the region where $\overline{x_3} = \sqrt{1-\overline{x_1}^2-\overline{x_2}^2}$ is invertible, one can write

$$\mathrm{d}\overline{x_3} = -\frac{\overline{x_1}}{\sqrt{1-\overline{x_1}^2-\overline{x_2}^2}}\mathrm{d}\overline{x_1} - \frac{\overline{x_2}}{\sqrt{1-\overline{x_1}^2-\overline{x_2}^2}}\mathrm{d}\overline{x_2}.$$

## VII. THE SPACE $\mathbb{R}^3_\lambda$ AS A LIMIT OF $q$-MINKOWSKI SPACE

In this section, we will express the noncommutative space $\mathbb{R}^3_\lambda$ as a spacelike surface of constant time in a certain scaling limit of the standard $q$-deformed Minkowski space $\mathbb{R}^{1,3}_q$ in Refs. 6, 16, and 15. This is defined in Ref. 15 as the algebra of $2\times 2$ braided (Hermitian) matrices $\mathrm{BM}_q(2)$ generated by 1 and

$$\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with the commutation relations

$$\begin{aligned} ba &= q^2 ab, \\ ca &= q^{-2} ac, \\ da &= ad, \\ bc &= cb + (1-q^{-2}) a(d-a), \\ db &= bd + (1-q^{-2}) ab, \\ cd &= dc + (1-q^{-2}) ca. \end{aligned} \tag{47}$$

If we choose a suitable set of generators, namely,

$$\tilde{t} = \frac{qd + q^{-1}a}{2}, \quad \tilde{x} = \frac{b+c}{2}, \quad \tilde{y} = \frac{b-c}{2i}, \quad \tilde{z} = \frac{d-a}{2},$$

then the braided determinant

$$\underline{\det}(\mathbf{u}) = ad - q^2 cb \tag{48}$$

can be written as

$$\underline{\det}(\mathbf{u}) = \frac{4q^2}{(q^2+1)^2} \tilde{t}^2 - q^2 \tilde{x}^2 - q^2 \tilde{y}^2 - \frac{2(q^4+1)q^2}{(q^2+1)^2} \tilde{z}^2 + 2q \left( \frac{q^2-1}{q^2+1} \right)^2 \tilde{t}\tilde{z}.$$

This expression, in the limit $q \to 1$, becomes the usual Minkowskian metric on $\mathbb{R}^{1,3}$. Here we will consider a different scaled limit related to the role of this algebra as braided enveloping algebra of a braided Lie algebra $\widetilde{\mathfrak{su}_q(2)}$ (see Ref. 10 for a recent treatment). This is such that we can still have a noncommutative space even when $q \to 1$. Defining new generators

$$x_+ = \frac{c}{(q-q^{-1})}, \quad x_- = \frac{b}{(q-q^{-1})}, \quad h = \frac{a-d}{(q-q^{-1})}, \quad t = \frac{qd + q^{-1}a}{c(q+q^{-1})}, \tag{49}$$

and considering the commutation relations (47), we have

$$\begin{aligned} [x_+, x_-] &= q^{-1} cth + q^{-1} \frac{(q-q^{-1})}{(q+q^{-1})} h^2, \\ q^{-2} h x_+ &= x_+ h + q^{-2}(q+q^{-1}) c x_+ t, \\ q^2 h x_- &= x_- h - (q+q^{-1}) c x_- t, \\ t x_\pm &= x_\pm t, \\ th &= ht. \end{aligned} \tag{50}$$

In the limit $q \to 1$, we obtain the commutation relations

$$[x_a, x_b] = \imath c t \epsilon_{abc} x_c, \quad [x_a, t] = 0, \tag{51}$$

of the so-called homogenized universal enveloping algebra $\widetilde{\mathrm{U}(\mathfrak{su}(2))}$, which we will denote by $\mathbb{R}_c^{1,3}$. Here $c$ is a parameter required by dimensional analysis (of dimension $ms^{-1}$). When $ct = \lambda$ we recover exactly the relations (28) of $\mathbb{R}_\lambda^3$. So the noncommutative space that we have studied in previous sections is the "slice" at a certain time of $\mathbb{R}_c^{1,3}$, which in turn is a contraction of $\mathbb{R}_q^{1,3}$. The possibility of these two $q \to 1$ limits where one gives a classical coordinate algebra and the other gives essentially its dual (an enveloping algebra) is called a "quantum-geometry duality transformation."

We now go further and also obtain the differential structure on $\mathbb{R}_\lambda^3$ via this scaling limit. Thus, the algebra $\mathbb{R}_q^{1,3} = \mathrm{BM}_q(2)$ has a standard $U_q(\mathfrak{su}(2))$-covariant noncommutative differential calculus whose commutation relations between basic one-forms and the generators of the algebra are given by[15]

