

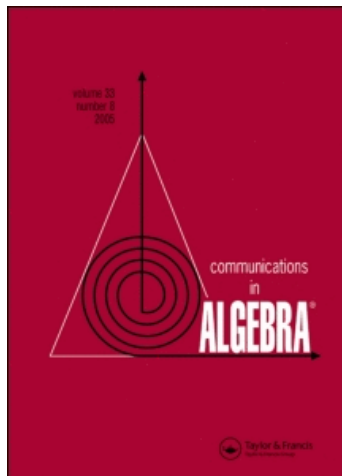
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On: 19 August 2010

Access details: Access Details: [subscription number 925989134]

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Communications in Algebra

Publication details, including instructions for authors and subscription information:

<http://www.informaworld.com/smpp/title~content=t713597239>

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Online publication date: 18 August 2010

To cite this Article Alves, Marcelo Muniz S. and Batista, Eliezer(2010) 'Enveloping Actions for Partial Hopf Actions', Communications in Algebra, 38: 8, 2872 — 2902

To link to this Article: DOI: 10.1080/00927870903095582

URL: <http://dx.doi.org/10.1080/00927870903095582>

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ENVELOPING ACTIONS FOR PARTIAL HOPF ACTIONS

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Motivated by partial group actions on unital algebras, in this article we extend many results obtained by Exel and Dokuchaev to the context of partial actions of Hopf algebras, according to Caenepeel and Jansen. First, we generalize the theorem about the existence of an enveloping action, also known as the globalization theorem. Second, we construct a Morita context between the partial smash product and the smash product related to the enveloping action. Third, we dualize the globalization theorem to partial coactions and finally, we define partial representations of Hopf algebras and show some results relating partial actions and partial representations.

Key Words: Partial coactions; Partial group actions; Partial Hopf actions; Partial representations; Smash products.

2000 Mathematics Subject Classification: Primary 16W30; Secondary 16S40, 16S35, 58E40.

1. INTRODUCTION

Partial group actions were first defined by Exel in the context of operator algebras and they turned out to be a powerful tool in the study of C^* -algebras generated by partial isometries on a Hilbert space [9]. The developments originated by the definition of partial group actions include crossed products [13], partial representations [6, 10] and soon this theme became an independent topic of interest in ring theory [7, 11]. Now, the results are formulated in a purely algebraic way, independent of the C^* algebraic techniques which originated them.

A partial action α of a group G on a (possibly non-unital) k -algebra A is a pair of families of sets and maps indexed by G , $\alpha = (\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G})$, where each D_g is an ideal of A and each α_g is an algebra isomorphism $\alpha : D_{g^{-1}} \rightarrow D_g$ satisfying the following conditions:

- (i) $D_e = A$ and $\alpha_e = I_A$;
- (ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ for every $g, h \in G$;
- (iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for every $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

Received July 28, 2008; Revised April 30, 2009. Communicated by M. Cohen.

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A first example of partial action is the following: If G acts on an algebra B by automorphisms and A is an ideal of B , then we have a partial action α on A in the following manner: letting β_g stand for the automorphism corresponding to g , take $D_g = A \cap \beta_g(A)$, and define $\alpha_g : D_{g^{-1}} \rightarrow D_g$ as the restriction of the automorphism β_g to $D_{g^{-1}}$.

Partial Hopf actions were motivated by an attempt to generalize the notion of partial Galois extensions of commutative rings, first introduced by Dokuchaev et al. in [8]. The first ideas towards partial Hopf actions were introduced by Caenepeel and de Groot in [3], using the concept of Galois coring. Afterwards, Caenepeel and Janssen defined partial actions and partial coactions of a Hopf algebra H on a unital algebra A using the notions of partial entwining structures [4]; in particular, partial actions of G determine partial actions of the group algebra kG in a natural way. In the same article, the authors also introduced the concept of partial smash product which, in the case of the group algebra kG , turns out to be the crossed product by a partial action $A \rtimes_{\alpha} G$. Further developments in the theory of partial Hopf actions were done by Lomp in [12], where the author pushed forward classical results of Hopf algebras concerning smash products, like the Blattner–Montgomery and Cohen–Montgomery theorems [14].

Certainly, the theory of partial actions of Hopf algebras remains as a huge landscape to be explored, and this present work intends to generalize some results for partial group actions to the context of partial Hopf actions. We divided this article as follows.

In Section 2, we prove the theorem of existence of an enveloping action for a partial Hopf action, i.e., we prove that if H is a Hopf algebra which acts partially on a unital algebra A , then there exists an H module algebra B such that A is isomorphic to a right ideal of B , and the restriction of the action of H to this ideal is equivalent to the partial action of H on A . The *uniqueness* of the enveloping action is treated separately; we introduce the concept of minimal enveloping action and prove the existence and uniqueness of such an action for every partial action. The question on the existence of enveloping actions for partial group actions arises naturally when we consider the basic example of partial action, that of the restriction of a global action of a group G on an algebra B to an ideal $A \trianglelefteq B$. Thus we may ask, which conditions on a partial action enable us to say that this partial action is a restriction of a global action? The first result concerning enveloping actions was proved in the context of C^* algebras in [1]; to this intent, the author used techniques of Fell Bundles and Hilbert C^* modules. A purely algebraic version of this theorem on enveloping actions only appeared in [7]. Basically, the same ideas for the proof in the group case are present in the Hopf algebraic case as we shall see later.

In Section 3, we show the existence of a Morita context between the partial smash product $A \# H$, where H is a Hopf algebra which acts partially on the unital algebra A , and the smash product $B \# H$, where B is an enveloping action of A . This result can also be found in [7] for the context of partial group actions.

In Section 4, we discuss the existence of an enveloping coaction of a Hopf algebra H on a unital algebra A . There, we dualize this partial coaction of H to a partial action of H^* (in fact, the finite dual H°), we take an enveloping action and then check whether the H° module B of the enveloping action is a rational module. If this occurs, one dualizes again to obtain a structure of H comodule algebra in B ; this is our enveloping coaction.

In Section 5, we introduce the notion of partial representation of a Hopf algebra. We show that, under certain conditions on the algebra H , the partial smash product $A \# H$ carries a partial representation of H .

2. ENVELOPING ACTIONS

2.1. Partial Hopf Actions

We recall that a left action of a Hopf algebra H on an algebra A is a linear mapping $\alpha : H \otimes A \rightarrow A$, which we will denote by $\alpha(h \otimes a) = h \triangleright a$, such that:

- (i) $h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$;
- (ii) $1 \triangleright a = a$;
- (iii) $h \triangleright (k \triangleright a) = hk \triangleright a$;
- (iv) $h \triangleright 1_A = \epsilon(h)1_A$.

We also say that A is an H module algebra. Note that (ii) and (iii) say that A is a left H -module.

In [4], Caenepeel and Jansen defined a weaker version of an action, called a partial action. A partial action of the Hopf algebra H on the algebra A is a linear mapping $\alpha : H \otimes A \rightarrow A$, denoted here by $\alpha(h \otimes a) = h \cdot a$, such that:

- (i) $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$;
- (ii) $1 \cdot a = a$;
- (iii) $h \cdot (g \cdot a) = \sum (h_{(1)} \cdot 1_A)((h_{(2)}g) \cdot a)$.

We will also call A a *partial H module algebra*. It is easy to see that every action is also a partial action.

As a basic example, consider a partial action α of a group G on an unital algebra A . Suppose that each D_g is also a unital algebra, that is, D_g is of the form $D_g = A1_g$; then there is a partial action of the group algebra kG on A which is defined on the basis elements by

$$g \cdot a = \alpha_g(a1_{g^{-1}}), \quad (1)$$

and extended linearly to all elements of kG . In order to see that this action satisfies the relations (i), (ii), and (iii) of the definition of partial action above, let us remember some facts about the partial action α . First, the elements $1_g \in D_g$ are central idempotents in the algebra A and are given by $1_g = g \cdot 1_A$, second, the unity of the ideal $D_g \cap D_h$ is the product $1_g 1_h$ and finally, since $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ and each α_g is an isomorphism, we have $\alpha_g(1_{g^{-1}} 1_h) = 1_g 1_{gh}$. Then, the action (1) satisfies:

$$\begin{aligned} g \cdot (ab) &= \alpha_g(ab1_{g^{-1}})\alpha_g(a1_{g^{-1}}b1_{g^{-1}}) \\ &= \alpha_g(a1_{g^{-1}})\alpha_g(b1_{g^{-1}}) = (g \cdot a)(g \cdot b), \\ e \cdot a &= \alpha_e(a1_{e^{-1}}) = I_A(a1_A) = a, \\ h \cdot (g \cdot a) &= \alpha_h(\alpha_g(a1_{g^{-1}})1_{h^{-1}}) \\ &= \alpha_h(\alpha_g(a1_{g^{-1}})1_g 1_{h^{-1}}) \end{aligned}$$

$$\begin{aligned}
 &= \alpha_h(\alpha_g(a1_{g^{-1}})\alpha_g(1_{g^{-1}}1_{g^{-1}h^{-1}})) \\
 &= \alpha_h(\alpha_g(a1_{g^{-1}}1_{g^{-1}h^{-1}})) \\
 &= \alpha_{hg}(a1_{g^{-1}}1_{g^{-1}h^{-1}}) \\
 &= \alpha_{hg}(a1_{g^{-1}h^{-1}})\alpha_{hg}(1_{g^{-1}}1_{g^{-1}h^{-1}}) \\
 &= \alpha_{hg}(a1_{g^{-1}h^{-1}})1_{hg}1_h = 1_h\alpha_{hg}(a1_{g^{-1}h^{-1}}) \\
 &= \alpha_h(1_A1_{h^{-1}})\alpha_{hg}(a1_{g^{-1}h^{-1}}) = (h \cdot 1_A)(hg \cdot a).
 \end{aligned}$$

Note that we have also proved that $h \cdot (g \cdot a) = (hg \cdot a)1_h = (hg \cdot a)(h \cdot 1_A)$. In general, it is not true that a partial action of kG induces automatically a partial group action of G . We mention that in [4] the authors consider a slight generalization of partial group actions, where the idempotents 1_g are not necessarily central and D_g is the right ideal $D_g = 1_g A$; in this case, it can be proven that there is a bijective correspondence between partial group actions and partial kG -actions on A .

2.2. Induced Partial Actions

There is an important class of examples of partial Hopf actions induced by total actions. This idea is motivated by the construction of a partial group action induced by a global action of a group G on an algebra B by automorphisms.

Let $\beta: G \times B \rightarrow B$ be an action of the group G on the algebra B by automorphisms, and let A be an ideal of B generated by a central idempotent 1_A . Define $D_g = A \cap \beta_g(A)$; then D_g is the ideal generated by the central idempotent $1_g = 1_A \beta_g(1_A)$.

The partial action $\alpha = (\{\alpha_g\}, \{D_g\})$ induced by β on A is

$$\alpha_g(a) = \beta_g(a) \quad \text{for } g \in G \text{ and } a \in D_{g^{-1}}.$$

This corresponds to a partial action of kG on A , given by

$$g \cdot a = \alpha_g(a1_{g^{-1}}).$$

Since

$$\alpha_g(a1_{g^{-1}}) = \beta_g(1_{g^{-1}}a) = \beta_g(1_A \beta_{g^{-1}}(1_A))\beta_g(a) = \beta_g(1_A)1_A \beta_g(a) = 1_A \beta_g(a),$$

one could also define the partial action by $g \cdot a = 1_A \beta_g(a)$ (or $g \cdot a = \beta_g(a)1_A$). This provides the idea for constructing induced partial actions in the Hopf case.

