A unified treatment of Katsura and Nekrashevych C^*-algebras.

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Outline

1. Katsura and Nekrashevych algebras
   - Katsura algebras
   - Nekrashevych algebras
2. The new construction
   - Groups acting on graphs
   - The algebra $\mathcal{O}_{G,E}$
3. $\mathcal{O}_{G,E}$ as groupoid C^*-algebra
4. Characterizing properties

Joint work with Ruy Exel (Departamento de Matemática, Universidade Federal de Santa Catarina, Florianópolis, Brasil),

R. EXEL, E. PARDO, *Representing Kirchberg algebras as inverse semigroup crossed products, arXiv:1303.6268v1 (2013),*

and

R. EXEL, E. PARDO, *Graphs, groups and self-similarity, arXiv:1307.1120v1 (2013).*

GOAL: Present a family of C*-algebras, generalizing certain algebras introduced by Katsura and Nekrashevych.

Realize these new algebras as groupoid C^*-algebras and as inverse semigroup crossed product C^*-algebras.

Use these pictures to characterize properties of the algebras: e.g. to be simple, or purely infinite.

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Definition

Let $N \in \mathbb{N} \cup \{\infty\}$, let $A \in M_N(\mathbb{Z}^+)$ and $B \in M_N(\mathbb{Z})$ be row-finite matrices. Define a set Ω_A by

$$\Omega_A := \{(i,j) \in \{1,2,\ldots,N\} \times \{1,2,\ldots,N\} \mid A_{i,j} \geq 1\}.$$

Fix the following condition:

(0) $\Omega_A(i) \neq \emptyset$ *for all* i*, and* $B_{i,j} = 0$ *for* $(i,j) \notin \Omega_A$.

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Define $\mathcal{O}_{A,B}$ to be the universal C^*-algebra generated by mutually orthogonal projections $\{q_i\}_{i=1}^N$, partial unitaries $\{u_i\}_{i=1}^N$ with $u_i u_i^* = u_i^* u_i = q_i$, and partial isometries $\{s_{i,j,n}\}_{(i,j)\in\Omega_A, n\in\mathbb{Z}}$ satisfying the relations:

(i) $s_{i,j,n}^* s_{i,j,n} = q_j$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$.

(ii) $q_i = \sum_{j\in\Omega_A(i)} \sum_{n=1}^{A_{i,j}} s_{i,j,n} s_{i,j,n}^*$ for all i.

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(iii) $s_{i,j,n} u_j = s_{i,j,n+A_{i,j}}$ and $u_i s_{i,j,n} = s_{i,j,n+B_{i,j}}$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$.

Now, the following facts holds:

1. $\mathcal{O}_{A,B}$ is separable, nuclear and in the UCT class.
2. Every Kirchberg algebra in the UCT class can be represented, up to isomorphism, by a $\mathcal{O}_{A,B}$.
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$X = \{a_1, \ldots, a_n\}$ finite alphabet. G group acting on the n-rooted tree on X by automorphisms.

$g(a_i a_j) = g(a_i) \cdot g_{|a_i}(a_j)$ defines a 1-cocycle on G:

$$\begin{array}{ccc} G \times X & \to & G \\ (g, a_i) & \mapsto & g_{|a_i} \end{array}$$

with $gh_{|a} = g_{|ha} \cdot h_{|a}$ and $g_{|ab} = (g_{|a})_{|b}$.

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Relation (iii) in definition is equivalent to

$$u_i s_{i,j,n} = s_{i,j,\widehat{n}} u_j^k$$

for unique $1 \leq \widehat{n} \leq A_{i,j}$ and $k \in \mathbb{Z}$ such that $n + B_{i,j} = \widehat{n} + kA_{i,j}$.

If E_A is a finite graph, then $u := \sum_{i=1}^{N} u_i$ is a unitary of $\mathcal{O}_{A,B}$.

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Action of $\mathbb{Z}$ on E_A: Given (i,j,n) with $1 \leq n \leq A_{ij}$, $l \in \mathbb{Z}$, define $l \cdot (i,j,n) = (i,j,\widehat{n})$ for the unique $1 \leq \widehat{n} \leq A_{i,j}$ and $k \in \mathbb{Z}$ such that $n + lB_{i,j} = \widehat{n} + kA_{i,j}$.

1-cocycle: For the above data,

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Katsura algebras rise from an action of $\mathbb{Z}$ on the graph E_A. But the action need not be faithful.