$$\begin{gathered}
a\,\mathrm{d}a = q^2(\mathrm{d}a)a,\\
a\,\mathrm{d}b = (\mathrm{d}b)a,\\
a\,\mathrm{d}c = q^2(\mathrm{d}c)a + (q^2-1)(\mathrm{d}a)c,\\
a\,\mathrm{d}d = (\mathrm{d}d)a + (q^2-1)(\mathrm{d}b)c + (q-q^{-1})^2(\mathrm{d}a)a,\\
b\,\mathrm{d}a = q^2(\mathrm{d}a)b + (q^2-1)(\mathrm{d}b)a,\\
b\,\mathrm{d}b = q^2(\mathrm{d}b)b,\\
b\,\mathrm{d}c = (\mathrm{d}c)b + (1-q^{-2})((\mathrm{d}d)a + (\mathrm{d}a)d) + (q-q^{-1})^2(\mathrm{d}b)c - (2-3q^{-2}+q^{-4})(\mathrm{d}a)a,\\
b\,\mathrm{d}d = (\mathrm{d}d)b + (q^2-1)(\mathrm{d}b)d + (q^{-2}-1)(\mathrm{d}b)a + (q-q^{-1})^2(\mathrm{d}a)b,\\
c\,\mathrm{d}a = (\mathrm{d}a)c,\\
c\,\mathrm{d}b = (\mathrm{d}b)c + (1-q^{-2})(\mathrm{d}a)a,\\
c\,\mathrm{d}c = q^2(\mathrm{d}c)c,\\
c\,\mathrm{d}d = q^2(\mathrm{d}d)c + (q^2-1)(\mathrm{d}c)a,\\
d\,\mathrm{d}a = (\mathrm{d}a)d + (q^2-1)(\mathrm{d}b)c + (q-q^{-1})^2(\mathrm{d}a)a,\\
d\,\mathrm{d}b = q^2(\mathrm{d}b)d + (q^2-1)(\mathrm{d}a)b,\\
d\,\mathrm{d}c = (\mathrm{d}c)d + (q^2-1)(\mathrm{d}d)c + (q-q^{-1})^2(\mathrm{d}c)a + (q^{-2}-1)(\mathrm{d}a)c,\\
d\,\mathrm{d}d = q^2(\mathrm{d}d)d + (q^2-1)(\mathrm{d}c)b + (q^{-2}-1)(\mathrm{d}b)c - (1-q^{-2})^2(\mathrm{d}a)a.
\end{gathered} \tag{52}$$

This is designed in the $q \to 1$ limit to give the usual commutative calculus on classical $\mathbb{R}^{1,3}$. In order to obtain a noncommutative calculus in our noncommutative scaled limit $q \to 1$, we have to also redefine the derivative operator by a scale factor

$$\mathbf{d} = (q-q^{-1})\mathrm{d}.$$

This scaled derivative gives the following expressions for the basic one-forms:

$$\begin{gathered}
\mathbf{d}x_+ = \mathrm{d}c = (q-q^{-1})\mathrm{d}x_+,\\
\mathbf{d}x_- = \mathrm{d}b = (q-q^{-1})\mathrm{d}x_-,\\
\mathbf{d}h = \mathrm{d}a - \mathrm{d}d = (q-q^{-1})\mathrm{d}h.
\end{gathered}$$

Define also the basic one-form

$$\theta = q\mathrm{d}d + q^{-1}\mathrm{d}a,$$

which allows us to write

$$\mathbf{d}t = \frac{(q-q^{-1})}{c(q+q^{-1})}\theta. \tag{53}$$

This new set of generators and basic one-forms satisfy the following relations:

$$\begin{aligned}
&x_+\mathbf{d}x_+ = q^2(\mathbf{d}x_+)x_+,\\
&x_+\mathbf{d}x_- = (\mathbf{d}x_-)x_+ + \frac{q^{-1}}{(q+q^{-1})}\theta ct + \frac{1}{(q+q^{-1})}(\mathbf{d}h)ct + \mathcal{O}(q-q^{-1}),\\
&x_+\mathbf{d}h = (\mathbf{d}h)x_+ - q\,\mathbf{d}x_+ + \mathcal{O}(q-q^{-1}),\\
&x_-\mathbf{d}x_+ = (\mathbf{d}x_+)x_- + \frac{q^{-3}}{(q+q^{-1})}\theta ct - \frac{(2-q^{-2})}{(q+q^{-1})}(\mathbf{d}h)ct + \mathcal{O}(q-q^{-1}),\\
&x_-\mathbf{d}x_- = q^2(\mathbf{d}x_-)x_-,\\
&x_-\mathbf{d}h = q^2(\mathbf{d}h)x_- + q^{-1}(\mathbf{d}x_-)ct + \mathcal{O}(q-q^{-1}),\\
&h\mathbf{d}x_+ = (\mathbf{d}x_+)h + q(\mathbf{d}x_+)ct + \mathcal{O}(q-q^{-1}),\\
&h\mathbf{d}x_- = (\mathbf{d}x_-h) - q(\mathbf{d}x_-)ct + \mathcal{O}(q-q^{-1}),\\
&h\mathbf{d}h = (\mathbf{d}h)h + \frac{2q}{(q+q^{-1})}\theta ct + \mathcal{O}(q-q^{-1}),\\
&x_+\theta = \theta x_+ + q^2(\mathbf{d}x_+)ct + \mathcal{O}(q-q^{-1}),\\
&x_-\theta = \theta x_- + q^2(\mathbf{d}x_-)ct + \mathcal{O}(q-q^{-1}),\\
&h\theta = \theta h + \frac{2q}{(q+q^{-1})}(\mathbf{d}h)ct + \mathcal{O}(q-q^{-1}),\\
&t\mathbf{d}x_+ = (\mathbf{d}x_+)t + \mathcal{O}(q-q^{-1}),\\
&t\mathbf{d}x_- = (\mathbf{d}x_-)t + \mathcal{O}(q-q^{-1}),\\
&t\mathbf{d}h = (\mathbf{d}h)t + \mathcal{O}(q-q^{-1}),\\
&t\theta = \theta t + \mathcal{O}(q-q^{-1}).
\end{aligned} \tag{54}$$

