A Ruelle Operator for continuous time Markov Chains

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Abstract

We consider a generalization of the Ruelle theorem for the case of continuous
time problems. We present a result which we believe is important for future use in
problems in Mathematical Physics related to $C^*$-Algebras.

We consider a finite state set $S$ and a stationary continuous time Markov Chain
$X_t, t \geq 0$, taking values on $S$. We denote by $\Omega$ the set of paths $w$ taking values
on $S$ (the elements $w$ are locally constant with left and right limits and are also
right continuous on $t$). We consider an infinitesimal generator $L$ and a stationary
vector $p_0$. We denote by $P$ the associated probability on $(\Omega, \mathcal{B})$. All functions $f$ we
consider below are in the set $L^\infty(P)$.

From the probability $P$ we define a Ruelle operator $L^t, t \geq 0$, acting on functions
$f : \Omega \to \mathbb{R}$ of $L^\infty(P)$. Given $V : \Omega \to \mathbb{R}$, such that is constant in sets of the form
$\{X_0 = c\}$, we define a modified Ruelle operator $\tilde{L}_V^t, t \geq 0$, in the following way: there exist a certain $f_V$ such that for each $t$ we consider the operator acting on $g$
given by

$$\tilde{L}_V^t(g)(w) = \left[ \frac{1}{f_V}L^t(e^{-\int_0^t (V \circ \Theta_s)(\cdot) ds} g f_V) \right](w)$$

We are able to show the existence of an eigenfunction $u$ and an eigen-probability
$\nu_V$ on $\Omega$ associated to $\tilde{L}_V^t, t \geq 0$.

We also show the following property for the probability $\nu_V$: for any integrable
g $\in L^\infty(P)$ and any real and positive $t$

$$\int e^{-\int_0^t (V \circ \Theta_s)(\cdot) ds} \left[ (\tilde{L}_V^t (g)) \circ \theta_t \right] d\nu_V = \int g d\nu_V$$

This equation generalize, for the continuous time Markov Chain, a similar one
for discrete time systems (and which is quite important for understanding the KMS
states of certain $C^*$-algebras).

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1 Introduction

We want to extend the concept of Ruelle operator to continuous time Markov Chains. In order to do that we need a probability a priori on paths. This fact is not explicit in the discrete time case (thermodynamic formalism) but it is necessary here.

We consider a continuous time stochastic process: the sample paths are functions of the positive real line \( \mathbb{R}_+ = \{ t \in \mathbb{R} : t \geq 0 \} \) taking values in a finite set \( S \) with \( n \) elements, that we denote by \( S = \{1, 2, \ldots, n\} \). Now, consider a \( n \) by \( n \) real matrix \( L \) such that:

1) \( 0 < -L_{ii} \), for all \( i \in S \),

2) \( L_{ij} \geq 0 \), for all \( i \neq j \), \( i \in S \),

3) \( \sum_{i=1}^n L_{ij} = 0 \) for all fixed \( j \in S \).

We point out that, by convention, we are considering column stochastic matrices and not line stochastic matrices (see [N] section 2 and 3 for general references).

We denote by \( P^t = e^{tL} \) the semigroup generated by \( L \). The left action of the semigroup can be identified with an action over functions from \( S \) to \( \mathbb{R} \) (vectors in \( \mathbb{R}^n \)) and the right action can be identified with action on measures on \( S \) (also vectors in \( \mathbb{R}^n \)).

The matrix \( e^{tL} \) is column stochastic, since from the assumptions on \( L \) it follows that

\[
(1, \ldots, 1) e^{tL} = (1, \ldots, 1)(I + tL + \frac{1}{2}t^2L^2 + \cdots) = (1, \ldots, 1).
\]

It is well known that there exist a vector of probability \( p_0 = (p_0^1, p_0^2, \ldots, p_0^n) \in \mathbb{R}^n \) such that \( e^{tL}(p_0) = P^t p_0 = p_0^0 \) for all \( t > 0 \). The vector \( p_0 \) is a right eigenvector of \( e^{tL} \). All entries \( p_0^i \) are strictly positive, as a consequence of hypothesis 1.

Now, let’s consider the space \( \Omega = \{1, 2, \ldots, n\}^{\mathbb{R}^+} \) of all functions from \( \mathbb{R}_+ \) to \( S \). In principle this seems to be enough for our purposes, but technical details in the construction of probability measures on such a space force us to use a restriction: we consider the space \( \Omega \subset \bar{\Omega} \) as the set of right-continuous functions from \( \mathbb{R}_+ \) to \( S \), which also have left limits in every \( t > 0 \). These functions are constant in intervals (closed in the left and open in the right). In this set we consider the sigma algebra \( \mathcal{B} \) generated by the cylinders of the form

\[
\{w_0 = a_0, w_{t_1} = a_1, w_{t_2} = a_2, \ldots, w_{t_r} = a_r\},
\]

where \( t_i \in \mathbb{R}_+, r \in \mathbb{Z}^+, a_i \in S \) and \( 0 < t_1 < t_2 < \ldots < t_r \). It is possible to endow \( \Omega \) with a metric, the Skorohod-Stone metric \( d \), which makes \( \Omega \) complete and separable ([EK] section 3.5), but the space is not compact.

Now we can introduce a continuous time version of the shift map as follows: we define for each fixed \( s \in \mathbb{R}_+ \) the \( \mathcal{B} \)-measurable transformation \( \Theta_s : \Omega \to \Omega \) given by \( \Theta_s(w_t) = w_{t+s} \) (we remark that \( \Theta_s \) is also a continuous transformation with respect to the Skorohod-Stone metric \( d \)).

