SELF-SIMILAR GRAPHS
A UNIFIED TREATMENT OF KATSURA AND NEKRASHEVYCH C*-ALGEBRAS

Ruy Exel and Enrique Pardo

Given a graph E, an action of a group G on E, and a G-valued cocycle φ on the edges of E, we define a C*-algebra denoted $\mathcal{O}_{G,E}$, which is shown to be isomorphic to the tight C*-algebra associated to a certain inverse semigroup $\mathcal{S}_{G,E}$ built naturally from the triple (G, E, φ). As a tight C*-algebra, $\mathcal{O}_{G,E}$ is also isomorphic to the full C*-algebra of a naturally occurring groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. We then study the relationship between properties of the action, of the groupoid and of the C*-algebra, with an emphasis on situations in which $\mathcal{O}_{G,E}$ is a Kirchberg algebra. Our main applications are to Katsura algebras and to certain algebras constructed by Nekrashevych from self-similar groups. These two classes of C*-algebras are shown to be special cases of our $\mathcal{O}_{G,E}$, and many of their known properties are shown to follow from our general theory.

1. Introduction.

The purpose of this paper is to give a unified treatment to two classes of C*-algebras which have been studied in the past few years from rather different points of view, namely Katsura algebras [**18**], and certain algebras constructed by Nekrashevych [**24**], [**26**] from self-similar groups.

The realization that these classes are indeed closely related, as well as the fact that they could be given a unified treatment, came to our mind as a result of our attempt to understand Katsura's algebras $\mathcal{O}_{A,B}$ from the point of view of inverse semigroups. The fact, proven by Katsura in [**18**], that all Kirchberg algebras in the UCT class may be described in terms of his $\mathcal{O}_{A,B}$ was, in turn, a strong motivation for that endeavor.

While studying $\mathcal{O}_{A,B}$, it slowly became clear to us that the two matricial parameters A and B, present in Katsura's construction, play very different roles. The reader acquainted with Katsura's work will easily recognize that the matrix A is destined to be viewed as the edge matrix of a graph, but it took us much longer to realize that B should be thought of as providing parameters for an action of the group $\mathbb{Z}$ on the graph given by A. In trying to understand these different roles, some interesting arithmetic popped up sparking a connection with the work done by Nekrashevych [**26**] on the C*-algebra $\mathcal{O}_{(G,X)}$ associated to a self-similar group (G, X).

Date: 27 September 2014.

2010 *Mathematics Subject Classification:* 46L05, 46L55.

Key words and phrases: Kirchberg algebra, Katsura algebra, Nekrashevych algebra, tight representation, inverse semigroup, groupoid, groupoid C*-algebra.

The first-named author was partially supported by CNPq. The second-named author was partially supported by PAI III grants FQM-298 and P11-FQM-7156 of the Junta de Andalucía and by the DGI-MICINN and European Regional Development Fund, jointly, through Project MTM2011-28992-C02-02.

While Nekrashevych's algebras contain a Cuntz algebra, Katsura's algebras contain a graph C*-algebra. This fact alone ought to be considered as a hint that self-similar groups lie in a much bigger class, where the group action takes place on the path space of a graph, rather than on a rooted tree (which, incidentally, is the path space of a bouquet of circles).

One of the first important applications of the idea of self-similarity in group theory is in constructing groups with exotic properties [**15**], [**16**]. Many of these are defined as subgroups of the group of all automorphisms of a tree. Having been born from automorphisms, it is natural that the theory of self-similar groups generally assumes that the group acts *faithfully* on its tree (see, e.g. [**26**: Definition 2.1]).

However, based on the example provided by Katsura's algebras, we decided that perhaps it is best to view the group on its own, the action being an extra ingredient.

The main idea behind self-similar groups, namely the equation

$$g(xw) = yh(w) \tag{1.1}$$

appearing in [**26**: Definition 2.1], and the subsequent notion of *restriction*, namely

$$g|_x := h,$$

depend on faithfulness, since otherwise the group element h appearing in (1.1) would not be unique and therefore will not be well defined as a function of g and x. Working with non-faithful group actions we were forced to postulate a functional dependence

$$h = \varphi(g, x),$$

and we were surprised to find that the natural properties expected of φ are that of a group cocycle.