In the limit $q\to 1$ we recover the relations (30) by setting $ct=\lambda$. Then the calculus on $\mathbb{R}^3_\lambda$ can be seen as the pull-back to the time slice of the scaled limit of the calculus on $q$-deformed Minkowski space. Unlike for usual $\mathbb{R}^3$, the d$t$ direction in our noncommutative case does not "decouple" and has remnant $\theta$. In other words, *the geometry of* $\mathbb{R}^3_\lambda$ *remembers that it is the pull-back of a relativistic theory*.

Finally, let us recall the action of the $q$-Lorentz group on the $\mathbb{R}^{1,3}_q$ and analyze its scaled limit when $q\to 1$. The appropriate $q$-Lorentz group can be written as the double cross coproduct $U_q(\mathfrak{su}(2))\bowtie U_q(\mathfrak{su}(2))$. The Hopf algebra $U_q(\mathfrak{su}(2))$ is the standard $q$-deformed Hopf algebra which we write explicitly as generated by 1, $X_+$, $X_-$ and $q^{\pm H/2}$ with

$$q^{\pm H/2}X_{\pm}q^{\mp H/2}=q^{\pm 1}X_{\pm}, \quad [X_+,X_-]=\frac{q^H-q^{-H}}{q-q^{-1}},$$

$$\Delta(X_\pm)=X_\pm\otimes q^{H/2}+q^{-H/2}\otimes X_\pm, \quad \Delta(q^{\pm H/2})=q^{\pm H/2}\otimes q^{\pm H/2},$$

$$\epsilon(X_\pm)=0, \quad \epsilon(q^{\pm H/2})=1,$$

$$S(X_\pm)=-q^{\pm 1}X_\pm, \quad S(q^{\pm H/2})=q^{\mp H/2}. \tag{55}$$

It is well known that one may also work with these generators in an R-matrix form

$$\mathbf{l}^+=\begin{pmatrix} q^{H/2} & 0 \\ q^{-1/2}(q-q^{-1})X_+ & q^{-H/2}\end{pmatrix}, \quad \mathbf{l}^-=\begin{pmatrix} q^{-H/2} & q^{1/2}(q^{-1}-q)X_- \\ 0 & q^{H/2}\end{pmatrix}, \tag{56}$$

and most formulas are usually expressed in terms of these matrices of generators. In particular, the $q$-Lorentz group has two mutually commuting copies of $U_q(\mathfrak{su}(2))$, so let us denote the generators of the first copy by $\mathbf{m}^\pm$ or $Y_\pm, G$ [related as for $\mathbf{l}^\pm$ and $X_\pm, H$ in (56)] and the generators of the second copy of $U_q(\mathfrak{su}(2))$ by $\mathbf{n}^\pm$ or $Z_\pm, T$ (similarly related). The actions on $\mathbb{R}_q^{1,3}$ are given in Ref. 15 in an R-matrix form

$$\mathbf{n}^{\pm k}{}_l \triangleright \mathbf{u}^i{}_j=\langle \mathbf{n}^{\pm k}{}_l, t^m{}_j\rangle \mathbf{u}^i{}_m, \quad \mathbf{m}^{\pm k}{}_l \triangleright \mathbf{u}^i{}_j=\langle S\mathbf{m}^{\pm k}{}_l, t^i{}_m\rangle \mathbf{u}^m{}_j. \tag{57}$$

Here $\langle S\mathbf{m}^{\pm k}{}_l, t^i{}_j\rangle$ and $\langle \mathbf{n}^{\pm k}{}_l, t^i{}_j\rangle$ are the $i,j$ matrix entries of the relevant functions of $Y_\pm, G$ and $Z_\pm, T$, respectively, in the Pauli matrix representation [as in (9) in other generators]. We need the resulting actions more explicitly, and compute them as

$$\frac{q^G-q^{-G}}{q-q^{-1}}\triangleright\begin{pmatrix} h & x_- \\ x_+ & t\end{pmatrix}=\begin{pmatrix} -\frac{2ct}{q-q^{-1}}-\frac{q-q^{-1}}{q+q^{-1}}h & -x_- \\ x_+ & \frac{q-q^{-1}}{q+q^{-1}}t-\frac{q-q^{-1}}{c(q+q^{-1})^2}h\end{pmatrix},$$

$$Y_+\triangleright\begin{pmatrix} h & x_- \\ x_+ & t\end{pmatrix}=\begin{pmatrix} -qx_+ & -\frac{qct}{q-q^{-1}}+\frac{h}{q+q^{-1}} \\ 0 & -\frac{q-q^{-1}}{c(q+q^{-1})}x_+\end{pmatrix}, \tag{58}$$

$$Y_-\triangleright\begin{pmatrix} h & x_- \\ x_+ & t\end{pmatrix}=\begin{pmatrix} q^{-1}x_- & 0 \\ -\frac{q^{-1}ct}{q-q^{-1}}-\frac{h}{q+q^{-1}} & -\frac{q-q^{-1}}{c(q+q^{-1})}x_-\end{pmatrix},$$