Proposition 1. *Let H be a Hopf algebra which acts on the algebra B , and let A be a right ideal of B with unity 1_A . Then H acts partially on A by*

$$h \cdot a = 1_A(h \triangleright a)$$

Proof. The first property is immediate. For the third, given $h, k \in H$ and $a \in A$,

$$\begin{aligned} h \cdot (k \cdot a) &= 1_A(h \triangleright (1_A(k \triangleright a))) \\ &= 1_A[\sum (h_{(1)} \triangleright 1_A)(h_{(2)} \triangleright (k \triangleright a))] \\ &= \sum 1_A(h_{(1)} \triangleright 1_A)((h_{(2)}k) \triangleright a) = (*) \end{aligned}$$

and since $1_A(h_{(1)} \triangleright 1_A) \in A$, it follows that $1_A(h_{(1)} \triangleright 1_A) = 1_A(h_{(1)} \triangleright 1_A)1_A$; therefore,

$$(*) = \sum 1_A(h_{(1)} \triangleright 1_A)1_A((h_{(2)}k) \triangleright a) = \sum (h_{(1)} \cdot 1_A)((h_{(2)}k) \cdot a).$$

The second property is proved in an analogous manner. \square

We say that the partial action $h \cdot a = 1_A(h \triangleright a)$ is the *partial action induced by B* . We mention again that in [4] the authors introduce a slightly more general concept of partial group action where the domains D_g are already taken as right ideals.

Although this proposition provides a method for constructing examples, it comes as a surprise that, in some cases, the induced partial action is total. As we have seen, every partial group action induces a partial kG action, and it is easy to define partial group actions that are not total actions; on the other hand, every induced partial action by an universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra \mathfrak{g} is total: there are no properly partial actions of Lie algebras. This result was discovered independently by us, in a previous version of this work (see [2]), and by an anonymous referee of [4].

Proposition 2 ([4]). *Let \mathfrak{g} be a Lie algebra over k and let A be a k -algebra. Every partial action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a total action.*

As another example of an induced partial action, let H_4 be the Sweedler 4-dimensional Hopf algebra, with $\beta = \{1, g, x, xg\}$ as basis over the field k , where $\text{char}(k) \neq 2$. The algebra structure is determined by the relations

$$g^2 = 1, \quad x^2 = 0 \quad \text{and} \quad xg = -gx.$$

The coalgebra structure is given by the coproducts

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x,$$

and counit $\epsilon(g) = 1$, $\epsilon(x) = 0$. The antipode S in H_4 reads

$$S(g) = g, \quad \text{and} \quad S(x) = xg.$$

A more suitable basis for the study of ideals of H_4 consists of the vectors $e_1 = (1 + g)/2$, $e_2 = (1 - g)/2$, $h_1 = xe_1$, $h_2 = xe_2$. The multiplication table of H_4 in this

new basis elements reads

	e_1	e_2	h_1	h_2
e_1	e_1	0	0	h_2
e_2	0	e_2	h_1	0
h_1	h_1	0	0	0
h_2	0	h_2	0	0

The expressions for the coproducts of this new basis, are

$$\begin{aligned}\Delta(e_1) &= e_1 \otimes e_1 + e_2 \otimes e_2, \\ \Delta(e_2) &= e_1 \otimes e_2 + e_2 \otimes e_1, \\ \Delta(h_1) &= e_1 \otimes h_1 - e_2 \otimes h_2 + h_1 \otimes e_1 + h_2 \otimes e_2, \\ \Delta(h_2) &= e_1 \otimes h_2 - e_2 \otimes h_1 + h_1 \otimes e_2 + h_2 \otimes e_1.\end{aligned}$$

The counit calculated in the elements of this new basis takes the values $\epsilon(e_1) = 1$ and $\epsilon(e_2) = \epsilon(h_1) = \epsilon(h_2) = 0$. Finally, the antipode is given by

$$S(e_1) = e_1, \quad S(e_2) = e_2, \quad S(h_1) = -h_2, \quad S(h_2) = h_1.$$

The Hopf algebra H_4 acts on itself in the canonical way by the left adjoint action, i.e., $h \triangleright k = \sum_{(h)} h_{(1)} k S(h_{(2)})$. Its action is summed up in the table below.

\triangleright	e_1	e_2	h_1	h_2
e_1	e_1	e_2	0	0
e_2	0	0	h_1	h_2
h_1	$h_1 - h_2$	$h_2 - h_1$	0	0
h_2	0	0	0	0

If $X \subset H_4$, denote by $\langle X \rangle$ the k -subspace generated by X . As one sees directly from the multiplication table, $e_1 H_4 = \langle e_1, h_2 \rangle$ and, since e_1 and h_2 do not commute, we have to kill the latter in order to get a right ideal with unity. But $\langle h_2 \rangle$ is an ideal of H_4 which, given the nature of the action, is also a H_4 -submodule. Hence $\bar{B} = H_4 / \langle h_2 \rangle$ is a H_4 -module algebra. In what follows, we denote $x + \langle h_2 \rangle \in \bar{B}$ by \bar{x} .

The map

$$a\bar{e}_1 + b\bar{e}_2 + c\bar{h}_1 \mapsto \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$$

is an algebra isomorphism. Now, the action of H_4 on \bar{B} is as follows:

\triangleright	\bar{e}_1	\bar{e}_2	\bar{h}_1
e_1	\bar{e}_1	\bar{e}_2	0
e_2	0	0	\bar{h}_1
h_1	\bar{h}_1	$-\bar{h}_1$	0
h_2	0	0	0

The subspace $A = \langle \bar{e}_1 \rangle$ is a right ideal with unity in \bar{B} . Hence, we have a partial action on A induced by the action on \bar{B} . This partial action is given by

$$e_1 \cdot \bar{e}_1 = \bar{e}_1, \quad e_2 \cdot \bar{e}_1 = h_1 \cdot \bar{e}_1 = h_2 \cdot \bar{e}_1 = 0.$$

Once again, it is easy to see that this partial action is in fact total. This happens because the subspace $J = \langle e_2, h_1, h_2 \rangle$ is an ideal of H_4 , and hence an H_4 -submodule of H_4 by the left adjoint action; therefore, H_4/J is an H_4 -module algebra. Since $H_4 = \langle e_1 \rangle \oplus J$ as a vector space, the projection of H_4 onto H_4/J induces an isomorphism of H -module algebras $A \cong H_4/J$. If one looks at the action of H_4 on A , one gets the same table as for the partial action of H_4 on H_4/J (via the natural identification of $e_1 + J$ with $e_1 + \langle h_2 \rangle$).

A truly partial action is obtained from the left action of $(kG)^*$ on kG . We recall that H^* is a Hopf algebra when $\dim H$ is finite, and H^* acts on H on the left by $h^* \rightharpoonup h = \sum h^*(h_{(2)})h_{(1)}$. Let G be a finite group, kG the group algebra, and let $(kG)^*$ act on kG in this manner. If N is a normal subgroup of G , then

$$e_N = \frac{1}{|N|} \sum_{n \in N} n$$

is a central idempotent of kG . Let A be the ideal $e_N kG$ (which is a unital algebra with $1_A = e_N$), and let $\beta^* = \{p_g; g \in G\} \subset (kG)^*$ be the dual basis of the canonical basis of kG . Given $x \in G$ and $p_g \in \beta^*$,

$$\begin{aligned} p_g \rightharpoonup (e_N x) &= \sum_{h \in G} (p_{gh^{-1}} \rightharpoonup e_N)(p_h \rightharpoonup x) \\ &= \sum_{h \in G} (p_{gh^{-1}} \rightharpoonup e_N) p_h(x) x \\ &= (p_{gx^{-1}} \rightharpoonup e_N) x \\ &= \frac{1}{|N|} \sum_{n \in N} p_{gx^{-1}}(n) nx, \end{aligned}$$

which is equal to $(1/|N|)g$ if $gx^{-1} \in N$, and is zero otherwise. Hence, if $gx^{-1} \in N$,

$$p_g \cdot (e_N x) = e_N(p_g \rightharpoonup e_N x) = (1/|N|)e_N g = (1/|N|)e_N x$$

and $p_g \cdot (e_N x) = 0$ otherwise. Therefore, if x_1, \dots, x_m is a complete set of representatives modulo N , then the matrix (a_{ij}) of p_g with respect to the basis $\beta_N = \{e_N x_1, \dots, e_N x_m\}$ of A is a diagonal matrix with all entries but one equal to zero, and this nonzero entry is $a_{ii} = 1/|N|$, where $gx_i^{-1} \in N$. In particular, $p_g \cdot e_N \neq \epsilon(p_g)e_N$ when $g \in N$, and this is a properly partial action.

We shall prove now that every partial action is induced.

2.3. Enveloping Actions

In the context of partial group actions, a natural question arises: under which conditions can a partial action of a group G on an algebra A be obtained, up to

equivalence, from a suitable restriction of a group action of G on an algebra B ? In other words, given a partial action $\alpha = \{\{\alpha_g\}_{g \in G}\{D_g\}_{g \in G}\}$ of G on A , we want to extend the isomorphisms $\alpha_g : D_{g^{-1}} \rightarrow D_g$ to automorphisms $\beta_g : B \rightarrow B$ of an algebra B , such that A is a subalgebra of B (in fact, an ideal) and such that this extension is the smallest possible (more precisely, we impose that $B = \cup_{g \in G} \beta_g(A)$). In this case, the partial action is said to be an *admissible restriction*. We say that an action β of G on B is an enveloping action of a partial action α of G on A if α is equivalent to an admissible restriction of β to an ideal of B .

In the context of partial group actions, it is proved that a partial action α of a group G on a unital algebra A admits an enveloping action if, and only if, each of the ideals $D_g \trianglelefteq A$ is a unital algebra. Moreover, if it exists, this enveloping algebra is unique up to equivalence (see [7, Theorem 4.5]). This is the result we generalize here in the context of partial actions of Hopf algebras.

Definition 1. Let A and B be two partial H -module algebras. We will say that a morphism of algebras $\theta : A \rightarrow B$ is a morphism of partial H -module algebras if $\theta(h \cdot a) = h \cdot \theta(a)$ for all $h \in H$ and all $a \in A$. If θ is an isomorphism, we say that the partial actions are equivalent.

Definition 2. Let B be an H -module algebra, and let A be a right ideal of B with unity 1_A . We will say that the induced partial action on A is admissible if $B = H \triangleright A$.

Definition 3. Let A be a partial H -module algebra. An enveloping action for A is a pair (B, θ) , where

- (1) B is a (not necessarily unital) H -module algebra;
- (2) The map $\theta : A \rightarrow B$ is a monomorphism of algebras;
- (3) The sub-algebra $\theta(A)$ is a right ideal in B ;
- (4) The partial action on A is equivalent to the induced partial action on $\theta(A)$;
- (5) The induced partial action on $\theta(A)$ is admissible.

We will show now that every partial H -action has an enveloping action. In [7], the authors consider the algebra $\mathcal{F}(G, A)$ of functions from G to A . Since there is a canonical algebra isomorphism from $\mathcal{F}(G, A)$ into $\text{Hom}_k(kG, A)$, it is reasonable to consider, in the Hopf case, the algebra $\text{Hom}_k(H, A)$ in place of $\mathcal{F}(G, A)$. We remind the reader that the product in $\text{Hom}_k(H, A)$ is the convolution product $(f * g)(h) = \sum f(h_{(1)})g(h_{(2)})$, and that H acts on this algebra on the left by

$$(h \triangleright f)(k) = f(kh),$$

where $h, k \in H$ and $f \in \text{Hom}_k(H, A)$.

Lemma 1. Let $\varphi : A \rightarrow \text{Hom}_k(H, A)$ be the map given by $\varphi(a)(k) = k \cdot a$.

