Partial Schur multiplier

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Abstract

The partial Schur multiplier $pM(G)$ of a group $G$ is a generalization of the classical Schur multiplier $M(G)$. While its classical version is a group, $pM(G)$ is a semilattice of abelian groups $pM(G)$ (called components), indexed by certain subsets $D \subseteq G \times G$. Each component $pM(G)$ consists of partially defined functions $\sigma: G \times G \to K$ having $D$ as its domain. Such functions are called partial factor sets of $G$ and are associated to the partial projective representations of $G$. This work discusses a specific component, $pM_{G\times G}(G)$, which is particularly interesting due to the fact that there are epimorphisms from it to any other component, assuming that $K$ is an algebraically closed field. We characterize this component for some families of groups, including the dihedral and dicyclic groups. This is a joint work with H. Pinedo.

1 Preliminaries

In [1] it was obtained the following decomposition for the partial Schur multiplier:

**Theorem 1.** The semigroups $pM(G)$ and $M(G)$ are semilattices of abelian groups

$$pM(G) = \bigcup_{D \subseteq G \times G} pM_{D}(G),$$

where $C(G)$ is a semilattice of subsets of $G \times G$, and the factor sets in $pM_{D}(G)$ have domain $D$.

The semilattice $(C(G), \cap)$ is formed by the subsets of $G \times G$ which are invariant under the action of a specific semigroup $\mathcal{T}$ (it is the same for every group $G$), which is generated by symbols $u, v$ and $t$ with relations

$$u^2 = v^2 = (uv)^2 = 1, \quad t^4 = 1, \quad ut = tu = 1, \quad t = 0.$$

This semigroup $\mathcal{T}$ contains a copy $\mathcal{G} = \{u, v \mid u^2 = v^2 = (uv)^2 = 1\}$ of the symmetric group $S_3$, and acts on $G \times G$ by $t(x, y) = (x, 1), \quad u(x, y) = (xy, y^{-1})$ and $v(x, y) = (y^{-1}, x^{-1})$.

Moreover, these values can be chosen arbitrarily in the effective orbits of $\mathcal{T}$, meaning that the number of effective orbits of $\mathcal{T}$ is completely determined by its values in the representatives of $\mathcal{T}$-orbits of the group $D_{2m}$, given by:

$$\sigma(a, b) = (\sigma(a, b))^{\pm 1} = (\sigma(a, b))^{\pm 3} = \left\{\begin{array}{ll}
1 & \text{if } m \mid n, \\
3 & \text{if } m \nmid n.
\end{array}\right.$$

4 Infinite cyclic group

On [3, Corollary 6.4], there was a description of $pM_{\mathbb{Z}}(G)$ for infinite cyclic groups $G$. We obtained an analogous for the infinite cyclic group.

**Lemma 5.** Any element of $pM_{\mathbb{Z}}(G)$ is uniquely determined by its values in pairs $(x, j) \in \mathbb{Z} \times \mathbb{N}$. Moreover, these values can be chosen arbitrarily in $K$.

**Proposition 2.** If $\sigma \in pM_{\mathbb{Z}}(G)$ and $\sigma \sim 1$ then $\sigma$ is uniquely determined by its values on the pairs

$$(a, a^k), \quad (a, b), \quad (a, a^k b), \quad (a^k, b), \quad (a^k, b^k), \quad (a^k, b^k) (ab)^k,$$

where $0 \leq k \leq m - 1$.

4.1 Direct product of two cyclic groups

**Proposition 3.** Let $G = C_m \times C_n$, $m, n \in \mathbb{N}$ and $\sigma \in pM_{\mathbb{Z}}(G)$ such that $\sigma \sim 1$. Then $\sigma$ is uniquely determined by its values on the pairs

$$(a, a^k b), \quad (a, a^k b^k), \quad (a, b), \quad (a^k, b), \quad (b, b^k), \quad (a^k, b^k) (ab)^k,$$

where $0 \leq k \leq m - 1$.

4.2 Free abelian groups

**Theorem 3.** Let $G = \mathbb{Z}^2$ and $\sigma \in pM_{\mathbb{Z}}(G)$ such that $\sigma \sim 1$. Then $\sigma$ is uniquely determined by its values on the pairs

$$(a, a^k b), \quad (a, a^k b^k), \quad (a, b), \quad (a^k, b), \quad (b, b^k), \quad (a^k, b^k) (ab)^k,$$

where $0 \leq k \leq m - 1$.

4.3 Euclidean domains

**Theorem 4.** Let $G = \mathbb{Q}^2$ and $\sigma \in pM_{\mathbb{Z}}(G)$ such that $\sigma \sim 1$. Then $\sigma$ is uniquely determined by its values on the pairs

$$(a, a^k b), \quad (a, a^k b^k), \quad (a, b), \quad (a^k, b), \quad (b, b^k), \quad (a^k, b^k) (ab)^k,$$

where $0 \leq k \leq m - 1$.

4.4 Abelian groups

**Theorem 5.** Let $G = \mathbb{Z}$ and $\sigma \in pM_{\mathbb{Z}}(G)$ such that $\sigma \sim 1$. Then $\sigma$ is uniquely determined by its values on the pairs

$$(a, a^k b), \quad (a, a^k b^k), \quad (a, b), \quad (a^k, b), \quad (b, b^k), \quad (a^k, b^k) (ab)^k,$$

where $0 \leq k \leq m - 1$.

**References**