$$\frac{q^T-q^{-T}}{q-q^{-1}}\triangleright\begin{pmatrix} h & x_- \\ x_+ & t\end{pmatrix}=\begin{pmatrix} \frac{2ct}{q-q^{-1}}+\frac{q-q^{-1}}{q+q^{-1}}h & -x_- \\ x_+ & -\frac{q-q^{-1}}{q+q^{-1}}t+\frac{q-q^{-1}}{c(q+q^{-1})^2}h\end{pmatrix},$$

$$Z_+\triangleright\begin{pmatrix} h & x_- \\ x_+ & t\end{pmatrix}=\begin{pmatrix} -x_+ & \frac{ct}{q-q^{-1}}+\frac{qh}{q+q^{-1}} \\ 0 & \frac{q(q-q^{-1})}{c(q+q^{-1})}x_+\end{pmatrix}, \tag{59}$$

$$Z_-\triangleright\begin{pmatrix} h & x_- \\ x_+ & t\end{pmatrix}=\begin{pmatrix} x_- & 0 \\ \dfrac{ct}{q-q^{-1}}-\dfrac{q^{-1}h}{q+q^{-1}} & \dfrac{q^{-1}(q-q^{-1})}{c(q+q^{-1})}x_-\end{pmatrix}.$$

We are now able to see that these actions (58) and (59) blow up in the limit $q\to 1$ because of some singular terms appearing in their expressions. Hence the scaling limit $\mathbb{R}_c^{1,3}$ is no longer Lorentz invariant.

On the other hand, we also have the same quantum group symmetry in an isomorphic form $BSU_q(2)\rtimes U_q(\mathfrak{su}(2))$ for $q\neq 1$, and this version survives. The braided algebra $BSU_q(2)$ here is simply the braided matrices $BM_q(2)$ with the additional condition $\underline{\det}(\mathbf{u})=1$ (i.e., geometrically, it is the mass-hyperboloid in $q$-Minkowski space). To be clear, the generators of $BSU_q(2)$ in this crossed product will be denoted by $\overline{\mathbf{u}}$ and the generators of $U_q(\mathfrak{su}(2))$ in this cross product will be denoted by $\mathbf{l}^{\pm}$ or $X_{\pm},H$ as before. The isomorphism with the $q$-Lorentz group in the form above is given by the assignments[15]

$$\overline{\mathbf{u}}\otimes 1\mapsto \mathbf{m}^+S(\mathbf{m}^-)\otimes 1,\quad 1\otimes \mathbf{l}^{\pm}\mapsto \mathbf{m}^{\pm}\otimes\mathbf{n}^{\pm}. \tag{60}$$

Under the isomorphism (60), the expressions (58) and (59) become the action of $BSU_q(2)\rtimes U_q(\mathfrak{su}(2))$ on $BM_q(2)$ given by

$$\overline{\mathbf{u}}\triangleright\mathbf{u}=\mathbf{m}^+S(\mathbf{m}^-)\triangleright\mathbf{u},\quad \mathbf{l}^{\pm}\triangleright\mathbf{u}=\mathbf{m}^{\pm}\triangleright(\mathbf{n}^{\pm}\triangleright\mathbf{u}).$$

On the generators (49) the action of $BSU_q(2)$ reads

$$\begin{aligned}
\overline{\mathbf{u}}^1{}_1\triangleright h&=-ct+qh-\frac{q(q-q^{-1})}{q+q^{-1}}h,\\
\overline{\mathbf{u}}^1{}_1\triangleright x_+&=qx_+,\\
\overline{\mathbf{u}}^1{}_1\triangleright x_-&=q^{-1}x_-,\\
\overline{\mathbf{u}}^1{}_1\triangleright t&=\frac{q^2+q^{-2}}{q+q^{-1}}t-\frac{(q-q^{-1})^2}{c(q+q^{-1})^2}h,\\
\overline{\mathbf{u}}^1{}_2\triangleright h&=q^{-2}(q-q^{-1})x_-\\
\overline{\mathbf{u}}^1{}_2\triangleright x_+&=-q^{-2}ct-\frac{q^{-1}(q-q^{-1})}{q+q^{-1}}h,\\
\overline{\mathbf{u}}^1{}_2\triangleright x_-&=0,\\
\overline{\mathbf{u}}^1{}_2\triangleright t&=-\frac{q-q^{-1}}{q+q^{-1}}t+\frac{q^{-1}(q-q^{-1})^2}{c(q+q^{-1})^2}h,\\
\overline{\mathbf{u}}^2{}_1\triangleright h&=-(q-q^{-1})x_+,\\
\overline{\mathbf{u}}^2{}_1\triangleright x_+&=0,\\
\overline{\mathbf{u}}^2{}_1\triangleright x_-&=-ct+\frac{q(q-q^{-1})}{q+q^{-1}}h,
\end{aligned}\tag{61}$$

$$
\begin{aligned}
\overline{\mathbf{u}}^2_1 \triangleright t &= -\frac{q(q-q^{-1})}{q+q^{-1}}\, t + \frac{(q-q^{-1})^2}{c(q+q^{-1})^2}\, h,\\
\overline{\mathbf{u}}^2_2 \triangleright h &= ct + qh - q^{-1}(q-q^{-1})ct - \frac{q^{-1}(q-q^{-1}) - q^{-2}(q-q^{-1})^2}{q+q^{-1}}\, h,\\
\overline{\mathbf{u}}^2_2 \triangleright x_+ &= q^{-1}x_+ + q^{-1}(q-q^{-1})^2 x_+ ,\\
\overline{\mathbf{u}}^2_2 \triangleright x_- &= q x_- ,\\
\overline{\mathbf{u}}^2_2 \triangleright t &= \frac{2t}{q+q^{-1}} + \frac{(q-q^{-1})^2}{q+q^{-1}}\, t\, \frac{(q-q^{-1})^2 - q^{-1}(q-q^{-1})^3}{c(q+q^{-1})^2}\, h.
\end{aligned}
\tag{62}
$$