For \( L \) and \( p_0 \) fixed as above we denote by \( P \) the probability on the sigma-algebra \( \mathcal{B} \) defined for cylinders by

\[
P(\{w_0 = a_0, w_{t_1} = a_1, \ldots, w_{t_r} = a_r\}) = P_{a_r \ldots a_{t-1}}^{t_r \ldots t_1} P_{a_1 a_0}^{t_1} p_0^{a_0}.
\]
For further details of the construction of this measure we refer the reader to [B].

The probability $P$ on $(\Omega, \mathcal{B})$ is stationary in the sense that for any integrable function $f$, and, any $s \geq 0$
\[ \int f(w)dP(w) = \int (f \circ \Theta_s)dP(w). \]

From now on the Stationary Process defined by $P$ is denoted by $X_t$ and all functions $f$ we consider are in the set $L^\infty(P)$.

There exist a version of $P$ such that for a set of full measure we have that all sample elements $w$ are locally constant on $t$, with left and right limits, and $w$ is right continuous on $t$. We consider from now on such probability $P$ acting on this space (see [EK] chapter 3).

From $P$ we are able to define a continuous time Ruelle operator $L^t$, $t > 0$, acting on functions $f : \Omega \rightarrow \mathbb{R}$ of $L^\infty(P)$. It’s also possible to introduce the endomorphism $\alpha_t : L^\infty(P) \rightarrow L^\infty(P)$ defined as
\[ \alpha_t(\varphi) = \varphi \circ \Theta_t , \quad \forall \varphi \in L^\infty(P). \]

We relate in the next section the conditional expectation with respect to the $\sigma$-algebras $\mathcal{F}_t^+$ with the operators $L^t$ and $\alpha_t$, as follows:
\[ [L^t(f)](\Theta_t) = E(f|\mathcal{F}_t^+). \]

Given $V : \Omega \rightarrow \mathbb{R}$, such that it is constant in sets of the form $\{X_0 = c\}$ (i.e., $V$ depends only on the value of $w(0)$), we consider a Ruelle operator family $\tilde{L}_V^t$, for all $t > 0$, given by
\[ \tilde{L}_V^t (g)(w) = \left[ \frac{1}{f_V} L^t(e^{\int_0^t (V \circ \Theta_s)(\cdot)ds} g f_V) \right](w), \]
for any given $g$, where $f_V$ is a fixed function.

We are able to show the existence of an eigen-probability $\nu_V$ on $\Omega$, for the family $\tilde{L}_V^t$, for all $t > 0$, such that satisfies:

**Theorem A.** For any integrable $g \in L^\infty(P)$ and any positive $t$
\[ \int \left[ \frac{1}{e^{\int_0^t (V \circ \Theta_s)(\cdot)ds} f_V} \right] [L^t \left( \left[ e^{\int_0^t (V \circ \Theta_s)(\cdot)ds} f_V \right] (g) \circ \Theta_t \right)] d\nu_V = \int g d\nu_V. \]

The above functional equation is a natural generalization (for continuous time) of the similar one presented in Theorem 7.4 in [EL1] and proposition 2.1 in [EL2].

In [EL1] and [EL2] the important probability in the Bernoulli space is an eigen-probability $\nu$ for the Ruelle operator associated to a certain potential $V = \log H : \{1, 2, ..., d\}^\mathbb{N} \rightarrow \mathbb{R}$. This probability $\nu$ satisfies: for any $m \in \mathbb{N}$, $g \in C(X)$,
\[ \int \lambda_m E_m(\lambda_m^{-1} g) d\nu = \int g d\nu, \]
certain above theorem. We point out, however, that we are able to show this theorem for a continuous time dynamical systems a characterization of KMS states by means of the concepts like pressure, entropy, etc.

to the Jones index.
on the sigma-algebra \( \sigma \) the expected projection (with respect to a initial probability \( P \)) is the shift on the Bernoulli space \( \{1, 2, \ldots, d\}^\mathbb{N} \). Here \( E_m(f) = E(f|\sigma^{-m}(\mathcal{B})) \) denotes the expected projection (with respect to a initial probability \( P \) on the Bernoulli space) on the sigma-algebra \( \sigma^{-m}(\mathcal{B}) \), where \( \mathcal{B} \) is the Borel sigma-algebra and \( \Lambda[m] \) is associated to the Jones index.

We refer the reader to [CL] for a Thermodynamic view of \( C^* \)-Algebras which include concepts like pressure, entropy, etc.. We believe it will be important in the analysis of certain \( C^* \) algebras associated to continuous time dynamical systems a characterization of KMS states by means of the above theorem. We point out, however, that we are able to show this theorem for a certain \( \rho \) just for a quite simple function \( V \) as above. In a forthcoming paper we will consider more general potentials \( V \).

One could consider a continuous time version of the \( C^* \)-algebra considered in [EL1]. We just give an idea of what we are talking about. Given the above defined \( P \) for each \( t > 0 \), denote by \( s_t : \mathcal{L}^2(P) \rightarrow \mathcal{L}^2(P) \), the Koopman operator, where for \( \eta \in \mathcal{L}^2(P) \) we define \( (s_t\eta)(x) = \eta(\theta_t(x)) \).

Another important class of linear operators is \( M_f : \mathcal{L}^2(P) \rightarrow \mathcal{L}^2(P) \), for a given fixed \( f \in C(\Omega) \), and defined by \( M_f(\eta)(x) = f(x)\eta(x) \), for any \( \eta \in \mathcal{L}^2(P) \). We assume that \( f \) is such defines an operator on \( \mathcal{L}^2(P) \) (remember that \( \Omega \) is not compact).