To be precise, the ingredients needed in our generalization of self-similar groups are: a countable discrete group G, an action

$$G \times E \to E$$

of G on a finite graph $E = (E^0, E^1, r, d)$, and a one-cocycle

$$\varphi : G \times E^1 \to G$$

for the action of G on the edges of E.

Starting with this data (which we assume satisfies a few other natural axioms) we construct an action of G on the space of finite paths E^* which satisfies the "self-similarity" equation

$$g(\alpha\beta) = (g\alpha)\big(\varphi(g, \alpha)\beta\big), \quad \forall\, g \in E, \quad \forall\, \alpha, \beta \in E^*.$$

Adopting a philosophy similar to that embraced by Katsura and Nekrashevych, we define a C*-algebra, denoted

$$\mathcal{O}_{G,E},$$

in terms of generators and relations inspired by the above group action. The study of $\mathcal{O}_{G,E}$ is, thus, the purpose of this paper.

Given a self-similar group (G, X), if we consider X as the set of edges of a graph with a single vertex, and if we define $\varphi(g, x) = g|_x$, then our $\mathcal{O}_{G,E}$ coincides with Nekrashevych's $\mathcal{O}_{(G,X)}$.

On the other hand, if we are given two integer $N \times N$ matrices A and B, with $A_{i,j} \geq 0$, for all i and j, we may form a graph E with vertex set $E^0 = \{1, 2, \ldots, N\}$ and with $A_{i,j}$ edges from vertex i to vertex j.

We may then use B to define an action of $\mathbb{Z}$ on E, by fixing all vertices and acting on the set of edges as follows: denote the set of edges in E from i to j by

$$\{e_{i,j,n} : 0 \leq n < A_{i,j}\}.$$

Given $m \in \mathbb{Z}$, and given an edge $e_{i,j,n}$, in order to define $\sigma_m(e_{i,j,n})$, we first perform the Euclidean division of $mB_{i,j} + n$ by $A_{i,j}$, say

$$mB_{i,j} + n = \hat{k}A_{i,j} + \hat{n}$$

with $0 \leq \hat{n} < A_{i,j}$. We then put

$$\sigma_m(e_{i,j,n}) := e_{i,j,\hat{n}},$$

so that the group element m permutes the $A_{i,j}$ edges from i to j in the same way that addition by $mB_{i,j}$, modulo $A_{i,j}$, permutes the integers $\{0, 1, \ldots, A_{i,j} - 1\}$.

The quotient $\hat{k}$ in the above Euclidean division also plays an important role, namely in the definition of the cocycle:

$$\varphi(m, e_{i,j,n}) := \hat{k}.$$

In possession of the graph, the action of $\mathbb{Z}$, and the cocycle φ constructed above, we apply our construction and we find that $\mathcal{O}_{G,E}$ is isomorphic to Katsura's $\mathcal{O}_{A,B}$.

So, both Nekrashevych's and Katsura's algebras become special cases of our construction. We therefore believe that the project of studying such group actions on path spaces as well as the corresponding algebras is of great importance.

Taking the first few steps in this direction we have been able to describe $\mathcal{O}_{G,E}$ as the C*-algebra of an étale groupoid $\mathcal{G}_{G,E}$, whose construction is remarkably similar to the groupoid associated to the relation of "tail equivalence with lag" on the path space, as described by Kumjian, Pask, Raeburn and Renault in [**20**].

The first similarity is that our groupoid $\mathcal{G}_{G,E}$ has the exact same unit space as the corresponding graph groupoid, namely the infinite path space. The second, and most surprising similarity is that $\mathcal{G}_{G,E}$ is also described by a *lag* function, except that the values of the lag are not integer numbers, as in [**20**], but lie in a slightly more complicated group, namely the semi-direct product of the *corona* group of G by the right shift automorphism (see below for precise definitions).