The action of $\mathrm{U}_q(\mathfrak{su}(2))$ is given by

$$
\begin{aligned}
\frac{q^H - q^{-H}}{q-q^{-1}} \triangleright \begin{pmatrix} h & x_- \\ x_+ & t \end{pmatrix} &= \begin{pmatrix} 0 & -(q+q^{-1})x_- \\ (q+q^{-1})x_+ & 0 \end{pmatrix},\\
X_+ \triangleright \begin{pmatrix} h & x_- \\ x_+ & t \end{pmatrix} &= \begin{pmatrix} -q(q^{1/2}+q^{-1/2})x_+ & q^{1/2}h \\ 0 & 0 \end{pmatrix},\\
X_- \triangleright \begin{pmatrix} h & x_- \\ x_+ & t \end{pmatrix} &= \begin{pmatrix} q^{-1/2}(q+q^{-1})x_- & 0 \\ -q^{-1/2}h & 0 \end{pmatrix}.
\end{aligned}
\tag{63}
$$

In the limit $q \to 1$, the crossed product $\mathrm{BSU}_q(2) \rtimes \mathrm{U}_q(\mathfrak{su}(2))$ becomes the double $D(\mathrm{U}(\mathfrak{su}(2)) = \mathbb{C}(\mathrm{SU}(2)) \rtimes \mathrm{U}(\mathfrak{su}(2))$ as studied in Sec. III. The elements $\overline{\mathbf{u}}^i_j$ become in the limit the $t^i_j$, and $X_\pm$ and $H$ become the usual $\mathfrak{su}(2)$ generators equivalent to the $J_a$ there. (More precisely, we should map $\overline{u}^i_j$ to $St^i_j$ for the action to become the right coregular one which we viewed in Sec. III as a left coaction.) Finally, this action of the double on $\mathrm{BM}_q(2)$ thus becomes in the scaling limit $q \to 1$ an action of $D(\mathrm{U}(\mathfrak{su}(2))$ on $\mathbb{R}^{1,3}_c$ in the form

$$[x_+, x_-] = 2cth, \quad [h, x_\pm] = \pm ctx_\pm ,$$

with the same change of variables to $x_a$ as in Sec. IV. The result is

$$M^i{}_j \triangleright t = 0, \quad M^i{}_j \triangleright x_a = \frac{ct}{2\lambda}\sigma_a{}^i{}_j , \quad J_a \triangleright t = 0, \quad J_a \triangleright x_a = \imath \epsilon_{abc} x_c .$$

This is consistent with the time slice $ct = \lambda$ and gives the action of the quantum double in Sec. III as in fact the nonsingular version of scaled limit of the $q$-Lorentz symmetry on the $q$-Minkowski space.

One can also analyze a different time slice of $\mathbb{R}^{1,3}_q$, namely, the quotient obtained by imposing the condition $ct = q^2 + q^{-2} - 1$. This algebra is the reduced braided algebra $\mathrm{BM}_q(2)^{\mathrm{red}}$, see Ref. 10, with commutation relations

$$
\begin{aligned}
x_+ x_- &= x_- x_+ + q^{-1}(q^2 + q^{-2} - 1)h + \frac{(q-q^{-1})}{(q+q^{-1})}h^2,\\
q^{-2}hx_+ &= x_+ h + q^{-2}(q^2+q^{-2}-1)(q+q^{-1})x_+ ,\\
q^2 hx_- &= x_- h - (q^2+q^{-2}-1)(q+q^{-1})x_- .
\end{aligned}
$$

This is also known in the literature as the ‘‘Witten algebra’’ [13,23] and in a scaled limit $q\to 1$ it likewise turns into the universal enveloping algebra $U(\mathfrak{su}(2))$. A calculus on this reduced algebra, however, is not obtained from the calculus given by relations (52); consistency conditions result in the vanishing of all derivatives $\mathrm{d}a$, $\mathrm{d}b$ and $\mathrm{d}c$ (note that the constraint on $t$ allows one to write $d$ in terms of the other generators).