In this way we can consider a \( C^* \)-algebra generated by the above defined operators (for all different values of \( t > 0 \)), then the concept of state, and finally given \( V \) and \( \beta \) we can ask about KMS states. There are several technical difficulties in the definition of the above \( C^* \)-algebra, etc... Anyway, at least formally, there is a need for finding \( \nu \) which is a solution of an equation of the kind we describe here. We need this in order to be able to obtain a characterization of KMS states by means of an eigen-probability for the continuous time Ruelle operator. This setting will be the subject of a future work. This was the motivation for our result.

With the operators \( \alpha \) and \( \mathcal{L} \) we can rewrite the theorem above as

\[
\rho_V(G_T^{-1}E_T(G_T\varphi)) = \rho_V(\varphi),
\]

for all \( \varphi \in \mathcal{L}^\infty \) and all \( T > 0 \), where, as usual, \( \rho_V(\varphi) = \int \varphi d\rho_V \), \( E_T = \alpha_T\mathcal{L}^T \) is in fact a projection on a subalgebra of \( \mathcal{B} \), and \( G_T : \Omega \rightarrow \mathbb{R} \) is given by

\[
G_T(x) = \exp\left( \int_0^T V(x(s))ds \right).
\]

For the map \( V : \Omega \rightarrow \mathbb{R} \), which is constant in cylinders of the form \( \{w_0 = i\} \), \( i \in \{1, 2, \ldots, n\} \), we denote by \( V_i \) the corresponding value. We also denote by \( V \) the diagonal matrix with the \( i \)-diagonal element equal to \( V_i \).
Now, consider $P_t^V = e^{t(L+V)}$. The classical Perron-Frobenius Theorem for such semigroup will be one of the main ingredients of our main proof.

As usual, we denote by $\mathcal{F}_s$ the sigma-algebra generated by $X_s$. We also denote by $\mathcal{F}_s^+$ the sigma-algebra generated $\sigma(\{X_u, s \leq u\})$. Note that a $\mathcal{F}_s^+$-measurable function $f(w)$ on $\Omega$ does depend of the value $w_s$.

We also denote by $I_A$ the indicator function of a measurable set $A$ in $\Omega$.

In a forthcoming paper we will analyze the case where the potential $V$ is of a more general type (not just $V : \Omega \to \mathbb{R}$, such that it's constant in sets of the form $\{X_0 = c\}$)

### 2 A continuous time Ruelle Operator

We present a quite general definition: let $\mathcal{X}$ and $X$ be a separable metric Radon spaces, $\hat{\mu}$ probability on $\mathcal{X}$, $\pi : \mathcal{X} \to X$ Borel measurable and $\mu = \pi_*\hat{\mu}$. Then there exists a Borel family of probabilities $\{\hat{\mu}_x\}_{x \in \mathcal{X}}$ on $\mathcal{X}$, uniquely determined $\mu$-a.e, such that,

1) $\hat{\mu}_x(\mathcal{X}\backslash \pi^{-1}(x)) = 0$, $\mu$-a.e;
2) $\int g(z)d\hat{\mu}(z) = \int_X \int_{\pi^{-1}(x)} g(z)d\hat{\mu}_x(z)d\mu(x)$.

We refer the reader to [AGS] for the proof. Here we will need a more simple version of this general result that can be obtained in an explicit form.

We consider the disintegration of $P$ given by the family of measures, indexed by the elements of $\Omega$ and $t > 0$ defined as follows: first, consider a sequence $0 = t_0 < t_1 < \cdots < t_2 < t_3 < \cdots < t_r$. Then for $w \in \Omega$ and $t > 0$ we have on cylinders:

$$
\mu_t^w([X_0 = a_0, \ldots, X_t = a_r]) = 
\begin{cases} 
\frac{1}{P_0^{w(t)}} P_{t-t_1}^{t_0} \cdots P_{a_{j+1}}^{w(t) a_j} P_{a_{j+1} a_j}^{a_{j+1}} P_0^{a_j} & \text{if } a_j = w(t_j), \ldots, a_r = w(t_r) \\
0 & \text{otherwise.}
\end{cases}
$$

**Proposition 2.1.** Under the above conditions on the sequence $0 = t_0 < t_1 < \cdots < t_{j-1} < t \leq t_j < \cdots < t_r$, we have that $\mu_t^w$ is the disintegration of $P$ along the fibers $\Theta_t^{-1}(.)$.

**Proof:** It is enough to show that for any integrable $f$

$$
\int \Omega f \, dP = \int \Omega \int_{\Theta_t^{-1}(w)} f(x) d\mu_t^w(x) dP(w).
$$

For doing that we can assume that $f$ is in fact the indicator of the cylinder $[X_0 = a_0, \ldots, X_t = a_r]$; then the right hand side becomes

$$
\int \int f d\mu_t^w(x) dP(w) = \int I_{[w(t_j) = a_j, \ldots, w(t_r) = a_r]} \frac{1}{P_0^{w(t)}} P_{t-t_1}^{t_0} \cdots P_{a_{j+1} a_j}^{w(t) a_{j+1}} P_0^{a_j} dP(w) =
\sum_{a=1}^{n} \int I_{[w(t) = a, w(t_j) = a_j, \ldots, w(t_r) = a_r]} \frac{1}{P_0^{w(t)}} P_{t-t_1}^{t_0} \cdots P_{a_{j+1} a_j}^{w(t) a_{j+1}} P_0^{a_j} dP(w) =
$$

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\[
\sum_{a=1}^{n} \frac{1}{p_{0}^a} P_{a_1a_0} P_{a_2a_1} \cdots P_{a_n} P_{a_1a_0} = P_{a_r} \cdots P_{a_0} = P([X_0 = a_0, \ldots, X_{t_r} = a_r]) = \int f dP.
\]

In the second inequality we use the fact that \( P \) is stationary.