We would like to stress that, like Nekrashevych's groupoid [**26**: Theorem 5.1], our groupoid $\mathcal{G}_{G,E}$ is constructed as a groupoid of germs. However, departing from Nekrashevych's techniques, we use Patterson's [**27**] notion of "germs", rather than the one employed in [**26**: Section 5]. While agreeing in many cases, such as when the action is topologically free (see below for the precise definition), the former has a much better chance of producing Hausdorff groupoids and, in our case, we are able to give a precise characterization of Hausdorffness in terms of a property we call *pseudo freeness* (see below for the precise definition).

The techniques we use to give $\mathcal{O}_{G,E}$ a groupoid model bear heavily on the theory of tight representations of inverse semigroups developed by the first named author in [**6**]. In particular, from our initial data we construct an abstract inverse semigroup $\mathcal{S}_{G,E}$ and show that $\mathcal{O}_{G,E}$ is the universal C*-algebra for tight representations of $\mathcal{S}_{G,E}$.

In another direction we again take inspiration from Nekrashevych [**24**] and give a description of $\mathcal{O}_{G,E}$ as a Cuntz-Pimsner algebra for a very natural correspondence M over the algebra

$$C(E^0) \rtimes G.$$

As a result we are able to prove that $\mathcal{O}_{G,E}$ is nuclear when G is amenable.

We briefly study the natural representation of the graph C*-algebra $C^*(E)$ [**30**] into $\mathcal{O}_{G,E}$, which turns out to be faithful. Also, we study the natural representation of the group G into $\mathcal{O}_{G,E}$, which turns out to be faithful when the triple (G, E, φ) satisfies pseudo freeness, but fails in general.

Simplicity of $\mathcal{O}_{G,E}$ is also discussed by using our description of this algebra as a groupoid C*-algebra and employing results from [**4**]. In doing so, it is crucial to determine when is $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ a Hausdorff, minimal essentially principal groupoid. To this end, we strongly rely on results obtained by both authors in [**12**] about characterization of minimality and essential irreducibility for the groupoid of germs of a general $*$-inverse semigroup. We then specialize these results to the particular context of the inverse semigroup $\mathcal{S}_{G,E}$. Hence, we characterize Hausdorffness of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ in terms of the existence of finitely many minimal strongly fixed paths (see below for a precise definition).

Also, we characterize minimality of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ in terms of weak G-transitivity of the graph (see Section 13 for a definition of this concept). We then obtain a natural generalization of the analog result obtained in [**9**] for Exel-Laca algebras.

We also show that being essentially principal is related to the topological freeness of the action of $\mathcal{S}_{G,E}$ on the infinite path space. In this sense, we obtain a characterization that relies on the existence of entries for any circuit of the graph, plus a formal condition which forces any element of G fixing open sets to be tighly related to the existence of suitable minimal strongly fixed paths.

Moreover, we give sufficient conditions on $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ to guarantee its local contractiveness (see e.g. [**1**] for a definition); this property turns out to be a consequence of essential principality, so that any simple algebra in the class $\mathcal{O}_{G,E}$ will be purely infinite simple.

With the machinery developped we are then able to give a characterization of simplicity (and so pure infinite simplicity) for $\mathcal{O}_{G,E}$ when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff.

Finally, we revisit the case of Nekrashevych and Katsura algebras, giving a picture of the properties enjoyed by these algebras that turns out to be more general than the ones given by Nekrashevych or Katsura.

Some of the results in the present paper appeared in the preprints [**10**] and [**11**], which in turn are to be replaced by the present work.

We would also like to mention [**13**] and [**32**], which are strongly related to the algebras we study here. In [**13**] conditions are given for $\mathcal{O}_{G,E}$ to be a partial crossed product and in [**32**] an interesting connection with Zappa-Szép products is made.

Part of this work was done during visits of the second named author to the Departamento de Matemática da Universidade Federal de Santa Catarina (Florianópolis, Brasil) and he would like to express his thanks to the host center for its warm hospitality. Both authors thank Benjamin Steinberg for interesting discussions on topological freeness of actions, and Hausdorffness of groupoids.