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To extend (σ, φ) to E^*, we need to fix

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We denote the data by (G, E, φ).

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$\mathcal{O}_{A,B}$ and $\mathcal{O}_{(G,X)}$ are examples of $\mathcal{O}_{G,E}$.

If the 1-cocycle is $\varphi(g,a) = g$ for all $g \in G, a \in E^1$, then $\mathcal{O}_{G,E} \cong C^*(E) \rtimes G$ (crossed product).

If the 1-cocycle is $\varphi(g,a) = 1$ for all $g \in G, a \in E^1$, then vertices are fixed by the action of G, and $\mathcal{O}_{G,E} \cong C^*(E)$.

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Outline

Fix a $*$-inverse subsemigroup $\mathcal{S}$ of $\mathcal{O}_{G,E}$.

$$\mathcal{S} := \{s_\alpha u_g s_\beta^* \mid g \in G, \alpha, \beta \in E^* \text{ with } d(\alpha) = g d(\beta)\}.$$

Def. Inverse semigroup

Define an abstract version $\mathcal{S}_{G,E}$ of $\mathcal{S}$:

$$\mathcal{S}_{G,E} := \{(\alpha, g, \beta) \mid g \in G, \alpha, \beta \in E^* \text{ with } d(\alpha) = g d(\beta)\},$$

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Proposition

$\pi : \mathcal{S}_{G,E} \to \mathcal{O}_{G,E}$ is the universal tight representation of $\mathcal{S}_{G,E}$.

Theorem

$$\mathcal{O}_{G,E} \cong C^*_{\text{tight}}(\mathcal{S}_{G,E}).$$

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Define the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.

The space $\mathcal{E}_{\text{tight}}$ of tight filters over the idempotent semilattice of $\mathcal{S}_{G,E}$ is homeomorphic to E^∞.

$\mathcal{S}_{G,E}$ acts (partially) on E^∞ as follows: If $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, $\omega = \beta\widehat{\omega} \in E^\infty$, then

$$(\alpha, g, \beta)\omega = \alpha(g\widehat{\omega}),$$

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$\mathcal{S}_{G,E}$ acts (partially) on E^∞ as follows: If $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, $\omega = \beta\widehat{\omega} \in E^\infty$, then

$$(\alpha, g, \beta)\omega = \alpha(g\widehat{\omega}),$$

where $g\widehat{\omega}$ is defined by recurrence over the action of g on $\widehat{\omega}_{|n}$ for all $n \in \mathbb{N}$.

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Consider the transformation groupoid $\mathcal{S}_{G,E} \times E^\infty$, and $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ the groupoid of germs of the action.

$C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ is universal for tight representations of $\mathcal{S}_{G,E}$.

Theorem

There is a $*$-isomorphism

$$\mathcal{O}_{G,E} \cong C^*_{tight}(\mathcal{S}_{G,E}) \cong C^*(\mathcal{G}_{tight}(\mathcal{S}_{G,E})) \cong C(E^\infty) \rtimes \mathcal{S}_{G,E}.$$

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Def. E^*-unitary

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Lemma

$\mathcal{S}_{G,E}$ is E^*-unitary iff holds (RF):

$$(g,\alpha) \in G \times E^* \text{ with } g\alpha = \alpha \text{ and } \varphi(g,\alpha) = 1 \text{ implies } g = 1.$$

Outline

1. Katsura and Nekrashevych algebras
   - Katsura algebras
   - Nekrashevych algebras
2. The new construction
   - Groups acting on graphs
   - The algebra $\mathcal{O}_{G,E}$
3. $\mathcal{O}_{G,E}$ as groupoid C^*-algebra
4. Characterizing properties

The dynamical approach lets us to deal with some questions in a more intuitive form.

Simplicity: If $\mathcal{G}$ is amenable & Hausdorff, then $C^*(\mathcal{G})$ simple iff $\mathcal{G}$ is minimal & essentially principal **[Brown-Clark-Farthing-Sims]**.

(RF) implies $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ Hausdorff.

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Def. Minimal and Irreducible

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1. *The matrix A is G-irreducible.*

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Theorem

Given the action of $\mathcal{S}_{G,E}$ on E^∞, the following are equivalent:

1. *The matrix A is G-irreducible.*
2. *The groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is minimal.*

Essentially principal is connected to topologically free, as follows:

Theorem

Let S be an E^*-unitary inverse semigroup, let τ be an action of S on a locally compact, Hausdorff space X, and let $\mathcal{G}$ be the corresponding groupoid of germs. Then $\mathcal{G}$ is essentially principal if and only if τ is topologically free.