The proof for a general \( f \) follows from standard arguments. \( \square \)

**Definition 2.2.** For \( t \) fixed we define the operator \( \mathcal{L}^t : \mathcal{L}^\infty(\Omega, P) \to \mathcal{L}^\infty(\Omega, P) \) as follows:

\[
\mathcal{L}^t(\varphi)(x) = \int_{y \in \Theta_t^{-1}(x)} \varphi(y) d\mu_t^x(y).
\]

**Remark 2.3.** The definition above can be rewritten as

\[
\mathcal{L}^t(\varphi)(x) = \int_{y \in D[0,t]} \varphi(yx) d\mu_t^x(yx),
\]

where the symbol \( yx \) means the concatenation of the path \( y \) with the translation of \( x \):

\[
xy(s) = \begin{cases} 
y(s) & \text{if } s \in [0,t) \\
x(s-t) & \text{if } s \geq t,
\end{cases}
\]

and, \( D[0,t) \) is the set of right-continuous functions from \( [0,t) \) to \( S \). This follows simply from the fact that, in this notation, \( \Theta_t^{-1}(x) \) is the set of \( y \in D[0,t) \).

Note that the value \( \lim_{s \to t} y(s) \) do not have to be necessarily equal to \( x(0) \).

In order to understand better the definitions above we apply the operator to some simple functions. For example, we can see the effect of \( \mathcal{L}^t \) on some indicator function of a given cylinder: consider the sequence \( 0 = t_0 < t_1 < \ldots < t_{j-1} < t \leq t_j < \ldots < t_{r} \) and then take \( f = I_{\{X_0 = a_0, X_{t_1} = a_1, \ldots, X_{t_r} = a_r\}} \). Then, for a path \( z \in \Omega \) such that \( z_{t_j-t} = a_j, \ldots, z_{t_r-t} = a_r \) (the future condition) we have

\[
\mathcal{L}^t(f)(z) = \frac{1}{p_0^{t_0}} P_{t_0}^{t-t_j-1} \cdots P_{t_{j-1}}^{t_{j-1}} P_{t_{j}^{t_{j+1}}} \cdots P_{a_1a_0} P_0^{a_0},
\]

otherwise (i.e., if the path \( z \) does not satisfy the condition above) we get \( \mathcal{L}^t(f)(z) = 0 \).

Note that if \( t_r < t \), then \( \mathcal{L}^t(f)(z) \) depends only on \( z_0 \). For example, if \( f = I_{\{X_0 = i_0\}} \) then

\[
\mathcal{L}^t(f)(z) = \int_{y \in D[0,t]} I_{\{X_0 = i_0\}}(yx) d\mu_t^x(yx) = \mu_t^x([X_0 = i_0]) = \frac{1}{p_0^{t_0}} P_{t_0,0}^{j_0} p_0^{i_0},
\]

In the case \( f = I_{\{X_0 = i_0, X_t = j_0\}} \), then \( \mathcal{L}^t(f)(z) = P_{t_0}^{j_0} P_{j_0,0}^{i_0} p_0^{i_0} \), if \( z_0 = j_0 \), and \( \mathcal{L}^t(f)(z) = 0 \) otherwise.

We describe bellow some properties of \( \mathcal{L}^t \).
Proposition 2.4. \( \mathcal{L}^t(1) = 1 \), where 1 is the function that maps every point in \( \Omega \) to 1.

Proof: Indeed,
\[
\mathcal{L}^t(1)(x) = \int_{y \in D(0,t)} 1(yx) \, d\mu_t^x(yx) = \int d\mu_t^x(yx) = \mu_t^x([X_t = x(0)]) = \\
\sum_{a=1}^{n} \mu_t^x([X_0 = a, X_t = x(0)]) = \frac{1}{p_0} \sum_{a=1}^{n} P_{x(0)}^a p_0^a = 1.
\]

We can also define the dual of \( \mathcal{L}^t \), denoted by \( (\mathcal{L}^t)^* \), acting on the measures. Then we get:

Proposition 2.5. For any positive \( t \) we have that \( (\mathcal{L}^t)^*(P) = (P) \).

Proof: For a fixed \( t \) we have that \( (\mathcal{L}^t)^*(P) = (P) \), because for any \( f \) of the form
\[
f = I_{\{X_0 = a_0,X_1 = a_1,\ldots,X_{r} = a_r\}}, \quad 0 = t_0 < t_1 < \ldots < t_{j-1} < t \leq t_j < \ldots < t_r,
\]
we get
\[
\int \mathcal{L}^t(f)(z) dP(z) = \sum_{b=1}^{n} \int_{\{X_0 = b\}} \mathcal{L}^t(f)(z) dP(z) = \\
\sum_{b=1}^{n} \int_{\{X_0 = b, X_{t_j-t} = a_j, \ldots, X_{t_r-t} = a_r\}} (z) dP(z) \frac{1}{p_0} P_{a_2 a_1}^{t_{j-1}} P_{a_1 a_0}^{t_1} p_0^a = \\
\sum_{b=1}^{n} P\{\{X_0 = b, X_{t_j-t} = a_j, X_{t_{j-1}+1-t} = a_{j+1}, \ldots, X_{t_{r-1}+1-t} = a_{r}\}\} \frac{1}{p_0} P_{a_2 a_1}^{t_{j-1}} P_{a_1 a_0}^{t_1} p_0^a = \\
\sum_{b=1}^{n} P_{a_2 a_1}^{t_{j-1}} P_{a_1 a_0}^{t_1} p_0^a = \\
\int f(w) dP(w).
\]