## VIII. QUANTUM MECHANICAL INTERPRETATION AND SEMICLASSICAL LIMIT OF $\mathbb{R}^3_\lambda$

Finally, we turn to the important question of how to relate expressions in the above noncommutative geometry to ordinary numbers in order to compare with experiment. We will first explain why a normal ordering postulate as proposed in Ref. 3 is not fully satisfactory and then turn to a quantum mechanical approach. Thus, one idea is to write elements of $\mathbb{R}^3_\lambda$ as: $f(x)$: where $f(x_1,x_2,x_3)$ is a classical function defined by a powerseries and : : denotes normal ordering when we use noncommutative variables $x_i$. If one sticks to this normal ordering, one can use it to compare classical with quantum expressions and express the latter as a strict deformation of the former controlled by the parameter $\lambda$ governing the noncommutativity in (28). This will extend to the rest of the geometry and allows an order-by-order analysis. For example, the noncommutative partial derivatives $\partial_a$ defined in (34) have the expressions to lowest order

$$\begin{aligned}
\partial_1{:}f(x){:} &= {:}\overline{\partial_1}f(x){:}+\frac{\imath\lambda}{2}\overline{\partial_2}\,\overline{\partial_3}f(x),\\
\partial_2{:}f(x){:} &= {:}\overline{\partial_2}f(x){:}-\frac{\imath\lambda}{2}\overline{\partial_1}\,\overline{\partial_3}f(x),\\
\partial_3{:}f(x){:} &= {:}\overline{\partial_3}f(x){:}+\frac{\imath\lambda}{2}\overline{\partial_2}\,\overline{\partial_2}f(x),\\
\frac{1}{c}\partial_0{:}f(x){:} &= \frac{\lambda}{4}((\overline{\partial_1})^2f(x)+(\overline{\partial_2})^2f(x)+(\overline{\partial_3})^2f(x)),
\end{aligned} \tag{64}$$

where $\overline{\partial_a}$ are the usual partial derivatives in classical variables and we do not write the normal ordering on expressions already $O(\lambda)$ since the error is higher order. Note that the expression for $(1/c)\,\partial_0$ is one order of $\lambda$ higher than the other partial derivatives, which is another way to see that this direction is an anomalous dimension originating in the quantization process. The physical problem here is that the normal ordering is somewhat arbitrary; for algebras such as (1) or for usual phase space, putting all $t$ to one side makes a degree of sense physically, as well as mathematically because the algebra is solvable. But in the simple case such as $\mathbb{R}^3_\lambda$, each of the $x_1$, $x_2$, $x_3$ should be treated equally. Or one could use other coordinates such as $x_-$, $h$, $x_+$ in keeping with the Lie algebra structure, etc.; all different ordering schemes giving a different form of the lowest order corrections and hence different predictions. Choosing a natural ordering is certainly possible but evidently would require further input into the model.

On the other hand, we can take a more quantum mechanical line and consider our algebra $\mathbb{R}^3_\lambda$ as, after all, a spin system. The main result of this section is to introduce ‘‘approximately classical’’ states’ $|j,\theta,\phi\rangle$ for this system inspired in part by the theorem of Penrose[19] for spin networks, although not directly related to that. Penrose considered networks labeled by spins and showed how to assign probabilities to them and conditions for when the network corresponds approximately to spin measurements oriented with relative angles $\theta,\phi$. In a similar spirit we consider the problem of reconstructing classical angles from the noncommutative geometry.

We let $V^{(j)}$ be the vector space which carries a unitary irreducible representation of spin $j \in \frac{1}{2}\mathbb{Z}_+$, generated by states $|j,m\rangle$, with $m=-j,\ldots,j$ such that

$$x_\pm|j,m\rangle=\lambda\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle,$$

$$h|j,m\rangle = 2\lambda m|j,m\rangle.$$

The projection of $\mathbb{R}^3_\lambda$ to an irreducible representation of spin $j$ is geometrically equivalent to a restriction to a fuzzy sphere,[5,14] because the value of the Casimir $x\cdot x$ is $\lambda^2 j(j+1)$ in this representation. We have discussed this in Sec. VI, where we set $x\cdot x=1$ and considered the algebra geometrically as such a fuzzy sphere under a quantization condition for $\lambda$. By contrast in this section we leave $x\cdot x$ unconstrained and consider the geometry of our noncommutative three-dimensional space $\mathbb{R}^3_\lambda$ as the sum of geometries on all fuzzy spheres with the $V^{(j)}$ representation picking out the one of radius $\sim\lambda j$. Thus we use the Peter–Weyl decomposition of $\mathbb{C}(SU_2)$ into matrix elements of irreducible representations regarded as functions on $SU_2$, which gives (up to some technical issues about completions) a similar decomposition for its dual as $\mathbb{R}^3_\lambda = \oplus_j \mathrm{End}(V^{(j)})$. This also underlies the spherical harmonics in Sec. III.

Next, for each fixed spin $j$ representation we look for normalized states $|j,\theta,\phi\rangle$ parametrized by $0\leqslant\theta\leqslant\pi$ and $0\leqslant\varphi\leqslant 2\pi$, such that

$$\langle j,\theta,\varphi|x_1|j,\theta,\varphi\rangle = r\sin\theta\cos\varphi,$$

$$\langle j,\theta,\varphi|x_2|j,\theta,\varphi\rangle = r\sin\theta\sin\varphi, \tag{65}$$

$$\langle j,\theta,\varphi|x_3|j,\theta,\varphi\rangle = r\cos\theta,$$

where $r$ is some constant (independent of $\theta,\phi$) which we do not fix. Rather, in the space of such states and possible $r\geqslant 0$, we seek to minimize the normalized variance

$$\delta = \frac{\langle x\cdot x\rangle - \langle x\rangle\cdot\langle x\rangle}{\langle x\rangle\cdot\langle x\rangle}, \tag{66}$$

where $\langle\ \rangle = \langle j,\theta,\phi|\ |j,\theta,\phi\rangle$ is the expectation value in our state and we regard $\langle x_a\rangle$ as a classical vector in the dot product. Thus we seek states which are "closest to classical." This is a constrained problem and leads us to the following states:

$$|j,\theta,\varphi\rangle = \sum_{k=1}^{2j+1} 2^{-j}\sqrt{\binom{2j}{k-1}}(1+\cos\theta)^{(j-k+1)/2}(1-\cos\theta)^{(k-1)/2}e^{\imath(k-1)\varphi}|j,j-k+1\rangle. \tag{67}$$