- (i) φ is a linear injective map and an algebra morphism.
- (ii) If $h \in H$ and $a \in A$ then $\varphi(1_A) * (h \triangleright \varphi(a)) = \varphi(h \cdot a)$.
- (iii) If $h \in H$ and $a, b \in A$ then $\varphi(b) * (h \triangleright \varphi(a)) = \varphi(b(h \cdot a))$.

Proof. It is easy to see that φ is linear, because the partial action is bilinear; since $\varphi(a)(1_H) = 1_H \cdot a = a$, it follows that it is also injective. Take $a, b \in A$ and $h \in H$, then we have

$$\begin{aligned}\varphi(ab)(h) &= h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b) \\ &= \sum \varphi(a)(h_{(1)})\varphi(b)(h_{(2)}) = \varphi(a) * \varphi(b)(h),\end{aligned}$$

for all $h \in H$. Therefore φ is multiplicative.

For the third claim, let $h, k \in H$ and $a, b \in A$; then

$$\begin{aligned}\varphi(b(h \cdot a))(k) &= k \cdot (b(h \cdot a)) = \sum (k_{(1)} \cdot b)(k_{(2)} \cdot (h \cdot a)) \\ &= \sum (k_{(1)} \cdot b)(k_{(2)} \cdot 1_A)(k_{(3)} h \cdot a) \\ &= \sum (k_{(1)} \cdot b)(k_{(2)} h \cdot a) \\ &= \sum \varphi(b)(k_{(1)})\varphi(a)(k_{(2)} h) \\ &= \sum \varphi(b)(k_{(1)})(h \triangleright \varphi(a))(k_{(2)}) \\ &= \varphi(b) * (h \triangleright \varphi(a))(k),\end{aligned}$$

$\forall k \in H$. Therefore, $\varphi(h \cdot a) = \varphi(1_A)(h \triangleright \varphi(a))$. The second item follows from this one, if we put $b = 1_A$. \square

This result suggests that the partial action on A is equivalent to an induced action on $\varphi(A)$, but $\varphi(A)$ must also be a right ideal of an H -module algebra; while this may not hold in $\text{Hom}_k(H, A)$, it will be true in a certain subalgebra.

Lemma 2. *Let B be an H -module algebra, $x, y \in B$ and $h, k \in H$. Then:*

- (i) $(h \triangleright x)y = \sum h_{(1)} \triangleright (x(S(h_{(2)})) \triangleright y)$;
- (ii) $(h \triangleright x)(k \triangleright y) = \sum h_{(1)} \triangleright (x(S(h_{(2)}))k \triangleright y)$.

A proof of (i) can be found in [5, Lemma 6.1.3] and (ii) is a straightforward consequence of (i).

Proposition 3. *Let $\varphi : A \rightarrow \text{Hom}_k(H, A)$ be as above and consider the H -submodule $B = H \triangleright \varphi(A)$.*

- (i) *B is an H -module subalgebra of $\text{Hom}_k(H, A)$.*
- (ii) *$\varphi(A)$ is a right ideal in B with unity $\varphi(1_A)$.*

Proof. (i) Clearly, B is a H -submodule of $\text{Hom}_k(H, A)$. Now, given $h \triangleright \varphi(a)$ and $k \triangleright \varphi(b) \in H \triangleright \varphi(A)$, we have

$$\begin{aligned}(h \triangleright \varphi(a))(k \triangleright \varphi(b)) &= \sum h_{(1)} \triangleright (\varphi(a)(S(h_{(2)}))k \triangleright \varphi(b)) \\ &= \sum h_{(1)} \triangleright \varphi(a(S(h_{(2)}))k \cdot b),\end{aligned}$$

and this shows that B is also a subalgebra.

(ii) This follows by Lemma 1, since $\varphi(b) * (h \triangleright \varphi(a)) = \varphi(b(h \cdot a))$. \square

Lemmas 1, 2 and Proposition 3 prove the existence of enveloping actions.

Theorem 1. *Let A be a partial H -module algebra and let $\varphi : A \rightarrow \text{Hom}_k(H, A)$ be the map given by $\varphi(a)(h) = h \cdot a$, and let $B = H \triangleright \varphi(A)$; then (B, φ) is an enveloping action of A .*

We will call (B, φ) the *standard* enveloping action of A .

A special case which will be useful for further results is the case when $\varphi(A)$ is a bilateral ideal of B . When this occurs, the element $\varphi(1_A)$ becomes automatically a central idempotent in B , and we have also the following result.

Proposition 4. *Let A be a partial H module algebra, and let $\varphi : A \rightarrow \text{Hom}_k(H, A)$ and $B = H \triangleright \varphi(A)$ be as before. Then $\varphi(A) \trianglelefteq B$ if and only if*

$$h \cdot (k \cdot a) = \sum (h_{(1)} k \cdot a) (h_{(2)} \cdot 1_A), \quad \forall a \in A, \quad \forall h, k \in H.$$

Proof. Suppose that $\varphi(A)$ is an ideal of B . We already know that $\forall k \in H$ and $\forall a \in A$, we have

$$\varphi(k \cdot a) = \varphi(1_A) * (k \triangleright \varphi(a)) = (k \triangleright \varphi(a)) * \varphi(1_A).$$

Then, these two functions coincide for all $h \in H$

$$\varphi(k \cdot a)(h) = (k \triangleright \varphi(a)) * \varphi(1_A)(h).$$

The left-hand side of the previous equality leads to

$$\varphi(k \cdot a)(h) = h \cdot (k \cdot a). \quad (2)$$

While the right-hand side gives

$$\begin{aligned} (k \triangleright \varphi(a)) * \varphi(1_A)(h) &= \sum (k \triangleright \varphi(a))(h_{(1)}) * \varphi(1_A)(h_{(2)}) \\ &= \sum \varphi(a)(h_{(1)} k) \varphi(1_A)(h_{(2)}) \\ &= \sum (h_{(1)} k \cdot a) (h_{(2)} \cdot 1_A). \end{aligned} \quad (3)$$

Combining the expressions (2) with (3), we have the result.

Conversely, suppose that $h \cdot (k \cdot a) = \sum (h_{(1)} k \cdot a) (h_{(2)} \cdot 1_A)$ holds for all $a \in A$ and $h, k \in H$. Equations (2) and (3) show that

$$\varphi(1_A)(k \triangleright \varphi(a)) = (k \triangleright \varphi(a)) \varphi(1_A)$$

for every $a \in A$ and $k \in H$, i.e., $\varphi(1_A)$ is a central idempotent in B ; therefore, $\varphi(A) = \varphi(1_A)B$ is an ideal in B . \square

In [7] the authors proved the uniqueness of the enveloping action for a partial action of a group on a unital algebra A . In this case, we have seen that the existence of an enveloping action depends on the fact that every ideal D_g is endowed with an unity 1_g . The idea to prove the uniqueness is to suppose that there exist two algebras B and B' with actions β and β' of the group G , respectively, and embeddings $\varphi : A \rightarrow B$ and $\varphi' : A \rightarrow B'$ such that the partial action of G on A is equivalent to an admissible restriction of both β and β' . Then one defines a map $\phi : B' \rightarrow B$ by $\phi(\beta'_g(\varphi'(a))) = \beta_g(\varphi(a))$. The main difficulty in this theorem is to prove that this map ϕ is well defined as a linear map. This is achieved by two results: first, for each $g \in G$ the subspace $\beta_g(\varphi(A))$ is an ideal with unity in B (the same occurring in B'), and second, the sum of a finite number of ideals with unity is also an ideal with unity.

In the Hopf algebra context, the situation is a bit different. Consider a Hopf algebra H acting partially on a unital algebra A , and let (B, θ) be an enveloping action. By definition of enveloping action, $\theta(A)$ must be a right ideal of B but, unless $h \in H$ is a grouplike element, it is no longer true that the subspaces $h \triangleright \theta(A)$ are right ideals of B ; neither is it true that the elements $h \triangleright \theta(1_A)$ behave as some kind of unity.

In fact, a simple example shows that one does not have uniqueness of the enveloping action unless stronger conditions have been assumed. Let us recall the partial action obtained from the adjoint action of the Sweedler Hopf algebra H_4 on itself. The partial action is constructed when we reduce to the quotient algebra $\bar{B} = H_4 / \langle h_2 \rangle$ and take the residual action restricted to the right ideal $A = \langle e_1 \rangle$. One enveloping action is given by (B, i) , where $B = H_4 \triangleright A$ and $i : A \rightarrow B$ is the inclusion. Note that $H_4 \triangleright A = \langle \bar{e}_1, \bar{h}_1 \rangle$, which is isomorphic to the algebra

$$\begin{bmatrix} k & 0 \\ k & 0 \end{bmatrix}.$$

Hence, $B = \langle \bar{e}_1, \bar{h}_1 \rangle$ is an enveloping algebra of A . Nevertheless, the induced action on A is total, as we have seen, and hence (A, Id_A) is a “smaller” enveloping action.

In order to clarify these matters we are going to relate each enveloping action of A with the standard enveloping action (B, φ) given in Theorem 1. Suppose that (B', θ) is an enveloping action for A . Define the map $\Phi : B' \rightarrow \text{Hom}_k(H, A)$ by

$$\Phi\left(\sum_{i=1}^n h_i \triangleright \theta(a_i)\right) = \sum_{i=1}^n h_i \triangleright \varphi(a_i),$$

where $\varphi : H \rightarrow A$ and the H -action on $\text{Hom}_k(H, A)$ are the same as before.

Theorem 2. *If (B', θ) is an enveloping action of A , then the map Φ is an H -module algebra morphism onto $B = H \triangleright \varphi(A)$.*

Proof. First we have to check that Φ is well defined as linear map. In order to do this, it is enough to prove that if $\sum_{i=1}^n h_i \triangleright \theta(a_i) = 0$, then $\sum_{i=1}^n h_i \triangleright \varphi(a_i) = 0$.

So, suppose that $x = \sum_{i=1}^n h_i \triangleright \theta(a_i) = 0$; then, for all $k \in H$,

$$0 = \theta(1_A)(k \triangleright x) = \theta(1_A) \sum_{i=1}^n k h_i \triangleright \theta(a_i) = \sum_{i=1}^n k h_i \cdot \theta(a_i) = \theta \left(\sum_{i=1}^n k h_i \cdot a_i \right)$$

and, since θ is injective, it follows that

$$\sum_{i=1}^n k h_i \cdot a_i = 0$$

for all $k \in H$. Hence

$$\Phi(x)(k) = \sum_{i=1}^n h_i \triangleright \varphi(a_i)(k) = \sum_{i=1}^n \varphi(a_i)(k h_i) = \sum_{i=1}^n k h_i \cdot a_i = 0$$

for all $k \in H$, which means that $\Phi(x) = 0$ and that Φ is well defined.

Φ is linear by construction; we have to show that it is an algebra morphism. Given $h, k \in H$ and $a, b \in A$,

$$\begin{aligned} \Phi((h \triangleright \theta(a))(k \triangleright \theta(b))) &= \Phi \left(\sum h_{(1)} \triangleright (\theta(a)(S(h_{(2)}k \triangleright \theta(b)))) \right) \\ &= \Phi \left(\sum h_{(1)} \triangleright \theta(a(S(h_{(2)}k \cdot b)) \right) \\ &= \sum h_{(1)} \triangleright \varphi(a(S(h_{(2)}k \cdot b)) \\ &= \sum h_{(1)} \triangleright (\varphi(a) * (S(h_{(2)}k \triangleright \varphi(b))) \\ &= \sum (h_{(1)} \triangleright \varphi(a)) * (h_{(2)} \triangleright (S(h_{(3)}k \triangleright \varphi(b))) \\ &= \sum (h_{(1)} \triangleright \varphi(a)) * (h_{(2)}S(h_{(3)}k \triangleright \varphi(b)) \\ &= (h \triangleright \varphi(a)) * (k \triangleright \varphi(b)) \\ &= \Phi(h \triangleright \theta(a)) * \Phi(k \triangleright \theta(b)). \end{aligned}$$

The fact that Φ is a morphism of left H modules is easily obtained by the definition of this map. And since

$$\sum_{i=1}^n h_i \triangleright \varphi(a_i) = \Phi \left(\sum_{i=1}^n h_i \triangleright \theta(a_i) \right),$$

Φ is a surjective map from B' onto $B = H \triangleright \varphi(A)$. □

Now, *injectivity* of Φ will follow only if whenever $\sum_{i=1}^n k h_i \cdot a_i = 0$ for all $k \in H$, then $\sum_{i=1}^n h_i \triangleright \theta(a_i) = 0$. This motivates the following definition.