Proposition 2.6. Given \( t \in \mathbb{R}_+ \), and the functions \( \varphi, \psi \in \mathcal{L}^\infty(P) \), then
\[
\mathcal{L}^t(\varphi \times (\psi \circ \Theta_t))(z) = \psi(z) \times \mathcal{L}^t(\varphi)(z).
\]

Proof: Using the formula of \( \mathcal{L}^t \) given by Remark 2.3 we get
\[
\mathcal{L}^t(\varphi(\psi \circ \Theta_t))(x) = \int_{i \in D(0,t)} \varphi(i x)(\psi \circ \Theta_t)(i x) d\mu_t^x(i) = \\
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\[
\psi(x) \int_{i \in D[0,t]} \varphi(ix) d\mu^x_t(i) = (\psi \mathcal{L}^t(\varphi))(x) = \psi(x) \mathcal{L}^t(\varphi)(x),
\]

since \(\psi \circ \Theta_t(ix) = \psi(x)\), independently of \(i\).

We just recall that the last proposition can be restated as

\[
\mathcal{L}^t(\varphi \alpha_t(\psi)) = \psi \mathcal{L}^t(\varphi).
\]

Then we get:

**Proposition 2.7.** \(\alpha_t\) is the adjoint of \(\mathcal{L}^t\) on \(L^2(P)\).

**Proof:** From last two propositions

\[
\int \mathcal{L}^t(f) g \, dP = \int \mathcal{L}^t(f \times (g \circ \Theta_t)) \, dP = \int f \times (g \circ \Theta_t) \, dP = \int f \alpha_t(g) \, dP,
\]

as claimed.

We want to obtain conditional expectations in a more explicit form. For a given \(f\), recall that the function \(Z(w) = E(f|\mathcal{F}^+_t)\) is the \(Z\) (almost everywhere defined) \(\mathcal{F}^+_t\)-measurable function such that for any \(\mathcal{F}^+_t\)-measurable set \(B\) we have \(\int_B Z(w) \, dP(w) = \int_B f(w) \, dP(w)\).

**Proposition 2.8.** The conditional expectation is given by

\[
E(f|\mathcal{F}^+_t)(x) = \int f \, d\mu^x_t.
\]

**Proof:** For \(t\) fixed, consider a \(\mathcal{F}^+_t\)-measurable set \(B\). Then we have

\[
\int_B \int f \, d\mu^w_t \, dP(w) = \int (I_B(w) \int f \, d\mu^w_t) \, dP(w) = \\
\int \int (f I_B) \, d\mu^w_t \, dP(w) = \int f(w) I_B(w) \, dP(w) = \int_B f \, dP.
\]

Now we can relate the conditional expectation with respect to the \(\sigma\)-algebras \(\mathcal{F}^+_t\) with the operators \(\mathcal{L}^t\) and \(\alpha_t\) as follows:

**Proposition 2.9.** \([\mathcal{L}^t(f)](\Theta_t) = E(f|\mathcal{F}^+_t)\).
Proof: This follows from the fact that for any 
\[ B = \{ X_{s_1} = b_1, X_{s_2} = b_2, ..., X_{s_u} = b_u \}, \]  
with \( t < s_1 < ... < s_u \), we have \( I_B = I_A \circ \Theta_t \) for some measurable \( A \) and
\[
\int_B \mathcal{L}^t(f)(\Theta_t(w))dP(w) = \int I_B(w)\mathcal{L}^t(f)(\Theta_t(w))dP(w) = \\
\int (I_A \circ \Theta_t)(w) \mathcal{L}^t(f)(\Theta_t(w))dP(w) = \int I_A(w)\mathcal{L}^t(f)(w)dP(w) = \\
\int I_A(\Theta_t(w)) f(w) dP(w) = \int_B f(w)dP(w).
\]

3 The modified operator Ruelle Operator associated to \( V \)

We consider \( V : \Omega \rightarrow \mathbb{R} \), such that is constant in sets of the form \( \{ X_0 = c \} \). We are interested in the operator obtained by the perturbation of the operator \( \mathcal{L}^t \) by \( V \).

Definition 3.1. We define \( G_t : \Omega \rightarrow \mathbb{R} \) as
\[
G_t(x) = \exp \left( \int_0^t V(x(s))ds \right)
\]

Definition 3.2. We define the \( G \)-weighted transfer operator \( \mathcal{L}_V^t : \mathcal{L}^\infty(\Omega, P) \rightarrow \mathcal{L}^\infty(\Omega, P) \) acting on measurable functions \( f \) (of the form \( f = I_{\{ X_0 = a_0, X_{s_1} = a_1, ..., X_{t_r} = a_r \}} \)) by
\[
\mathcal{L}_V^t(f)(w) := \mathcal{L}^t(G_t f) = \\
\mathcal{L}^t(e^{\int_0^t V(\Theta_s)ds} f) = \sum_{b=1}^n \mathcal{L}^t(e^{\int_0^t V(\Theta_s)ds} I_{\{ X_i = b \}} f)(w).
\]