These obey $\langle j,\theta,\phi|j,\theta,\phi\rangle = 1$ and (65)–(66) with

$$r = \sqrt{\langle x\rangle\cdot\langle x\rangle} = \lambda j, \qquad \delta = \frac{1}{j}. \tag{68}$$

We see that in these states the "true radius" $|\langle x\rangle|$ is $\lambda j$. The square root of the Casimir does not give this true radius since it contains also the uncertainty expressed in the variance of the position operators, but the error $\delta$ vanishes as $j\to\infty$. Thus the larger the representation, the more the geometry resembles to the classical.

We can therefore use these states $|j,\theta,\phi\rangle$ to convert noncommutative geometric functions $f(x)$ into classical ones in spherical polar coordinates defined by

$$\langle f\rangle(r,\theta,\phi) \equiv \langle j,\theta,\phi|f(x)|j,\theta,\phi\rangle, \tag{69}$$

where $r=\lambda j$ is the effective radius. If we start with a classical function $f$ and insert noncommutative variables in some order, then $\langle f(x)\rangle$ (which depends on the ordering) looks more and more like $f(\langle x\rangle)$ as $j\to\infty$ and $\lambda\to 0$ with the product fixed to an arbitrary $r$. As an example, the noncommutative spherical harmonics $Y_l^m$ in Sec. III are already ordered in such a way that replacing the noncommutative variables by the expectation values $\langle x_a\rangle$ gives something proportional to the classical spherical harmonics. On the other hand, $\langle Y_l^m \rangle$ vanish for $l > 2j$ and only approximate the classical ones for lower $l$. Moreover, in view of the above, we expect

$$\langle \partial_i f \rangle = \overline{\partial_i} \langle f \rangle + O\left(\lambda, \frac{1}{j}\right), \tag{70}$$

where $r = j\lambda$ and $\overline{\partial_i}$ are the classical derivatives in the polar form

$$\overline{\partial_1} = \sin\theta\cos\varphi \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\theta\sin\varphi\frac{\partial}{\partial\varphi},$$

$$\overline{\partial_2} = \sin\theta\sin\varphi \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\sin\theta\cos\varphi\frac{\partial}{\partial\varphi},$$

$$\overline{\partial_3} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta},$$

where we understand $\partial/\partial r = (1/\lambda)(\partial/\partial j)$ on expectation values computed as functions of $j$. More precisely, one should speak in terms of the joint limit as explained above with $\lambda j = r$ a continuous variable in the limit. We note finally that the star product for $\mathbb{R}^3_\lambda$ as in Ref. 11 suggests that it should be possible to extend such a semiclassical analysis to all orders.

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## APPENDIX: 2-D AND 3-D CALCULI ON $\mathbb{R}^3_\lambda$

It might be asked why we need to take a four-dimensional calculus on $\mathbb{R}^3_\lambda$ and not a smaller one. In fact, bicovariant differential calculi on enveloping algebras $\mathrm{U}(\mathfrak{g})$ such as $\mathbb{R}^3_\lambda \cong \mathrm{U}(\mathfrak{su}(2))$ have been essentially classified[18] and in this appendix we look at some of the other possibilities for our model. In general the co-irreducible calculi (i.e., having no proper quotients) are labeled by pairs $(V_\rho, \Lambda)$, with $\rho: \mathrm{U}(\mathfrak{g}) \to \mathrm{End} V_\rho$ an irreducible representation of $\mathrm{U}(\mathfrak{g})$ and $\Lambda$ a ray in $\mathrm{V}_\rho$. In order to construct an ideal in ker $\epsilon$, take the map

$$\rho_\Lambda : \mathrm{U}(\mathfrak{g}) \to V_\rho, \quad h \mapsto \rho(h) \cdot \Lambda.$$

It is easy to see that ker $\rho_\Lambda$ is a left ideal in ker $\epsilon$. Then, if $\rho_\Lambda$ is surjective, the space of one-forms can be identified with $V_\rho = \ker \epsilon / \ker \rho_\Lambda$. The general commutation relations are

$$av = va + \rho(a) \cdot v, \tag{A1}$$

and the derivative for a general monomial $\xi_1 \cdots \xi_n$ is given by the expression

$$\mathrm{d}(\xi_1 \cdots \xi_n) = \sum_{k=1}^{n} \sum_{\sigma \in S_{(n,k)}} \rho_\Lambda(\xi_{\sigma(1)} \cdots \xi_{\sigma(k)}) \xi_{\sigma(k+1)} \cdots \xi_{\sigma(n)},$$

the sum being over all $(n,k)$ shuffles.