Definition 4. Let A be a partial H -module algebra. An enveloping action (B, θ) of A is minimal if, for every H -submodule M of B , $\theta(1_A)M = 0$ implies $M = 0$.

This concept does not appear in the theory of partial group actions because, in this case, *every enveloping action is minimal*. In fact, consider $H = kG$, B a H -module

algebra, A a right ideal with unity in B such that $B = kG \triangleright A$. Suppose that $\sum_i g_i \triangleright a_i \in B$ satisfies

$$h \triangleright \left(\sum_i g_i \triangleright a_i \right) = \sum_i h g_i \triangleright a_i = 0, \quad \forall h \in G.$$

Let $(h \triangleright b) \in B$, with $h \in G$. Then, by Lemma 2

$$(h \triangleright b) \left(\sum_i g_i \triangleright a_i \right) = h \triangleright \left(b \left(\sum_i (h^{-1} g_i \cdot a_i) \right) \right) = 0.$$

By the results of Exel and Dokuchaev [7], the ideal $J = (g_1 \triangleright A) + \cdots + (g_n \triangleright A)$ has a unity; since $B(\sum_i g_i \triangleright a_i) = 0$, we conclude that $\sum_i g_i \triangleright a_i = 0$.

On the other hand, in the case of the action of H_4 on $\bar{B} = H_4 / \langle h_2 \rangle$, we had $A = \langle e_1 \rangle$, $B = H_4 \triangleright A = \langle e_1, h_1 \rangle$, and $h_1 = h_1 \triangleright e_1$; then $\Phi(B) = \varphi(A)$, because $\Phi(h_1) = h_1 \triangleright \varphi(e_1)$ is the zero function. This shows that an enveloping action does not need to be minimal. However, regardless of A or H , the standard enveloping action is always minimal.

Lemma 3. *Let $\varphi : A \rightarrow \text{Hom}_k(H, A)$ be as above and consider the H -submodule $B = H \triangleright \varphi(A)$. Then, (B, φ) is a minimal enveloping action of A .*

Proof. It is enough to check that the minimality condition holds for cyclic submodules. Let $M = H \triangleright (\sum_{i=1}^n h_i \triangleright \varphi(a_i))$, and suppose that $\varphi(1_A) * M = 0$. This means that

$$0 = \varphi(1_A) * \left(\sum_{i=1}^n k h_i \triangleright \varphi(a_i) \right) = \sum_{i=1}^n \varphi(k h_i \cdot a_i) = \varphi \left(\sum_{i=1}^n k h_i \cdot a_i \right)$$

for each $k \in H$. Since φ is injective, $\sum_{i=1}^n k h_i \cdot a_i = 0$ for every $k \in H$. But then

$$\sum_{i=1}^n h_i \triangleright \varphi(a_i)(k) = \sum_{i=1}^n \varphi(a_i)(k h_i) = \sum_{i=1}^n \varphi(k h_i \cdot a_i) = 0$$

for each $k \in H$, and we conclude that $\sum_{i=1}^n h_i \triangleright \varphi(a_i) = 0$. \square

By Theorem 2 and Lemma 3, we conclude with the following theorem.

Theorem 3. *Every partial H -module algebra has a minimal enveloping action, and any two minimal enveloping actions of A are isomorphic as H -module algebras. Moreover, if (B', θ) is an enveloping action, then there is a morphism of H -module algebras of B' onto a minimal enveloping action.*

Proof. Since the standard enveloping action (B, φ) is a minimal action and $\Phi : B' \rightarrow \text{Hom}_k(H, A)$ is a H -module algebra isomorphism onto B , we just have to prove that it is injective if (B', θ) is minimal. If $\Phi(\sum_{i=1}^n h_i \triangleright \theta(a_i)) = 0$, then

$$\sum_{i=1}^n h_i \triangleright \varphi(a_i)(k) = \sum_{i=1}^n k h_i \cdot a_i = 0$$

for each $k \in H$, and hence $0 = \theta(\sum_{i=1}^n kh_i \cdot a_i) = \theta(1_A)[k \triangleright (\sum_{i=1}^n h_i \triangleright \theta(a_i))]$ for every $k \in H$; hence, $\theta(1_A)M = 0$ for $M = H \triangleright (\sum_{i=1}^n h_i \triangleright \theta(a_i)) = 0$, implying that $M = 0$ and hence that $\sum_{i=1}^n h_i \triangleright \theta(a_i) = 0$. \square

3. A MORITA CONTEXT

In the reference [7], the authors showed that if a partial group action α of G on a unital algebra A admits an enveloping action β of the same group on an algebra B , then the partial crossed product $A \rtimes_{\alpha} G$ is Morita equivalent to the crossed product $B \rtimes_{\beta} G$. They proved this result by constructing a Morita context between these two crossed products.

We recall that a Morita context is a six-tuple $(R, S, M, N, \tau, \sigma)$ where:

- a) R and S are rings;
- b) M is an $R - S$ bimodule;
- c) N is an $S - R$ bimodule;
- d) $\tau : M \otimes_S N \rightarrow R$ is a bimodule morphism;
- e) $\sigma : N \otimes_R M \rightarrow S$ is a bimodule morphism,

such that

$$\tau(m_1 \otimes n)m_2 = m_1\sigma(n \otimes m_2), \quad \forall m_1, m_2 \in M, \quad \forall n \in N, \quad (4)$$

and

$$\sigma(n_1 \otimes m)n_2 = n_1\tau(m \otimes n_2), \quad \forall m \in M, \quad \forall n_1, n_2 \in N. \quad (5)$$

It is well-known that if the morphisms τ and σ are surjective, then the categories ${}_R\text{Mod}$ and ${}_S\text{Mod}$ are equivalent (in this case, one says that R and S are *Morita equivalent*).

In the Hopf algebra case, we shall see that when a Hopf algebra H acts partially on a unital algebra A and the enveloping action (B, θ) is such that $\theta(A)$ is an ideal of B , then the partial smash product $\underline{A\#H}$ is Morita equivalent to the smash product $B\#H$. Just remembering [4], the partial smash product is defined as follows: $A \otimes H$ is a (possibly nonunital) algebra with the product

$$(a \otimes h)(b \otimes k) = \sum a(h_{(1)} \cdot b) \otimes h_{(2)}k,$$

and the partial smash product is the unital subalgebra of $A \otimes H$ given by

$$\begin{aligned} \underline{A\#H} &= (1_A \otimes 1_H)(A \otimes H)(1_A \otimes 1_H) \\ &= (A \otimes H)(1_A \otimes 1_H), \end{aligned}$$

that is, the subalgebra generated by elements of the form

$$x = \sum a(h_{(1)} \cdot 1_A) \otimes h_{(2)}, \quad \forall a \in A, \quad \forall h \in H.$$

Although the definition is quite different from that of the partial crossed product $A \rtimes_{\alpha} G$, it can be proved that $A \rtimes_{\alpha} G$ is isomorphic to $\underline{A\#kG}$.

Our aim in this section is to construct a Morita context between the partial smash product $\underline{A\#H}$ and the smash product $B\#H$, where B is an enveloping action for the partial left action $\cdot : H \otimes A \rightarrow A$. For this purpose, we will embed the partial smash product into the smash product.

Lemma 4. *If a Hopf algebra H acts partially on a unital algebra A and (B, θ) is an enveloping action, then there is an algebra monomorphism from the partial smash product $\underline{A\#H}$ into the smash product $B\#H$.*

Proof. Define first a map

$$\begin{aligned}\tilde{\Phi} : A \times H &\rightarrow B\#H. \\ (a, h) &\mapsto \theta(a)\#h\end{aligned}$$

It is easy to see that $\tilde{\Phi}$ is a bilinear map and then, by the universal property of the tensor product $A \otimes H$, $\tilde{\Phi}$ induces the k -linear map

$$\begin{aligned}\Phi : A \otimes H &\rightarrow B\#H. \\ a \otimes h &\mapsto \theta(a)\#h\end{aligned}$$

Now, let us check that Φ is a morphism of algebras:

$$\begin{aligned}\Phi((a \otimes h)(b \otimes k)) &= \Phi\left(\sum a(h_{(1)} \cdot b) \otimes h_{(2)}k\right) \\ &= \sum \theta(a(h_{(1)} \cdot b))\#h_{(2)}k \\ &= \sum \theta(a)(h_{(1)} \triangleright \theta(b))\#h_{(2)}k \\ &= (\theta(a)\#h)(\theta(b)\#k) = \Phi(a \otimes h)\Phi(b \otimes k).\end{aligned}$$

Next, we must verify that Φ is injective. For this, take

$$x = \sum_{i=1}^n a_i \otimes h_i \in \ker \Phi,$$

and, without loss of generality, choose $\{a_i\}_{i=1}^n$ to be linearly independent. As θ is injective, we conclude that the elements $\theta(a_i) \in B$ are linearly independent. For each $f \in H^*$, we have

$$\sum_{i=1}^n \theta(a_i)f(h_i) = 0,$$

and then, by the linear independence of $\{\theta(a_i)\}$, we get $f(h_i) = 0$, $\forall f \in H^*$. Therefore, $h_i = 0$ for $i = 1, \dots, n$, which implies that $x = 0$ and that Φ is injective.

As the partial smash product $\underline{A\#H}$ is a subalgebra of $A \otimes H$, then, it is injectively mapped into $B\#H$ by Φ . A typical element of the image of the partial smash product is

$$\Phi((a \otimes h)(1_A \otimes 1_H)) = \Phi(a \otimes h)\Phi(1_A \otimes 1_H)$$

$$\begin{aligned}
 &= (\theta(a)\#h)(\theta(1_A)\#1_H) \\
 &= \sum \theta(a)(h_{(1)} \triangleright \theta(1_A))\#h_{(2)}. \quad \square
 \end{aligned}$$

Take $M = \Phi(A \otimes H) = \{\sum_{i=1}^n \theta(a_i)\#h_i; a_i \in A, n \in \mathbb{N}\}$, and take N as the subspace of $B\#H$ generated by the elements $\sum(h_{(1)} \triangleright \theta(a))\#h_{(2)}$, with $h \in H$ and $a \in A$, i.e., $N = (1_A \otimes H)\Phi(A \otimes 1)$. We will take H as a Hopf algebra with invertible antipode (which means that H is Co-Frobenius as a coalgebra, see for example [5, Section 5.4]).

Proposition 5. *Let H be a Hopf algebra with invertible antipode, A a partial H -module algebra, and suppose that $\theta(A)$ is an ideal of B ; then M is a right $B\#H$ module and N is a left $B\#H$ module.*

Proof. In order to see that M is a right $B\#H$ module, let $\theta(a)\#h \in M$ and $b\#k \in B\#H$. Then

$$(\theta(a)\#h)(b\#k) = \sum \theta(a)(h_{(1)} \triangleright b)\#h_{(2)}k,$$

which lies in $\Phi(A \otimes H)$ because $\theta(A)$ is a right ideal in B .