We will show next the existence of an eigenfunction and an eigen-measure for such operator \( \mathcal{L}_V^t \). First we need the following:

Theorem (Perron-Frobenius for continuous time). ([S] page 111) Given \( L, p_0 \) and \( V \) as above, there exists

a) a unique positive function \( u_V : \Omega \rightarrow \mathbb{R} \), constant equal to the value \( u_V^i \) in each cylinder \( X_0 = i, i \in \{1, 2, .., n\} \), (sometimes we will consider \( u_V \) as a vector in \( \mathbb{R}^n \)).
b) a unique probability vector $\mu_V$ in $\mathbb{R}^n$ (a probability over over the set \{1, 2, ..., n\} such that $\mu_V(\{i\}) > 0, \forall i$), that is,

$$\sum_{i=1}^{n} (u_V)_i (\mu_V)_i = 1,$$

c) a real positive value $\lambda(V)$, such that for any positive $s$

$$e^{-s\lambda(V)} u_V e^{s(L+V)} = u_V.$$  

d) Moreover, for any $v = (v_1, ..., v_n) \in \mathbb{R}^n$

$$\lim_{t \to \infty} e^{-t\lambda(V)} v e^{t(L+V)} = (\sum_{i=1}^{n} v_i (\mu_V)_i) u_V,$$

e) for any positive $s$

$$e^{-s\lambda(V)} e^{s(L+V)} \mu_V = \mu_V.$$  

From property e) it follows that

$$(L + V) \mu_V = \lambda(V) \mu_V,$$

or

$$(L + V - \lambda(V) I) \mu_V = 0.$$  

From c) it follows that

$$u_V (L + V) = \lambda(V) u_V,$$

or

$$u_V (L + V - \lambda(V) I) = 0.$$  

We point out that e) means that for any positive $t$ we have $(P^t_V)^* \mu_V = e^{\lambda(V)t} \mu_V$.

Note that when $V = 0$, then $\lambda(V) = 0$, $\mu_V = \mu^0$ and $u_V$ is constant equal to 1.

Now we return to our setting: for each $i_0$ and $t$ fixed one can consider the probability $\mu_{i_0}^t$ defined over the sigma-algebra $\mathcal{F}_t^{-} = \sigma(\{X_s | s \leq t\})$ with support on $\{X_0 = i_0\}$ such that for cylinder sets with $0 < t_1 < ... < t_r \leq t$

$$\mu_{i_0}^t (\{X_0 = i_0, X_{t_1} = a_1, ..., X_{t_{r-1}} = a_{r-1}, X_t = j_0\}) = P_{j_0a_r}^{t-t_r} \cdots P_{a_2a_1}^{t_{t_{r-2}-t_1}} P_{a_1i_0}^{t_1}.$$  

The probability $\mu_{i_0}^t$ is not stationary.

We denote by $Q(j, i)_t$ the $i, j$ entry of the matrix $e^{t(L+V)}$, that is $(e^{t(L+V)})_{j,i}$.

It is known ([K] page 52 or [S] Lemma 5.15) that

$$Q(j_0, i_0)_t = E_{(X_0 = i_0)}\{e^{\int_0^t (V \circ \Theta_s)(w)dw}; X(t) = j_0\} = \ldots.$$
\[
\int I_{\{X_t = j_0\}} e^{\int_0^t (V \circ \Theta_s)(w) \, ds} d\mu_{j_0}^t(w).
\]

For example,
\[
\int I_{\{X_t = j_0\}} e^{\int_0^t (V \circ \Theta_s)(w) \, ds} dP = \sum_{i=1,2,\ldots,n} Q(j_0, i) p_0^i.
\]

In the particular case where \( V \) is constant equal 0, then \( p^0 = \mu_V \) and \( \lambda(V) = 0 \).

We denote by \( f = f_V \), where \( f(w) = f(w(0)) \), the density of probability \( \mu_V \) in \( S \) with respect to the probability \( p^0 \) in \( S \).

Therefore, \( \int f dp^0 = 1 \).

**Proposition 3.3.** \( f_V(w) = \frac{\mu_V(w)}{p^0(w)} = \frac{(\mu_V)_{w(0)}}{(p^0)_{w(0)}} \), \( f_V : \Omega \to \mathbb{R} \), is an eigenfunction for \( L_V^t \) with eigenvalue \( e^{t \lambda(V)} \).

**Proof:** Note that \( \frac{\mu_V}{p^0} = \sum_{c=1}^n \frac{\mu_V(c)}{p^0(c)} I_{\{X_0 = c\}} \).

For a given \( w \), denote \( w(0) \) by \( j_0 \), then conditioning
\[
L_V^t \left( \frac{\mu_V}{p^0} \right)(w) = \sum_{c=1}^n \sum_{b=1}^n L_V^t \left( \frac{\mu_V(c)}{p^0(c)} I_{\{X_0 = c\}} I_{\{X_t = b\}} \right)(w).
\]

Consider \( c \) fixed, then for \( b = j_0 \) we have
\[
L_V^t \left( I_{\{X_0 = c\}} I_{\{X_t = j_0\}} \right)(w) = \frac{Q(j_0, c)_t p_0^c}{p^0_{j_0}},
\]
and for \( b \neq j_0 \), we have \( L_V^t \left( I_{\{X_0 = c\}} I_{\{X_t = b\}} \right)(w) = 0 \).