2. Groups acting on graphs.

Let $E = (E^0, E^1, r, d)$ be a directed graph, where E^0 denotes the set of *vertices*, E^1 is the set of *edges*, r is the *range* map, and d is the *source*, or *domain* map.

By definition, a *source* in E is a vertex $x \in E^0$, for which $r^{-1}(x) = \varnothing$. Thus, when we say that a graph has *no sources*, we mean that $r^{-1}(x) \neq \varnothing$, for all $x \in E^0$.

By an *automorphism* of E we shall mean a bijective map

$$\sigma : E^0 \,\dot\cup\, E^1 \to E^0 \,\dot\cup\, E^1$$

such that $\sigma(E^i) \subseteq E^i$, for $i = 0, 1$, and moreover such that $r \circ \sigma = \sigma \circ r$, and $d \circ \sigma = \sigma \circ d$, on E^1. It is evident that the collection of all automorphisms of E forms a group under composition.

By an action of a group G on a graph E we shall mean a group homomorphism from G to the group of all automorphisms of E.

If X is any set, and if σ is an action of a group G on X, we shall say that a map

$$\varphi : G \times X \to G$$

is a *one-cocycle* for σ, when

$$\varphi(gh, x) = \varphi\big(g, \sigma_h(x)\big)\varphi(h, x), \tag{2.1}$$

for all $g, h \in G$, and all $x \in X$. In this case, plugging $g = h = 1$, above, we see that necessarily

$$\varphi(1, x) = 1, \tag{2.2}$$

for every x.

2.3. ▶ Standing Hypothesis. Throughout this work we shall let G be a countable discrete group, E be a finite graph with no sources, σ be an action of G on E, and

$$\varphi : G \times E^1 \to G$$

be a one-cocycle for the restriction of σ to E^1, which moreover satisfies

$$\sigma_{\varphi(g,e)}(x) = \sigma_g(x), \quad \forall\, g \in G, \quad \forall\, e \in E^1, \quad \forall\, x \in E^0. \tag{2.3.1}$$

The assumptions that E is finite and has no sources will in fact only be used in the next section and it could probably be removed by using well known graph C*-algebra techniques.

By a *path* in E of *length* $n \geq 1$ we shall mean any finite sequence of the form

$$\alpha = \alpha_1 \alpha_2 \ldots \alpha_n,$$

where $\alpha_i \in E^1$, and $d(\alpha_i) = r(\alpha_{i+1})$, for all i (this is the usual convention when treating graphs from a categorical point of view, in which functions compose from right to left). The *range* of α is defined by

$$r(\alpha) = r(\alpha_1),$$

while the *source* of α is defined by

$$d(\alpha) = d(\alpha_n).$$

A vertex $x \in E^0$ will be considered a path of length zero, in which case we set $r(x) = d(x) = x$.

For every integer $n \geq 0$ we denote by E^n the set of all paths in E of length n (this being consistent with the already introduced notations for E^0 and E^1). Finally, we denote by E^* the sets of all finite paths, and by $E^{\leq n}$ the set of all paths of length at most n, namely

$$E^* = \bigcup_{k \geq 0} E^k, \quad \text{and} \quad E^{\leq n} = \bigcup_{k=0}^{n} E^k.$$

We will often employ the operation of *concatenation* of paths. That is, if (and only if) α and β are paths such that $d(\alpha) = r(\beta)$, we will denote by $\alpha\beta$ the path obtained by juxtaposing α and β.

In the special case in which α is a path of length zero, the concatenation $\alpha\beta$ is allowed if and only if $\alpha = r(\beta)$, in which case we set $\alpha\beta = \beta$. Similarly, when $|\beta| = 0$, then $\alpha\beta$ is defined iff $d(\alpha) = \beta$, and then $\alpha\beta = \alpha$.

We would now like to describe a certain extension of σ and φ to finite paths.