Now, to prove that N is a left $B\#H$ module, let $\sum(h_{(1)} \triangleright \theta(a))\#h_{(2)}$ be a generator of N :

$$\begin{aligned}
 &(b\#k)\left(\sum(h_{(1)} \triangleright \theta(a))\#h_{(2)}\right) \\
 &= \sum b(k_{(1)}h_{(1)} \triangleright \theta(a))\#k_{(2)}h_{(2)} \\
 &= \sum b(\epsilon(k_{(1)}h_{(1)})k_{(2)}h_{(2)} \triangleright \theta(a))\#k_{(3)}h_{(3)} \\
 &= \sum (\epsilon(k_{(1)}h_{(1)})1_H \triangleright b)(k_{(2)}h_{(2)} \triangleright \theta(a))\#k_{(3)}h_{(3)} \\
 &= \sum ((k_{(2)}h_{(2)}S^{-1}(k_{(1)}h_{(1)})) \triangleright b)(k_{(3)}h_{(3)} \triangleright \theta(a))\#k_{(4)}h_{(4)} \\
 &= \sum (k_{(2)}h_{(2)} \triangleright (S^{-1}(k_{(1)}h_{(1)}) \triangleright b))(k_{(3)}h_{(3)} \triangleright \theta(a))\#k_{(4)}h_{(4)} \\
 &= \sum (k_{(2)}h_{(2)} \triangleright ((S^{-1}(k_{(1)}h_{(1)}) \triangleright b)\theta(a)))\#k_{(3)}h_{(3)}.
 \end{aligned}$$

Each term $(S^{-1}(k_{(1)}h_{(1)}) \triangleright b)\theta(a)$ is in $\theta(A)$ because $\theta(A)$ is an ideal of B . It follows that N is a left $B\#H$ ideal. \square

We can define a left $\underline{A\#H}$ module structure on M and a right $\underline{A\#H}$ module structure on N induced by the monomorphism Φ , that is,

$$\left(\sum a(h_{(1)} \cdot 1_A) \otimes h_{(2)}\right) \blacktriangleright (\theta(b)\#k) = \left(\sum \theta(a)(h_{(1)} \triangleright \theta(1_A))\#h_{(2)}\right)(\theta(b)\#k),$$

and

$$\begin{aligned}
 &\left(\sum k_{(1)} \triangleright \theta(b)\#k_{(2)}\right) \blacktriangleleft \left(\sum a(h_{(1)} \cdot 1_A) \otimes h_{(2)}\right) \\
 &= \left(\sum k_{(1)} \triangleright \theta(b)\#k_{(2)}\right)\left(\sum \theta(a)(h_{(1)} \triangleright \theta(1_A))\#h_{(2)}\right).
 \end{aligned}$$

Proposition 6. *Under the same hypotheses of the previous proposition, M is indeed a left $\underline{A\#H}$ module with the map \blacktriangleright , and N is indeed a right $\underline{A\#H}$ module with the map \blacktriangleleft .*

Proof. Let us first verify that $\underline{A\#H} \blacktriangleright M \subseteq M$.

$$\begin{aligned} \left(\sum a(h_{(1)} \cdot 1_A) \otimes h_{(2)} \right) \blacktriangleright (\theta(b)\#k) &= \left(\sum \theta(a)(h_{(1)} \triangleright \theta(1_A))\#h_{(2)} \right) (\theta(b)\#k) \\ &= \sum \theta(a)(h_{(1)} \cdot \theta(1_A))(h_{(2)} \triangleright \theta(b))\#h_{(3)}k \\ &= \sum \theta(a)(h_{(1)} \cdot \theta(1_A))(h_{(2)} \cdot \theta(b))\#h_{(3)}k \\ &= \sum \theta(a)(h_{(1)} \cdot \theta(1_A)\theta(b))\#h_{(2)}k \\ &= \sum \theta(a)(h_{(1)} \cdot \theta(b))\#h_{(2)}k, \end{aligned}$$

which lies inside M because $\theta(A)$ is a right ideal of B .

Now, we have to verify that $N \blacktriangleleft \underline{A\#H} \subseteq N$:

$$\begin{aligned} \left(\sum k_{(1)} \triangleright \theta(b)\#k_{(2)} \right) \blacktriangleleft \left(\sum a(h_{(1)} \cdot 1_A) \otimes h_{(2)} \right) &= \left(\sum k_{(1)} \triangleright \theta(b)\#k_{(2)} \right) \left(\sum \theta(a)(h_{(1)} \triangleright \theta(1_A))\#h_{(2)} \right) \\ &= \sum (k_{(1)} \triangleright \theta(b))(k_{(2)} \triangleright (\theta(a)(h_{(1)} \triangleright \theta(1_A)))\#k_{(3)}h_{(2)}) \\ &= \sum (k_{(1)} \triangleright \theta(b))(k_{(2)} \triangleright \theta(a))(k_{(3)} \triangleright (h_{(1)} \triangleright \theta(1_A)))\#k_{(4)}h_{(2)} \\ &= \sum (k_{(1)} \triangleright \theta(ba))(k_{(2)}h_{(1)} \triangleright \theta(1_A))\#k_{(3)}h_{(2)} \\ &= \sum (k_{(1)}\epsilon(h_{(1)}) \triangleright \theta(ba))(k_{(2)}h_{(2)} \triangleright \theta(1_A))\#k_{(3)}h_{(3)} \\ &= \sum (k_{(1)}h_{(2)}S^{-1}(h_{(1)}) \triangleright \theta(ba))(k_{(2)}h_{(3)} \triangleright \theta(1_A))\#k_{(3)}h_{(4)} \\ &= \sum (k_{(1)}h_{(2)} \triangleright (S^{-1}(h_{(1)}) \triangleright \theta(ba)))(k_{(2)}h_{(3)} \triangleright \theta(1_A))\#k_{(3)}h_{(4)} \\ &= \sum k_{(1)}h_{(2)} \triangleright ((S^{-1}(h_{(1)}) \triangleright \theta(ba))\theta(1_A))\#k_{(2)}h_{(3)} \\ &= \sum k_{(1)}h_{(2)} \triangleright \theta(S^{-1}(h_{(1)}) \cdot (ba))\#k_{(2)}h_{(3)} \\ &= \sum (kh_{(2)})_{(1)} \triangleright \theta(S^{-1}(h_{(1)}) \cdot (ba))\#(kh_{(2)})_{(2)}, \end{aligned}$$

where in the last two lines we used the fact that

$$\theta(t \cdot x) = \theta(1_A)(t \triangleright \theta(x)) = (t \triangleright \theta(x))\theta(1_A),$$

which holds because $\theta(1_A)$ is a central idempotent. \square

The last ingredient for a Morita context is the definition of two bimodule morphisms

$$\tau : M \otimes_{B\#H} N \rightarrow \underline{A\#H} \cong \Phi(\underline{A\#H}) \subseteq B\#H$$

and

$$\sigma : N \otimes_{\underline{A\#H}} M \rightarrow B\#H.$$

As M , N , and $\underline{A\#H}$ are viewed as subalgebras of $B\#H$, these two maps can be taken as the usual multiplication on $B\#H$. The associativity of the product assures us that these maps are bimodule morphisms and satisfy the associativity conditions (4) and (5). The following theorem shows us that the maps τ and σ are indeed well defined, and furthermore, that they are surjective, proving the Morita equivalence between these two smash products.

Theorem 4. *Let H be a Hopf algebra with invertible antipode, A a partial H -module algebra, (B, θ) a unital enveloping action, and suppose that $\theta(A)$ is an ideal of B ; let M and N be the bimodules defined above. Then $(\underline{A\#H}, B\#H, M, N, \tau, \sigma)$ is a strict Morita context.*

Proof. We have already shown that $(\underline{A\#H}, B\#H, M, N, \tau, \sigma)$ is a Morita context. We still have to show that τ and σ are surjective, or, equivalently, $MN = \Phi(\underline{A\#H})$ and $NM = B\#H$. Let us see first that $MN \subseteq \Phi(\underline{A\#H})$:

$$\begin{aligned} & (\theta(a)\#h)\left(\sum(k_{(1)} \triangleright \theta(b))\#k_{(2)}\right) \\ &= \sum \theta(a)h_{(1)} \triangleright (k_{(1)} \triangleright \theta(b))\#h_{(2)}k_{(2)} \\ &= \sum \theta(a)(h_{(1)}k_{(1)} \triangleright \theta(b))\#h_{(2)}k_{(2)} \\ &= \sum \theta(a)(h_{(1)}k_{(1)} \triangleright \theta(b))(h_{(2)}k_{(2)} \triangleright \theta(1_A))\#h_{(3)}k_{(3)} \\ &= \sum \theta(a(h_{(1)}k_{(1)} \cdot b))((h_{(2)}k_{(2)})_{(1)} \triangleright \theta(1_A))\#(h_{(2)}k_{(2)})_{(2)}, \end{aligned}$$

which is an element of $\Phi(\underline{A\#H})$.

Since

$$\sum \theta(a)(h_{(1)} \triangleright \theta(1_A))\#h_{(2)} = (\theta(a)\#h)(\theta(1_A)\#1_H)$$

and $\theta(1_A)\#1_H \in N$, it follows that $MN = \Phi(\underline{A\#H})$.

In order to prove $NM = B\#H$, we just have to show that every element of the form $(h \triangleright \theta(a))\#k$ is in NM , because this is a generating set for $B\#H$ as a vector space. We claim that

$$(h \triangleright \theta(a))\#k = \sum((h_{(1)} \triangleright \theta(a))\#h_{(2)})(\theta(1_A)\#S(h_{(3)})k).$$

This can be easily seen as follows:

$$\begin{aligned} & \sum((h_{(1)} \triangleright \theta(a))\#h_{(2)})(\theta(1_A)\#S(h_{(3)})k) \\ &= \sum((h_{(1)} \triangleright \theta(a))(h_{(2)} \triangleright \theta(1_A))\#h_{(3)}S(h_{(4)})k) \\ &= \sum((h_{(1)} \triangleright \theta(a)\theta(1_A))\#\epsilon(h_{(2)})k) \end{aligned}$$

$$\begin{aligned}
&= \left(\left(\sum h_{(1)} \epsilon(h_{(2)}) \right) \triangleright \theta(a) \right) \# k \\
&= (h \triangleright \theta(a)) \# k.
\end{aligned}$$

Therefore, $NM = B \# H$. \square

In conclusion, we have constructed a Morita context for the two smash products and proved that this Morita context gives us a Morita equivalence between these two algebras.

4. ENVELOPING COACTIONS

Following [4], we define a partial right coaction of a Hopf algebra H on a algebra A to be a linear map $\bar{\rho} : A \rightarrow A \otimes H$ such that

$$\begin{aligned}
1) \quad & \bar{\rho}(ab) = \bar{\rho}(a)\bar{\rho}(b), \quad \forall a, b \in A, \\
2) \quad & (I \otimes \epsilon)\bar{\rho}(a) = a, \quad \forall a \in A, \\
3) \quad & (\bar{\rho} \otimes I)\bar{\rho}(a) = (\bar{\rho}(1_A) \otimes 1_H)((I \otimes \Delta)\bar{\rho}(a)), \quad \forall a \in A.
\end{aligned} \tag{6}$$

We will denote

$$\bar{\rho}(a) = \sum a^{(0)} \otimes a^{(1)}.$$

Note that, in this notation, we can rewrite the items 1), 2), and 3) of (6) as

$$\begin{aligned}
1) \quad & \sum (ab)^{(0)} \otimes (ab)^{(1)} = a^{(0)}b^{(0)} \otimes a^{(1)}b^{(1)}, \\
2) \quad & \sum a^{(0)}\epsilon(a^{(1)}) = a, \\
3) \quad & \sum a^{(0)(0)} \otimes a^{(0)(1)} \otimes a^{(1)} = \sum 1_A^{(0)}a^{(0)} \otimes 1_A^{(1)}a^{(1)}_{(1)} \otimes a^{(1)}_{(2)}.
\end{aligned}$$

The simplest example of a partial coaction can be given as a restriction of a coaction of H on a right H comodule algebra B .