Finally, for any \( t > 0 \)
\[
L_V^t \left( \frac{\mu_V}{p^0} \right)(w) = \sum_{c=1}^n \frac{\mu_V(c)}{p^0(c)} \frac{Q(j_0, c)_t p_0^c}{p^0_{j_0}} = \sum_{c=1}^n \frac{\mu_V(c)}{p^0(c)} (e^{(L+V)}c)_{j_0,c} \frac{p_0^c}{p^0_{j_0}} = e^{t \lambda(V)} \left( \frac{\mu_V}{p^0} \right)_{j_0} = e^{t \lambda(V)} \left( \mu_V \right)(w),
\]

because \( e^{t(L+V)}(\mu_V) = e^{t \lambda(V)}(\mu_V) \).

Therefore, for any \( t > 0 \) the function \( f_V = \frac{\mu_V}{p^0} \) (that depends only on \( w(0) \)) is an eigenfunction for the operator \( L_V^t \) associated to the eigenvector \( e^{t \lambda(V)} \).

The above result shows that the eigenfunction for the Ruelle operator associated to the potential \( V \) is the Radon-Nykodin derivative for \( \mu_V \) with respect to \( p^0 \);
Definition 3.4. Consider for each $t$ the operator acting on $g$ given by

$$\hat{\mathcal{L}}_V^t(g)(w) = \left[ \frac{1}{f_V} \mathcal{L}_V(e^{\mu(V-V_0-L)}(\cdot)V)(\cdot) ds \right] f_V(g) \right](w)$$

From the above $\hat{\mathcal{L}}_V^t(1) = 1$, for all positive $t$.

We present some examples: note that by conditioning, if $g = I_{\{X_0 = a_0, X_{t_1} = a_1, X_{t_2} = a_2, X_t = a_3\}}$, with $0 < t_1 < t_2 < t$, then

$$\hat{\mathcal{L}}_V^t(g)(w) = \frac{1}{\mu_V(a_3)} \frac{p_{a_3}^0}{\mu_V(a_3)} e_{a_3 a_2}^{(t-t_2)} e_{a_2 a_1}^{(t_1-t_2)} e_{a_1 a_0}^{(t_1-t)} \frac{\mu_V(a_0)}{p_{a_0}^0} =$$

$$\frac{1}{\mu_V(a_3)} e_{a_3 a_2}^{(t-t_2)} e_{a_2 a_1}^{(t_1-t_2)} e_{a_1 a_0}^{(t_1-t)} \mu_V(a_0),$$

for $w$ such that $w_0 = a_3$, and $\hat{\mathcal{L}}_V^t(g)(w) = 0$ otherwise.

Moreover, for $g = I_{\{X_0 = a_0, X_{t_1} = a_1, X_{t_2} = a_2\}}$, with $0 < t_1 < t < t_2$, then

$$\hat{\mathcal{L}}_V^t(g)(w) = \frac{1}{\mu_V(a_2)} e_{a_2 a_1}^{(t-t_1)} e_{a_1 a_0}^{(t_1-t)} \mu_V(a_0),$$

for $w$ such that $w_0 = a$, $w_{t_2-t} = a_2$, and $\hat{\mathcal{L}}_V^t(g)(w) = 0$ otherwise.

Consider now the dual operator $(\hat{\mathcal{L}}_V^t)^*$. For $t$ fixed consider the transformation in the set of measures $\mu$ on $\Omega$ given by $(\hat{\mathcal{L}}_V^t)^*(\mu) = \nu$.

Theorem 1. For each $t$ there exists a probability measure $\nu_V$ on $(\Omega, \mathcal{B})$ which is fixed for such transformation $(\hat{\mathcal{L}}_V^t)^*$. The probability $\nu_V$ does not depend on $t$.

Proof:

We have to show that there exists $\nu_V$ such that for all $t > 0$ and for all $g$ we have

$$\int \hat{\mathcal{L}}_V^t(g) \, d\nu_V = \int g \, d\nu_V.$$

Remember that, $\hat{\mathcal{L}}_V^t(1) = 1$, therefore, if $\mu$ is a probability, then $(\hat{\mathcal{L}}_V^t)^*(\mu) = \nu$ is also a probability.

Denote by $\nu = \nu_V$ the probability obtained in the following way, for

$$g = I_{\{X_0 = a_0, X_{t_1} = a_1, X_{t_2} = a_2, \ldots, X_{t_s-1} = a_{r-1}, X_r = a_r\}},$$

with $0 < t_1 < t_2 < \ldots < t_s < \ldots < t_r$, we define

$$\int g(w) \, d\nu(w) = e_{a_0 a_{r-1}}^{(t_r-t_{r-1})} \ldots e_{a_{r-1} a_1}^{(t_1-t_2)} e_{a_2 a_1}^{(t_1-t_2)} e_{a_1 a_0}^{(t_1-t)} \mu_V(a_0).$$

It is easy to see that these probability transitions satisfy the Kolmogorov compatibility conditions. In section 4.1 and in Theorem 1.1 (same section) in [EK] it is described these
conditions. We point out that the space we consider is separable and complete according to chapter 3 in the same book.