Def. Essentially principal

Def. Topologically free

Thus, we can deal with the problem from the point of view of topological freeness. And the result we get is

Theorem

Let $(G, E; \varphi)$ with (RF). Consider the action of $\mathcal{S}_{G,E}$ on E^∞, and let $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ the associated groupoid. The following are equivalent:

1. (i) *The graph E_A satisfies Condition (Lgen).*

▸ Def. Condition (Lgen)

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1. (i) *The graph E satisfies Condition (Lgen).*
   (ii) *Given $g \in G$ and $\omega \in E^\infty$ fixed by g, then for every $n \geq 1$ there exists $k_n \geq n$ and $\alpha \in E^*$ with $d(\alpha) = d(\omega_{k_n})$ such that $\varphi(g, \omega_{|k_n}) \cdot \alpha \neq \alpha$.*
2. *The groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is essentially principal.*

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Now, we are ready to characterize simplicity.

Theorem

Let $(G, E; \varphi)$ with (RF), G amenable, A the adjacency matrix of E. Then the following are equivalent:

1. (i) *The matrix A is G-irreducible.*
   (ii) *The graph E satisfies Condition (Lgen).*
   (iii) *Given $g \in G$ and $\omega \in E^\infty$ fixed by g, then for every $n \geq 1$ there exists $k_n \geq n$ and $\alpha \in E^*$ with $d(\alpha) = d(\omega_{k_n})$ such that $\varphi(g, \omega_{|k_n}) \cdot \alpha \neq \alpha$.*
2. *$\mathcal{O}_{G,E}$ is simple.*

A unified treatment of Katsura and Nekrashevych C^*-algebras.

Enrique Pardo

Universidad de Cádiz

Partial Actions and Representations Symposium
Gramado (Brasil), May 13, 2014.

$C^*(E)$

E row-finite graph. $C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_x \mid x \in E^0\}$ and partial isometries $\{s_a \mid a \in E^1\}$ satisfying the relations:

1. $s_a^* s_b = \delta_{a,b} p_{d(a)}$.
2. $p_x = \sum_{a \in r^{-1}(x)} s_a s_a^*$ for any $x \in E^0$ non source.

Return

Inverse semigroup

A semigroup $\mathcal{S}$ is an inverse semigroup if for any $s \in \mathcal{S}$ exists a unique $s^* \in \mathcal{S}$ such that $ss^*s = s$ and $s^*ss^* = s^*$.

Return

E^*-unitary

A $*$-inverse semigroup $\mathcal{S}$ is E^*-unitary if whenever $s \geq e$ and $e = e^2$, then $s = s^2$.

Return

Minimal

A groupoid $\mathcal{G}$ is said to be minimal if the only invariant open subsets of $\mathcal{G}^{(0)}$ are the empty set and $\mathcal{G}^{(0)}$ itself.

Irreducible

If S is an inverse semigroup, and τ is an action by (partial) homeomorphisms on a topological space X, then we say that X is irreducible if it has no proper open invariant subsets.

Return

G-irreducible

If E graph, A the adjacency matrix of E, G group acting on E^0, then A is G-irreducible if for every $x, y \in E^0$ there exist $g \in G$ and $n \in \mathbb{N}$ such that $(A^n)_{gx,y} \neq 0$.

Return

Essentially principal

Let $\mathcal{G}$ be a locally compact, Hausdorff, étale groupoid. Then, $\mathcal{G}$ is essentially principal if the interior of the isotropy group bundle

$$\mathcal{G}' = \{\gamma \in \mathcal{G} : d(\gamma) = t(\gamma)\}$$

is contained in $\mathcal{G}^{(0)}$.

Return

Topologically free

Let S be an E^*-unitary inverse semigroup, and let τ be an action of S on a topological space X. We say that the action is topologically free if, for every $s \in S \setminus E(S)$, the interior of the set of fixed points for s is empty.

Return

$\omega \in E^\infty$ is a generalizad cycle if there exists $\alpha \in E^*$ and $(g_k)_{k\geq 1} \subseteq G$ such that $\omega = \alpha(g_1\alpha)(g_2\alpha)\cdots(g_k\alpha)\cdots$.

Condition (Lgen)

E satisfies Condition (Lgen) if every generalized cycle has an entry.

Return