Proposition 7. *Let B be a right H -comodule algebra with coaction $\rho : B \rightarrow B \otimes H$. If A is a right ideal of B with a unity 1_A , then the map*

$$\bar{\rho}(a) = (1_A \otimes 1_H)\rho(a) = \sum 1_A a^{(0)} \otimes a^{(1)}$$

defines a partial right coaction of H on A .

Proof. For the first item of (6) we have, for all $a \in A$,

$$\begin{aligned}
\bar{\rho}(ab) &= (1_A \otimes 1_H)(\rho(ab)) = (1_A \otimes 1_H)(\rho(a)\rho(b)) \\
&= \left(\sum 1_A a^{(0)} \otimes a^{(1)} \right) \left(\sum b^{(0)} \otimes b^{(1)} \right) \\
&= \sum 1_A a^{(0)}b^{(0)} \otimes a^{(1)}b^{(1)}
\end{aligned}$$

$$\begin{aligned}
 &= \sum 1_A a^{(0)} 1_A b^{(0)} \otimes a^{(1)} b^{(1)} \\
 &= \left(\sum 1_A a^{(0)} \otimes a^{(1)} \right) \left(\sum 1_A b^{(0)} \otimes b^{(1)} \right) \\
 &= \bar{\rho}(a) \bar{\rho}(b),
 \end{aligned}$$

where we used $1_A b = 1_A b 1_A \in A$ in the fourth line.

Item 2) is easily established for every $a \in A$,

$$(I \otimes \epsilon) \bar{\rho}(a) = \sum 1_A a^{(0)} \epsilon(a^{(1)}) = 1_A a = a.$$

Finally, checking item 3), we have

$$\begin{aligned}
 (\bar{\rho} \otimes I) \bar{\rho}(a) &= \sum (1_A \otimes 1_H \otimes I) (\rho \otimes I) (1_A \otimes 1_H) (a^{(0)} \otimes a^{(1)}) \\
 &= \sum (1_A \otimes 1_H \otimes 1_H) ((1_A a)^{(0)} \otimes (1_A a)^{(1)} \otimes a^{(2)}) \\
 &= \sum (1_A \otimes 1_H \otimes 1_H) (1_A^{(0)} a^{(0)(0)} \otimes 1_A^{(1)} a^{(0)(1)} \otimes a^{(1)}) \\
 &= (1_A \otimes 1_H \otimes 1_H) (\rho(1_A) \otimes 1_H) ((\rho \otimes I) \left(\sum a^{(0)} \otimes a^{(1)} \right)) \\
 &= (1_A 1_A^{(0)} 1_A \otimes 1_A^{(1)} 1_H \otimes 1_H) ((I \otimes \Delta) \left(\sum a^{(0)} \otimes a^{(1)} \right)) \\
 &= (1_A \otimes 1_H \otimes 1_H) (\rho(1_A) \otimes 1_H) ((I \otimes \Delta) ((1_A \otimes 1_H) \rho(a))) \\
 &= (\bar{\rho}(1_A) \otimes I) (I \otimes \Delta) \bar{\rho}(a).
 \end{aligned}$$

Therefore, $\bar{\rho}$ is a partial coaction. \square

In [12] the author proved that if a finite dimensional Hopf algebra H acts partially on a unital algebra A , then there exists a partial coaction of the dual Hopf algebra H^* on A by the coaction

$$\bar{\rho}(a) = \sum_{i=1}^n (b_i \cdot a) \otimes p_i,$$

where $\{b_i\}_{i=1, \dots, n}$ is a basis for H and $\{p_i\}_{i=1, \dots, n}$ is its dual basis in H^* . In fact, one can push forward this result to a more general context. We recall the definition of a pairing of Hopf algebras.

Definition 5. A pairing between two Hopf algebras H_1 and H_2 is a linear map

$$\begin{aligned}
 \langle, \rangle : H_1 \otimes H_2 &\rightarrow k \\
 h \otimes f &\mapsto \langle h, f \rangle
 \end{aligned}$$

such that:

- (i) $\langle hk, f \rangle = \langle h \otimes k, \Delta(f) \rangle$;
- (ii) $\langle h, fg \rangle = \langle \Delta(h), f \otimes g \rangle$;
- (iii) $\langle h, 1_{H_2} \rangle = \epsilon(h)$;
- (iv) $\langle 1_{H_1}, f \rangle = \epsilon(f)$.

A pairing is said to be nondegenerate if the following conditions hold:

- (1) If $\langle h, f \rangle = 0, \forall f \in H_2$ then $h = 0$;
- (2) If $\langle h, f \rangle = 0, \forall h \in H_1$ then $f = 0$.

Let H_1 and H_2 be two Hopf algebras dually paired with a nondegenerate pairing, and suppose also that H_1 acts partially on an algebra A in such a way that for all $a \in A$ the subspace $H_1 \cdot a$ has finite dimension (this is a requirement that is analogous to the case of rational modules). Then we have the following result.

Theorem 5. *Let H_1 and H_2 be two Hopf algebras dually paired with a nondegenerate pairing, and suppose that H_1 acts partially on an algebra A in such a manner that $\dim(H_1 \cdot a) < \infty$ for all $a \in A$. Then H_2 coacts partially on A with a coaction $\bar{\rho} : A \rightarrow A \otimes H_2$ defined by*

$$(I \otimes h)\bar{\rho}(a) = h \cdot a, \quad \forall h \in H_1,$$

where $I \otimes h : A \otimes H_2 \rightarrow A$ is given by

$$(I \otimes h) \left(\sum_{i=1}^n a_i \otimes f_i \right) = \sum_{i=1}^n a_i \langle h, f_i \rangle.$$

Proof. The condition that $\dim(H_1 \cdot a) < \infty, \forall a \in A$, implies that given $a \in A$, there exist elements $a_1, \dots, a_n \in A$ and $\lambda_1, \dots, \lambda_n \in H_1^*$ such that

$$h \cdot a = \sum_{i=1}^n \lambda_i(h) a_i,$$

where we may choose the elements a_i linearly independent. The map $T_a : H \rightarrow A$, given by $T_a(h) = h \cdot a$, has a kernel of codimension n , and there is an n -dimensional subspace V_a of H_1 such that $H_1 = V_a \oplus \ker(T_a)$. Choose a basis $\{h_1, \dots, h_n\}$ of V_a such that $T_a(h_i) = a_i$ for each i ; since the pairing is nondegenerate, there are $f_1, \dots, f_n \in H_2$ such that $\langle h_i, f_j \rangle = \delta_{i,j}$. It follows that for all $h \in H_1$, we have

$$h \cdot a = \sum_{i=1}^n \langle h, f_i \rangle a_i.$$

Define then $\bar{\rho} : A \rightarrow A \otimes H_2$ as

$$\bar{\rho}(a) = \sum_{i=1}^n a_i \otimes f_i.$$

For each $h \in H_1$,

$$(I \otimes h)\bar{\rho}(a) = \sum_{i=1}^n a_i \langle h, f_i \rangle = h \cdot a.$$

We have to verify itens 1), 2), and 3) of (6). For the first item, let $a, b \in A$ such that

$$\bar{\rho}(a) = \sum_{i=1}^n a_i \otimes f_i, \quad \bar{\rho}(b) = \sum_{j=1}^m b_j \otimes g_j.$$

Then, for every $h \in H_1$, we have

$$\begin{aligned} (I \otimes h)\bar{\rho}(ab) &= h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b) \\ &= \sum_{(h)} \left(\sum_{i=1}^n a_i \langle h_{(1)}, f_i \rangle \right) \left(\sum_{j=1}^m b_j \langle h_{(2)}, g_j \rangle \right) \\ &= \sum_{(h)} \sum_{i=1}^n \sum_{j=1}^m a_i b_j \langle h_{(1)}, f_i \rangle \langle h_{(2)}, g_j \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \langle h, f_i g_j \rangle \\ &= (I \otimes h) \sum_{i=1}^n \sum_{j=1}^m a_i b_j \otimes f_i g_j \\ &= (I \otimes h) \bar{\rho}(a) \bar{\rho}(b). \end{aligned}$$

As this equality is true for all $h \in H_1$ and the pairing is nondegenerate, then we can conclude that

$$\bar{\rho}(ab) = \bar{\rho}(a) \bar{\rho}(b).$$

For the second item, we use the equality involving the pairing

$$\epsilon(f) = \langle 1_{H_1}, f \rangle,$$

then, $\forall a \in A$,

$$(I \otimes \epsilon)\bar{\rho}(a) = \sum_{i=1}^n a_i \langle 1_{H_1}, f_i \rangle = 1_{H_1} \cdot a = a.$$

Finally, for item 3), given any $h, k \in H_1$, we have

$$\begin{aligned} (I \otimes h \otimes k)(\bar{\rho} \otimes I)\bar{\rho}(a) &= (I \otimes h \otimes k) \sum_{i=1}^n \bar{\rho}(a_i) \otimes f_i \\ &= \sum_{i=1}^n (I \otimes h) \bar{\rho}(a_i) \langle k, f_i \rangle \\ &= \sum_{i=1}^n (h \cdot a_i) \langle k, f_i \rangle = h \cdot \left(\sum_{i=1}^n a_i \langle k, f_i \rangle \right) = h \cdot (k \cdot a) \end{aligned}$$

$$\begin{aligned}
&= \sum (h_{(1)} \cdot 1_A)(h_{(2)}k \cdot a) \\
&= \sum ((I \otimes h_{(1)})\bar{\rho}(1_A))((I \otimes h_{(2)}k)\bar{\rho}(a)) \\
&= \sum ((I \otimes h_{(1)})\bar{\rho}(1_A)) \left(\sum_{i=1}^n a_i \langle h_{(2)}k, f_i \rangle \right) \\
&= \sum ((I \otimes h_{(1)})\bar{\rho}(1_A)) \left(\sum_{i=1}^n a_i \left(\sum \langle h_{(2)}, f_{i(1)} \rangle \langle k, f_{i(2)} \rangle \right) \right) \\
&= (I \otimes k) \left(\sum ((I \otimes h_{(1)}) \otimes I)(\bar{\rho}(1_A) \otimes 1_{H_1}) \right) \\
&\quad \times \left(\sum_{i=1}^n a_i \left(\sum \langle h_{(2)}, f_{i(1)} \rangle \otimes f_{i(2)} \right) \right). \tag{7}
\end{aligned}$$

If we write $\bar{\rho}(1_A) = \sum_{j=1}^p e_j \otimes \varepsilon_j$, then the last equality in (7) reads

$$\begin{aligned}
&(I \otimes k) \left(\sum ((I \otimes h_{(1)}) \otimes I)(\bar{\rho}(1_A) \otimes 1_{H_1}) \right) \left(\sum_{i=1}^n a_i \left(\sum \langle h_{(2)}, f_{i(1)} \rangle \otimes f_{i(2)} \right) \right) \\
&= (I \otimes k) \left(\sum (I \otimes h_{(1)}) \otimes I \left(\sum_{j=1}^p e_j \otimes \varepsilon_j \otimes 1_{H_1} \right) \right) \left(\sum_{i=1}^n a_i \left(\sum \langle h_{(2)}, f_{i(1)} \rangle \otimes f_{i(2)} \right) \right) \\
&= (I \otimes k) \left(\sum \left(\sum_{j=1}^p e_j \langle h_{(1)}, \varepsilon_j \rangle \otimes 1_{H_1} \right) \right) \left(\sum_{i=1}^n a_i \left(\sum \langle h_{(2)}, f_{i(1)} \rangle \otimes f_{i(2)} \right) \right) \\
&= (I \otimes k) \left(\sum_{j=1}^p \sum_{i=1}^n e_j a_i \sum \langle h_{(1)}, \varepsilon_j \rangle \langle h_{(2)}, f_{i(1)} \rangle \otimes f_{i(2)} \right) \\
&= (I \otimes k) \left(\sum_{j=1}^p \sum_{i=1}^n e_j a_i \sum \langle h, \varepsilon_j f_{i(1)} \rangle \otimes f_{i(2)} \right) \\
&= (I \otimes k)(I \otimes h \otimes I) \left(\sum_{j=1}^p \sum_{i=1}^n e_j a_i \otimes \left(\sum \varepsilon_j f_{i(1)} \otimes f_{i(2)} \right) \right) \\
&= (I \otimes h \otimes k) \left(\sum_{j=1}^p e_j \otimes \varepsilon_j \otimes 1_{H_1} \right) \left(\sum_{i=1}^n a_i \otimes \Delta(f_i) \right) \\
&= (I \otimes h \otimes k)(\bar{\rho}(1_A) \otimes 1_{H_1})((I \otimes \Delta)\bar{\rho}(a)).
\end{aligned}$$

As this equality is valid for every $h, k \in H_1$, and because of the nondegeneracy of the pairing, we conclude that

$$(\bar{\rho} \otimes I)\bar{\rho}(a) = (\bar{\rho}(1_A) \otimes 1_{H_1})((I \otimes \Delta)\bar{\rho}(a)),$$

$\forall a \in A$.