In order to show that \( \nu \) is a probability we have to use the fact that \( \sum_{c \in \mathcal{S}} \mu_V(c) = 1 \). For example, \( \int I_{\{X_0 = c\}} \, d\nu = \mu_V(c) \). Moreover,

\[
\int 1 \, d\nu = \sum_c \sum_a \int I_{\{X_1 = c, X_0 = a\}} \, d\nu = \sum_c \sum_a e^{t(L-V-I)} \mu_V(a) = \sum_c \mu_V(c) = 1.
\]

Suppose \( t \) is such that \( 0 < t_1 < t_2 < \ldots < t_{s-1} < t \leq t_s < \ldots < t_r \), then

\[
z(w) = \hat{L}_V^t(g)(w) = \frac{1}{\mu_V(a)} e^{(t-t_{s-1})(L+V-I)} \ldots e^{(t_{t_1} - t_1)(L+V-I)} e^{L(I)(L+V-I)} \mu_V(a),
\]

for \( w \) such that \( w(0) = a, w_{t_{s-1}} = a_s, w_{t_{s+1} - t} = a_{s+1}, \ldots, w_{t_r - t} = a_r \), and \( \hat{L}_V^t(g)(w) = 0 \) otherwise. Note that \( z(w) = \hat{L}_V^t(g)(w) \) depends only on \( w_0, w_{t_{s-1}}, w_{t_{s+1} - t}, \ldots, w_{t_r - t} \).

We have to show that for any \( g \) we have \( \int g \, d\nu = \int \hat{L}_V^t(g) \, d\nu \).

Now,

\[
\int z(w) \, d\nu(w) = \int \sum_{c \in \mathcal{S}} I_{\{X_0 = c, X_{t_{s-1}} = a_s, X_{t_{s+1} - t} = a_{s+1}, \ldots, X_{t_r - t} = a_r\}} z(w) \, d\nu(w) =
\]

\[
\sum_{c \in \mathcal{S}} \nu(\{X_0 = c, X_{t_{s-1}} = a_s, X_{t_{s+1} - t} = a_{s+1}, \ldots, X_{t_r - t} = a_r\})
\]

\[
\frac{1}{\mu_V(c)} e^{(t-t_{s-1})(L+V-I)} \ldots e^{(t_{t_1} - t_1)(L+V-I)} e^{L(I)(L+V-I)} \mu_V(a) =
\]

\[
\sum_{c \in \mathcal{S}} e^{(t_r - t_{r-1})(L+V-I)} \ldots e^{(t_{s+1} - t_s)(L+V-I)} \mu_V(c)
\]

\[
\frac{1}{\mu_V(c)} e^{(t-t_{s-1})(L+V-I)} \ldots e^{(t_{t_1} - t_1)(L+V-I)} e^{L(I)(L+V-I)} \mu_V(a) =
\]

\[
\sum_{c \in \mathcal{S}} e^{(t_r - t_{r-1})(L+V-I)} \ldots e^{(t_{s+1} - t_s)(L+V-I)}
\]

\[
\int g \, d\nu.
\]

The claim for the general \( g \) follows from the above result. Therefore, \( (\hat{L}_V^t)^*(\nu_V) = \nu_V \).

\[\square\]
**Definition 3.5.** Consider the stationary probability $\rho_V = f_V \nu_V$ on $\Omega$. We call it the equilibrium state for $V$.

**Definition 3.6.** We call the probability $\nu_V$ on $\Omega$ the Gibbs state for $V$.

**Proposition 3.7.** For any integrable $f, g \in \mathcal{L}^\infty(P)$ and any positive $t$

$$
\int \mathcal{L}_V^t(f) \ g \ d\nu_V = \int \mathcal{L}_V^t(f \circ \theta_t) \ d\nu_V = \int f \circ \theta_t \ d\nu_V.
$$

Now we can prove our main result.

**Theorem A.** For any integrable $g \in \mathcal{L}^\infty(P)$ and any positive $t$

$$
\int e^{-\int_0^t (V \circ \theta_s) \ ds} \left[ \left( \frac{1}{f_V} \mathcal{L}_V^t (e^{\int_0^t (V \circ \theta_s) \ ds} g \ f_V) \right) \circ \theta_t \right] \ d\nu_V = \int g \ d\nu_V.
$$

**Proof:**

Note first that if $w$ is such that $w(0) = c$, then

$$
\mathcal{L}_V^t(f_V(w)) = \sum_j \mathcal{L}_V^t(I_{\{X_0=j\}}) \ f_V(j) = \sum_j \mathcal{L}_V^t(I_{\{X_0=j\}}) \frac{\mu_V(j)}{p_j^0} = \sum_j \frac{1}{p_c^0} P_{cj} \frac{\mu_V(j)}{p_j^0} = \sum_j \frac{1}{p_c^0} P_{cj} \mu_V(j) = \frac{\mu_V(c)}{p_c^0} = f_V(w).
$$

Therefore, for all $w$

$$
\frac{1}{f_V} \mathcal{L}_V^t(f_V(w)) = 1.
$$

Finally,

$$
\int e^{-\int_0^t (V \circ \theta_s) \ ds} \left[ \left( \frac{1}{f_V} \mathcal{L}_V^t (e^{\int_0^t (V \circ \theta_s) \ ds} g \ f_V) \right) \circ \theta_t \right] \ d\nu_V = \\
\int e^{-\int_0^t (V \circ \theta_s - \lambda) \ ds} \left[ \left( \frac{1}{f_V} \mathcal{L}_V^t (e^{\int_0^t (V \circ \theta_s - \lambda) \ ds} g \ f_V) \right) \circ \theta_t \right] \ d\nu_V = \\
\int \mathcal{L}_V^t (e^{-\int_0^t (V \circ \theta_s - \lambda) \ ds} e^{\int_0^t (V \circ \theta_s - \lambda) \ ds} f_V) \ d\nu_V = \\
\int \mathcal{L}_V^t (e^{-\int_0^t (V \circ \theta_s - \lambda) \ ds} f_V g) \ d\nu_V = \\
\int \mathcal{L}_V^t (f_V) \ d\nu_V = \int gd\nu_V.
$$
Bibliography