2.4. Proposition. *Under the assumptions of (2.3) there exists a unique pair (σ^*, φ^*), formed by an action σ^* of G on E^* (viewed simply as a set), and a one-cocycle φ^* for σ^*, such that, for every $n \geq 0$, every $g \in G$, and every $x \in E^0$, one has that:*

(i) $\sigma^*_g = \sigma_g$, *on* $E^{\leq 1}$,

(ii) $\varphi^*(g, x) = g$,

(iii) $\varphi^* = \varphi$, *on* $G \times E^1$,

(iv) $\sigma^*_g(E^n) \subseteq E^n$,

(v) $r \circ \sigma^*_g = \sigma_g \circ r$, *on* E^n,

(vi) $d \circ \sigma^*_g = \sigma_g \circ d$, *on* E^n,

(vii) $\sigma_{\varphi^*(g,\alpha)}(x) = \sigma_g(x)$, *for all* $\alpha \in E^n$,

(viii) σ^*_1 *is the identity*[1] *on* E^n,

(ix) $\sigma^*_g(\alpha\beta) = \sigma^*_g(\alpha)\ \sigma^*_{\varphi^*(g,\alpha)}(\beta)$, *provided* α *and* β *are finite paths with* $\alpha\beta \in E^n$,

(x) $\varphi^*(g, \alpha\beta) = \varphi^*\big(\varphi^*(g,\alpha), \beta\big)$, *provided* α *and* β *are finite paths with* $\alpha\beta \in E^n$.

Proof. Initially notice that, once (v), (vi) and (vii) are proved, the concatenation of the paths "$\sigma^*_g(\alpha)$" and "$\sigma^*_{\varphi^*(g,\alpha)}(\beta)$", appearing in (ix), is permitted because

$$r\big(\sigma^*_{\varphi^*(g,\alpha)}(\beta)\big) \overset{\text{(v)}}{=} \sigma_{\varphi^*(g,\alpha)}(r(\beta)) \overset{\text{(vii)}}{=} \sigma_g(r(\beta)) = \sigma_g\big(d(\alpha)\big) \overset{\text{(vi)}}{=} d\big(\sigma^*_g(\alpha)\big).$$

For every g in G, define σ^*_g on $E^{\leq 1}$ to coincide with σ_g. Also, define φ^* on $G \times E^{\leq 1}$ by (ii) and (iii). It is then clear that (i–iii) hold and it is easy to see that the remaining properties (iv–x) hold for all $n \leq 1$.

We shall complete the definitions of σ^* and φ^* by induction, so we assume that $m \geq 1$, that

$$\sigma^*_g : E^{\leq m} \to E^{\leq m}$$

is defined for all g in G, that

$$\varphi^* : G \times E^{\leq m} \to G,$$

is defined, and that (i–x) hold for all $n \leq m$. We then define

$$\sigma^*_g : E^{m+1} \to E^{m+1}$$

for all g in G, and

$$\varphi^* : G \times E^{m+1} \to G,$$

by induction as follows. Given $\alpha \in E^{m+1}$, write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^m$, and put

$$\sigma^*_g(\alpha) = \sigma_g(\alpha')\sigma^*_{\varphi(g,\alpha')}(\alpha''), \quad \text{and} \quad \varphi^*(g,\alpha) = \varphi^*\big(\varphi(g,\alpha'), \alpha''\big). \tag{2.4.1}$$

A quick analysis, as done in the first paragraph of this proof, shows that the concatenation of "$\sigma_g(\alpha')$" and "$\sigma^*_{\varphi(g,\alpha')}(\alpha'')$", appearing above, is permitted. We next verify (iv–x), substituting $m + 1$ for n.

We have that the length of $\sigma^*_g(\alpha)$, as defined above, is clearly $1+m$, thus proving (iv). With respect to (v) we have that

$$r\big(\sigma^*_g(\alpha)\big) = r\big(\sigma_g(\alpha')\big) = \sigma_g\big(r(\alpha')\big) = \sigma_g\big(r(\alpha)\big).$$

As for (vi), notice that

$$d\big(\sigma^*_g(\alpha)\big) = d\big(\sigma^*_{\varphi(g,\alpha')}(\alpha'')\big) = \sigma_{\varphi(g,\alpha')}\big(d(\alpha'')\big) = \sigma_g\big(d(\alpha'')\big) = \sigma_g\big(d(\alpha)\big).$$

[1] This is evidently already included in the statement that σ^* is an action, but we repeat it here to aid our proof by induction.