We explore some examples of co-irreducible calculi for the universal enveloping algebra $\mathbb{R}^3_\lambda$, generated by $x_\pm$ and $h$ satisfying the commutation relations (28). First, let us analyze the three-dimensional, co-irreducible calculus on $\mathbb{R}^3_\lambda$ by taking $V_\rho = \mathbb{C}^3$, with basis

$$e_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In this basis, the representation $\rho$ takes the form

$$\rho(x_+)=\lambda\begin{pmatrix}0&2&0\\0&0&1\\0&0&0\end{pmatrix},\quad \rho(x_-)=\lambda\begin{pmatrix}0&0&0\\1&0&0\\0&2&0\end{pmatrix},\quad \rho(h)=\lambda\begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix}.$$

We choose, for example, $\Lambda=e_0$. The space of one-forms will be generated by the vectors $e_+$, $e_-$ and $e_0$. The derivatives of the generators of the algebra are given by

$$\mathrm{d}x_+=\lambda^{-1}\rho(x_+)\cdot e_0=2e_+,\quad \mathrm{d}x_-=\lambda^{-1}\rho(x_-)\cdot e_0=2e_-,\quad \mathrm{d}h=\lambda^{-1}\rho(h)\cdot e_0=0.$$

The commutation relations between the basic one-forms and the generators can be deduced from (A1) giving

$$\begin{aligned}x_+e_+&=e_+x_+,\\ x_+e_0&=e_0x_++2\lambda e_+,\\ x_+e_-&=e_-x_++\lambda e_0,\\ x_-e_+&=e_+x_-+\lambda e_0,\\ x_-e_0&=e_0x_-+2\lambda e_-,\\ x_-e_-&=e_-x_-,\\ he_+&=e_+h+2\lambda e_+,\\ he_0&=e_0h,\\ he_-&=e_-h-2\lambda e_-.\end{aligned}\tag{A2}$$

The expression for the derivative of a general monomial $x_+^a x_-^b h^c$ is

$$\begin{aligned}\mathrm{d}(x_+^a x_-^b h^c)=&2ae_+x_+^{a-1}x_-^bh^c+2be_-x_+^ax_-^{b-1}h^c+2\lambda abe_0x_+^{a-1}x_-^{b-1}h^c\\ &+4\lambda^2a(a-1)be_+x_+^{a-2}x_-^{b-1}h^c.\end{aligned}\tag{A3}$$

We define the exterior algebra by skew-symmetrizing and, using similar methods as in Sec. IV, we compute the cohomologies as

$$H^0=\mathbb{C}[h],\quad H^1=e_0\mathbb{C}[h],\quad H^2=H^3=\{0\}.$$

This calculus is a three-dimensional calculus but we have introduced an isotropy by choosing $\Lambda$, and related to this all functions of $h$ are killed by d, which is why the cohomology is large. This is why we do not take this calculus even though it has the "obvious" dimension. There is the same problem if we choose any other direction $\Lambda$.

We can also have a two-dimensional coirreducible calculus on $\mathrm{U}(\mathfrak{su}(2))$ using then $V_\rho=\mathbb{C}^2$, with basis

$$e_1=\begin{pmatrix}1\\0\end{pmatrix},\quad e_2=\begin{pmatrix}0\\1\end{pmatrix}.$$

In this basis, the representation $\rho$ takes the form

$$\rho(x_+)=\lambda\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \rho(x_-)=\lambda\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \rho(h)=\lambda\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Choosing $\Lambda=e_1$, the space of one-forms will be generated by $e_1$ and $e_2$ and the derivatives of the generators of the algebra are given by

$$\mathrm{d}x_+=\lambda^{-1}\rho(x_+)\cdot e_1=0,\quad \mathrm{d}x_-=\lambda^{-1}\rho(x_-)\cdot e_1=e_2,\quad \mathrm{d}h=\lambda^{-1}\rho(h)\cdot e_1=e_1.$$

The commutation relations between the basic one-forms and the generators are then

$$\begin{aligned}x_+e_1&=e_1x_+,\\ x_+e_2&=e_2x_++\lambda e_1,\\ x_-e_1&=e_1x_-+\lambda e_2,\\ x_-e_2&=e_2x_-,\\ he_1&=e_1h+\lambda e_1,\\ he_2&=e_2h-\lambda e_2.\end{aligned}\tag{A4}$$

And the derivative of a monomial $x_-^a h^b x_+^c$ is given by

$$\mathrm{d}(x_-^a h^b x_+^c)=e_1\left(\sum_{i=0}^{b}\binom{b}{i}\lambda^{i-1}x_-^a h^{b-i}x_+^c\right)+e_2\left(\sum_{i=0}^{b}(-1)^i\binom{b}{i}\lambda^i a x_-^{a-1}h^{b-i}x_+^c\right).\tag{A5}$$

The cohomology of this calculus comes out as

$$H^0=\mathbb{C}[x_+],\quad H^1=H^2=\{0\}.$$

Here again d vanishes on all functions of $x_+$, which is related to our choice of $\Lambda$. On the other hand, this calculus motivates us similarly to take for $\rho$ the tensor product of the spin $\frac{1}{2}$ representations and its dual. In this tensor product representation there is a canonical choice of $\Lambda$, namely the $2\times 2$ identity matrix. This solves the anisotropy and kernel problems and this is the calculus that we have used on $\mathbb{R}^3_\lambda$ as the natural choice in our situation. The above spinorial ones are coirreducible quotients of it.

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