Therefore, the map $\bar{\rho}$ is indeed a partial coaction. □

A special case is when we consider the finite dual of a Hopf algebra H ,

$$H^\circ = \{f \in H^* \mid \exists I \trianglelefteq H, f(I) = 0, \dim H/I < \infty\}.$$

We say that H° separates points in H if the following condition holds:

$$f(h) = 0, \quad \forall h \in H \Rightarrow f = 0.$$

In a similar manner, we say that H° is dense in H^* if the following condition holds:

$$f(h) = 0, \quad \forall f \in H^\circ \Rightarrow h = 0.$$

These two conditions allow us to define a nondegenerate pairing between H and H° , and therefore, by the previous theorem, we have the following result.

Corollary 1. *Let H be a Hopf algebra which acts partially on a unital algebra A such that for each $a \in A$ the subspace $H \cdot a \subseteq A$ is finite dimensional. If the finite dual H° of H separates points in H and is dense in H^* , then there is a partial coaction of H° on A given by*

$$(I \otimes h)\bar{\rho}(a) = h \cdot a, \quad \forall a \in A, \quad \forall h \in H.$$

Conversely, suppose that a Hopf algebra H_1 coacts partially on a unital algebra A . Suppose also that there exists a pairing (not necessarily nondegenerate) between the Hopf algebras H_1 and H_2 . Then we define a map

$$\begin{aligned} \bar{\cdot} : H_2 \times A &\rightarrow A \\ (f, a) &\mapsto \sum a^{(0)} \langle a^{(1)}, f \rangle, \end{aligned}$$

where $\bar{\rho}(a) = \sum a^{(0)} \otimes a^{(1)}$ is the expression of the partial coaction of H_1 on A . It is easy to see that this map is bilinear; by the universal property of the tensor product, we can define a linear map

$$\begin{aligned} \cdot : H_2 \otimes A &\rightarrow A \\ f \otimes a &\mapsto \sum a^{(0)} \langle a^{(1)}, f \rangle. \end{aligned}$$

Proposition 8. *The map \cdot defined above is indeed a partial action of H_2 on A .*

Proof. Let $f \in H_2$ and $a, b \in A$, then

$$\begin{aligned} f \cdot (ab) &= \sum (ab)^{(0)} \langle (ab)^{(1)}, f \rangle \\ &= \sum a^{(0)} b^{(0)} \langle a^{(1)} b^{(1)}, f \rangle \\ &= \left(\sum a^{(0)} \langle a^{(1)}, f_{(1)} \rangle \right) \left(\sum b^{(0)} \langle b^{(1)}, f_{(2)} \rangle \right) \\ &= \sum (f_{(1)} \cdot a) (f_{(2)} \cdot b). \end{aligned}$$

Now, let $a \in A$, we have

$$\begin{aligned} 1_{H_2} \cdot a &= \sum a^{(0)} \langle a^{(1)}, 1_{H_2} \rangle \\ &= \sum a^{(0)} \epsilon(a^{(1)}) = a. \end{aligned}$$

Finally, for each $f, g \in H_2$ and $a \in A$, and writing $\bar{\rho}(1_A) = \sum 1^{(0)} \otimes 1^{(1)}$, we have

$$\begin{aligned} f \cdot (g \cdot a) &= f \cdot \left(\sum a^{(0)} \langle a^{(1)}, g \rangle \right) \\ &= \sum a^{(0)(0)} \langle a^{(0)(1)}, f \rangle \langle a^{(1)}, g \rangle \\ &= \sum 1^{(0)} a^{(0)} \langle 1^{(1)} a^{(1)}_{(1)}, f \rangle \langle a^{(1)}_{(2)}, g \rangle \\ &= \sum 1^{(0)} a^{(0)} \langle 1^{(1)}, f_{(1)} \rangle \langle a^{(1)}_{(1)}, f_{(2)} \rangle \langle a^{(1)}_{(2)}, g \rangle \\ &= \left(\sum 1^{(0)} \langle 1^{(1)}, f_{(1)} \rangle \right) \left(\sum a^{(0)} \langle a^{(1)}, f_{(2)} g \rangle \right) \\ &= \sum (f_{(1)} \cdot 1_A) ((f_{(2)} g) \cdot a). \end{aligned}$$

These three properties show that \cdot is in fact a partial action of H_2 on A . \square

A natural question is whether it is possible or not to define an enveloping coaction for a partial coaction of a Hopf algebra H on a unital algebra A . The basic idea is to use the previous proposition to define a partial action of the finite dual H° of the Hopf algebra H on A , then take an enveloping action (B, θ) of this action, and finally, considering the case when H° separates points, to analyse if $\dim(H^\circ \triangleright \theta(a)) < \infty$, $\forall a \in A$ (i.e., if B is a rational left H° module); if this holds, one defines back a coaction $\rho : B \rightarrow B \otimes H$ by the equation

$$(I \otimes f)\rho(b) = f \triangleright b, \quad \forall f \in H^\circ \quad \forall b \in B.$$

Proposition 9. *Let H be a Hopf algebra such that its finite dual H° is dense and separates points. Suppose that H coacts partially on a unital algebra A with coaction $\bar{\rho}$ and an enveloping action, (B, θ) , of the partial action of H° on A is a rational left H° module. Then the map $\theta : A \rightarrow B$ intertwines the partial coaction of H on A and the induced partial coaction of H on B given by*

$$\tilde{\rho}(b) = (\theta(1_A) \otimes 1_H)\rho(b), \quad \forall b \in B.$$

Proof. We have to show that $(\theta \otimes I) \circ \bar{\rho} = \tilde{\rho} \circ \theta$. Take $a \in A$ and $f \in H^\circ$, then

$$\begin{aligned} (I \otimes f)(\theta \otimes I)\bar{\rho}(a) &= (I \otimes f) \left(\sum \theta(a^{(0)}) \otimes a^{(1)} \right) \\ &= \sum \theta(a^{(0)}) \langle a^{(1)}, f \rangle \\ &= \theta \left(\sum a^{(0)} \langle a^{(1)}, f \rangle \right) \\ &= \theta(f \cdot a) = f \cdot \theta(a) \end{aligned}$$

$$\begin{aligned}
 &= \theta(1_A)(f \triangleright \theta(a)) = \theta(1_A)((I \otimes f)\rho(\theta(a))) \\
 &= (I \otimes f)((\theta(1_A) \otimes 1_H)\rho(\theta(a))) \\
 &= (I \otimes f)\tilde{\rho}(\theta(a)).
 \end{aligned}$$

As this identity holds for every $f \in H^\circ$ and H° separates points in H , then $(\theta \otimes I)\tilde{\rho}(a) = \tilde{\rho}(\theta(a))$, $\forall a \in A$. Therefore, the map θ intertwines these two partial coactions. \square

Certainly, the conditions on the existence of enveloping coactions are quite restrictive but, at least, one class of examples can be given where this occurs, namely, the finite dimensional Hopf algebras.

Proposition 10. *A partial coaction of a finite dimensional Hopf algebra H on a unital algebra A always admits an enveloping coaction.*

Proof. Let $\bar{\rho} : A \rightarrow A \otimes H$ be the coaction. As H is finite dimensional, its finite dual H° is simply the dual vector space H^* . The condition that the finite dual separate points is also automatically satisfied in finite dimension. Define the partial action of H^* on A by

$$f \cdot a = (I \otimes f)\bar{\rho}(a).$$

As it came from a partial coaction, it is easy to see that $\dim(H^* \cdot a) < \infty$, $\forall a \in A$. Choose a basis $\{a_i\}_{i=1}^n$ for the subspace $H^* \cdot a$. Then we can prove that there are elements $h_i \in H$ for $i = 1, \dots, n$ such that

$$f \cdot a = \sum_{i=1}^n \langle h_i, f \rangle a_i.$$

Let (B, φ) be the standard enveloping action for this partial action on A (remember that $B \subseteq \text{Hom}_k(H^*, A)$). Now, we have to verify whether the subspace $H^* \triangleright \varphi(a) \subseteq B$ is finite dimensional. Take $f, g \in H^*$, then

$$\begin{aligned}
 (f \triangleright \varphi(a))(g) &= \varphi(a)(gf) = gf \cdot a \\
 &= \sum_{i=1}^n \langle h_i, gf \rangle a_i \\
 &= \sum_{i=1}^n \langle f \rightharpoonup h_i, g \rangle a_i.
 \end{aligned}$$

Then, for each $f \in H^*$, we can see that

$$f \triangleright \varphi(a) \in \text{Hom}_k(H^*, \text{span}\{a_i \mid i = 1, \dots, n\}) \cong \bigoplus_{i=1}^n H.$$

As the space $\bigoplus_{i=1}^n H$ is finite dimensional, then $H \triangleright \varphi(a)$ is also finite dimensional.

Therefore, B is a rational left H^* module, which allows to define a coaction of H on B . \square

5. PARTIAL REPRESENTATIONS

Partial representations of groups were first introduced by Exel in [10] and became a powerful tool to investigate the action of semigroups on algebras. A partial representation of a group G is a map $\pi : G \rightarrow A$ on a unital algebra A such that

$$\begin{aligned} 1) \quad & \pi(e) = 1_A, \\ 2) \quad & \pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}), \quad \forall g, h \in G, \\ 3) \quad & \pi(g^{-1})\pi(g)\pi(h) = \pi(g^{-1})\pi(gh), \quad \forall g, h \in G. \end{aligned} \quad (8)$$

In the reference [7], the authors showed that partial actions and partial representations of a group are intimately related. By one hand, if we have a partial action of G on a unital algebra A such that each ideal D_g has unit 1_g , then there is a partial representation of the group G on the partial crossed product $A \rtimes_\alpha G$ given by

$$\pi(g) = 1_g \delta_g.$$

On the other hand, if there is a partial representation $\pi : G \rightarrow A$, it is possible to show that A is isomorphic to a partial crossed product $\bar{A} \rtimes_\alpha G$, where \bar{A} is an abelian subalgebra of A generated by the elements of the form $\varepsilon_g = \pi(g)\pi(g^{-1})$, $\forall g \in G$ and the partial action is given as follows: the ideals D_g are $D_g = \varepsilon_g \bar{A}$ and $\alpha_g : D_{g^{-1}} \rightarrow D_g$ given by $\alpha_g(x) = \pi(g)x\pi(g^{-1})$, $\forall x \in D_{g^{-1}}$.

Proposition 11. *Let α be a partial action of G on A such that every idempotent 1_g is central. Then the map $\pi : G \rightarrow \text{End}_k(A)$ given by*

$$\pi(g)(a) = g \cdot a = \alpha_g(a1_{g^{-1}})$$

defines a partial representation of G .

Proof. It is easy to see that the first item of (8) holds, because for each $a \in A$

$$\pi(e)(a) = 1 \cdot a = a.$$

Therefore, $\pi(g) = I = 1_{\text{End}_k(A)}$. As we have shown in the beginning, if the idempotents $1_g = g \cdot 1_A$ are central, then $k \cdot (l \cdot a) = (k \cdot 1_A)(kl \cdot a) = (kl \cdot a)(k \cdot 1_A)$. Hence,

$$\begin{aligned} \pi(g^{-1})\pi(gh)(a) &= g^{-1} \cdot (gh \cdot a) \\ &= (g^{-1} \cdot 1_A)(h \cdot a) \\ &= (g^{-1} \cdot 1_A)(g^{-1}g \cdot 1_A)(h \cdot a) \end{aligned}$$

$$\begin{aligned}
 &= (g^{-1} \cdot (g \cdot (h \cdot a))) \\
 &= \pi(g^{-1})\pi(g)\pi(h)(a).
 \end{aligned}$$

Since this occurs $\forall a \in A$, we conclude that $\pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1})$.

In a similar manner,

$$\begin{aligned}
 \pi(gh)\pi(h^{-1})(a) &= gh \cdot (h^{-1} \cdot a) \\
 &= (g \cdot a)(gh \cdot 1_A) \\
 &= (g \cdot 1_A a)(gh \cdot 1_A) \\
 &= (g \cdot 1_A)(g \cdot a)(gh \cdot 1_A) \\
 &= (g \cdot 1_A)(ghh^{-1} \cdot a)(gh \cdot 1_A) \\
 &= (g \cdot 1_A)(gh \cdot (h^{-1} \cdot a)) \\
 &= g \cdot (h \cdot (h^{-1}a)) \\
 &= \pi(g)\pi(h)\pi(h^{-1})(a).
 \end{aligned}$$

As this equality holds $\forall a \in A$ then $\pi(g^{-1})\pi(g)\pi(h)(a) = \pi(g^{-1})\pi(gh)(a)$. Therefore, π is a partial representation of G . \square

Inspired in this previous example, let us try to define a partial representation of a Hopf algebra H . In what follows, we assume that H is a Hopf algebra with invertible antipode.

Proposition 12. *Let H be a Hopf algebra, with invertible antipode, which acts partially on a unital algebra A . Then the map*

$$\begin{aligned}
 \pi : H &\rightarrow \text{End}_k(A), \\
 h &\mapsto \pi(h)
 \end{aligned}$$

given by $\pi(h)(a) = h \cdot a$, satisfies:

$$\begin{aligned}
 \pi(1_H) &= I, \\
 \sum \pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k) &= \sum \pi(S^{-1}(h_{(2)}))\pi(h_{(1)}k).
 \end{aligned}$$

Proof. The first identity is quite obvious. In order to prove the second equality, take any element $a \in A$, then

$$\begin{aligned}
 \sum \pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k)(a) &= \sum S^{-1}(h_{(2)}) \cdot (h_{(1)} \cdot (k \cdot a)) \\
 &= \sum (S^{-1}(h_{(3)}) \cdot 1_A)(S^{-1}(h_{(2)})h_{(1)} \cdot (k \cdot a)) \\
 &= (S^{-1}(h) \cdot 1_A)(1_H \cdot (k \cdot a)) \\
 &= (S^{-1}(h) \cdot 1_A)(k \cdot a),
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum \pi(S^{-1}(h_{(2)}))\pi(h_{(1)}k)(a) &= \sum S^{-1}(h_{(2)}) \cdot (h_{(1)}k \cdot a) \\ &= \sum (S^{-1}(h_{(3)}) \cdot 1_A)(S^{-1}(h_{(2)})h_{(1)}k \cdot a) \\ &= (S^{-1}(h) \cdot 1_A)(k \cdot a).\end{aligned}$$

□

With this result we can propose the following definition.

Definition 6. Let H be a Hopf algebra with invertible antipode. A partial representation of H on a unital algebra B is a linear map

$$\begin{aligned}\pi : H &\rightarrow B, \\ h &\mapsto \pi(h)\end{aligned}$$

such that

$$\begin{aligned}1) \quad &\pi(1_H) = 1_B, \\ 2) \quad &\sum \pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k) = \sum \pi(S^{-1}(h_{(2)}))\pi(h_{(1)}k), \forall h, k \in H.\end{aligned}\tag{9}$$

Unlike the group case, partial representations of Hopf algebras are not symmetrical with relation to the left and right side. This is not totally unexpected, since we are working with right ideals; we mention, however, that the situation does not seem to improve much if one imposes that $\varphi(A)$ is an ideal in the enveloping action (B, φ) .

In the group case, it is known that if G acts partially on a unital algebra A such that every D_g is unital, then there is a partial representation of G on the partial crossed product. The same occurs with the partial smash product, that is, if H acts partially on A , then there is a partial representation of H on $\underline{A\#H}$.

Theorem 6. Let H be a Hopf algebra with invertible antipode which acts partially on a unital algebra A . Then the linear map

$$\begin{aligned}\pi : H &\rightarrow \underline{A\#H}, \\ h &\mapsto \sum (h_{(1)} \cdot 1_A) \otimes h_{(2)}\end{aligned}$$

is a partial representation of H .

Proof. The first item in (9) can be easily derived, since

$$\pi(1_H) = (1_H \cdot 1_A) \otimes 1_H = 1_A \otimes 1_H,$$

which is the unit in $\underline{A\#H}$.

For the second item in the definition, take $h, k \in H$, then

$$\begin{aligned}\sum \pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k) &= \left(\sum (S^{-1}(h_{(4)}) \cdot 1_A) \otimes S^{-1}(h_{(3)}) \right) \left(\sum (h_{(1)} \cdot 1_A) \otimes h_{(2)} \right) \left(\sum (k_{(1)} \cdot 1_A) \otimes k_{(2)} \right)\end{aligned}$$

$$\begin{aligned}
 &= \left(\sum (S^{-1}(h_{(5)}) \cdot 1_A) (S^{-1}(h_{(4)}) \cdot (h_{(1)} \cdot 1_A)) \otimes S^{-1}(h_{(3)}) h_{(2)} \right) \left(\sum (k_{(1)} \cdot 1_A) \otimes k_{(2)} \right) \\
 &= \left(\sum (S^{-1}(h_{(3)}) \cdot 1_A) (S^{-1}(h_{(2)}) \cdot (h_{(1)} \cdot 1_A)) \otimes 1_H \right) \left(\sum (k_{(1)} \cdot 1_A) \otimes k_{(2)} \right) \\
 &= \left(\sum (S^{-1}(h_{(4)}) \cdot 1_A) (S^{-1}(h_{(3)}) \cdot 1_A) (S^{-1}(h_{(2)}) h_{(1)} \cdot 1_A) \otimes 1_H \right) \left(\sum (k_{(1)} \cdot 1_A) \otimes k_{(2)} \right) \\
 &= ((S^{-1}(h) \cdot 1_A) \otimes 1_H) \left(\sum (k_{(1)} \cdot 1_A) \otimes k_{(2)} \right) \\
 &= \sum (S^{-1}(h) \cdot 1_A) (k_{(1)} \cdot 1_A) \otimes k_{(2)}.
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 &\sum \pi(S^{-1}(h_{(2)})) \pi(h_{(1)} k) \\
 &= \left(\sum (S^{-1}(h_{(4)}) \cdot 1_A) \otimes S^{-1}(h_{(3)}) \right) \left(\sum (h_{(1)} k_{(1)} \cdot 1_A) \otimes h_{(2)} k_{(2)} \right) \\
 &= \sum (S^{-1}(h_{(5)}) \cdot 1_A) (S^{-1}(h_{(4)}) \cdot (h_{(1)} k_{(1)} \cdot 1_A)) \otimes S^{-1}(h_{(3)}) h_{(2)} k_{(2)} \\
 &= \sum (S^{-1}(h_{(3)}) \cdot 1_A) (S^{-1}(h_{(2)}) \cdot (h_{(1)} k_{(1)} \cdot 1_A)) \otimes k_{(2)} \\
 &= \sum (S^{-1}(h_{(4)}) \cdot 1_A) ((S^{-1}(h_{(3)}) \cdot 1_A) (S^{-1}(h_{(2)}) h_{(1)} k_{(1)} \cdot 1_A)) \otimes k_{(2)} \\
 &= \sum (S^{-1}(h) \cdot 1_A) (k_{(1)} \cdot 1_A) \otimes k_{(2)}.
 \end{aligned}$$

□

ACKNOWLEDGMENTS

The authors would like to thank to Edson R. Álvares and Eduardo O. C. Hoefel for fruitful discussions. The first author (M.M.S.A.) would like to thank Virgínia S. Rodrigues for her fundamental role in establishing the UFPR-UFSC Hopf Seminars. The second author (E.B.) would like to thank the Math Department of UFPR and its staff for their kind hospitality.

REFERENCES

- [1] Abadie, F. (2003). Enveloping actions and Takai duality for partial action. *J. Funct. Anal.* 197:14–67.
- [2] Alves, M. M. S., Batista, E. Enveloping actions for partial Hopf actions. arXiv:0805.4805v2 (This is a previous version of this article, which contains the whole proof of Proposition 2, omitted in this final version).
- [3] Caenepeel, S., de Groot, E. (2005). Galois corings applied to partial Galois theory. In: Kalla, S. L., Chawla, M. M., eds. *Proceedings of the International Conference on Mathematics and Its Applications, ICMA 2004*. Kuwait: Kuwait University, pp. 117–134.
- [4] Caenepeel, S., Janssen, K. (2008). Partial (co)actions of Hopf algebras and partial Hopf–Galois theory. *Comm. Algebra* 36:2923–2946.
- [5] Dăscălescu, S., Năstăsescu, C., Raianu, Ş. (2001). *Hopf Algebras: An Introduction*. New York: Marcel Dekker Inc.
- [6] Dokuchaev, M., Zhukavets, N. (2004). On finite degree partial representations of groups. *J. of Algebra* 274:309–334.
- [7] Dokuchaev, M., Exel, R. (2005). Associativity of crossed products by partial actions, enveloping actions and partial representations. *Trans. Amer. Math. Soc.* 357:1931–1952.

- [8] Dokuchaev, M., Ferrero, M., Paques, A. (2007). Partial actions and Galois theory. *J. Pure and Appl. Algebra* 208:77–87.
- [9] Exel, R. (1994). Circle actions on C^* -algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences. *J. Funct. Anal.* 122:361–401.
- [10] Exel, R. (1998). Partial actions of groups and actions of semigroups. *Proc. Am. Math. Soc.* 126:3481–3494.
- [11] Ferrero, M., Lazzarin, J. (2008). Partial actions and partial skew group rings. *J. Algebra* 319:5247–5264.
- [12] Lomp, C. (2008). Duality for partial group actions. *Int. El. J. Algebra* 4:53–62.
- [13] Quigg, J. C., Raeburn, I. (1997). Characterizations of crossed products by partial actions. *J. Operator Theory* 37:311–340.
- [14] Montgomery, S. (1993). *Hopf Algebras and Their Actions on Rings*. Providence, Rhode Island: Amer. Math. Soc.