Given $x \in E^0$, we have that

$$\sigma_{\varphi^*(g,\alpha)}(x) = \sigma_{\varphi^*(\varphi(g,\alpha'),\alpha'')}(x) = \sigma_{\varphi(g,\alpha')}(x) = \sigma_g(x),$$

taking care of (vii).

The verification of (viii) is done as follows: for $\alpha = \alpha'\alpha''$, as in (2.4.1), one has

$$\sigma_1^*(\alpha) = \sigma_1^*(\alpha'\alpha'') = \sigma_1(\alpha')\sigma^*_{\varphi(1,\alpha')}(\alpha'') \overset{(2.2)}{=} \sigma_1(\alpha')\sigma_1^*(\alpha'') = \alpha'\alpha'' = \alpha.$$

In order to prove (ix), pick paths α in E^k and β in E^l, where $k+l = m+1$, and such that $d(\alpha) = r(\beta)$.

We leave it for the reader to verify (ix) in the easy case in which $k = 0$, that is, when α is a vertex. The case $k = 1$ is also easy as it is nothing but the definition of σ_g^* given in (2.4.1). So we may assume that $k \geq 2$.

Writing $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{k-1}$, we then have that $\alpha\beta = \alpha'\alpha''\beta$, and hence, by definition,

$$\sigma_g^*(\alpha\beta) = \sigma_g(\alpha')\sigma^*_{\varphi(g,\alpha')}(\alpha''\beta) = \sigma_g(\alpha')\sigma^*_{\varphi(g,\alpha')}(\alpha'')\ \sigma^*_{\varphi^*(\varphi(g,\alpha'),\alpha'')}(\beta) =$$

$$= \sigma_g^*(\alpha'\alpha'')\ \sigma^*_{\varphi^*(g,\alpha'\alpha'')}(\beta).$$

We remark that, in last step above, one should use the induction hypothesis in case $k \leq m$, and the definitions of σ^* and φ^*, when $k = m+1$.

To verify (x) we again pick paths α in E^k and β in E^l, where $k+l = m+1$, and such that $d(\alpha) = r(\beta)$. We once more leave the easy case $k = 0$ to the reader and observe that the case $k = 1$ follows from the definition of φ^*.

We may then suppose that $k \geq 2$, so we write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{k-1}$. Then

$$\varphi^*(g,\alpha\beta) = \varphi^*(g,\alpha'\alpha''\beta) = \varphi^*\big(\varphi(g,\alpha'),\alpha''\beta\big) = \varphi^*\Big(\varphi^*\big(\varphi(g,\alpha'),\alpha''\big),\beta\Big) =$$

$$= \varphi^*\Big(\varphi^*(g,\alpha'\alpha''),\beta\Big) = \varphi^*\Big(\varphi^*(g,\alpha),\beta\Big).$$

Let us now prove that σ^* is in fact an action of G on E^n. We begin by proving that $\sigma_g^*\sigma_h^* = \sigma_{gh}^*$ on E^n, for every g and h in G, which we do by induction on n.

This follows immediately from the hypothesis for $n \leq 1$, so let us assume that $n \geq 2$. Given $\alpha \in E^n$, write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{n-1}$. Then

$$\sigma_g^*\big(\sigma_h^*(\alpha)\big) = \sigma_g^*\big(\sigma_h^*(\alpha'\alpha'')\big) = \sigma_g^*\big(\sigma_h(\alpha')\sigma_{\varphi(h,\alpha')}(\alpha'')\big) =$$

$$= \sigma_g\big(\sigma_h(\alpha')\big)\sigma^*_{\varphi(g,\sigma_h(\alpha'))}\big(\sigma_{\varphi(h,\alpha')}(\alpha'')\big) = \sigma_{gh}(\alpha')\sigma^*_{\varphi(g,\sigma_h(\alpha'))\varphi(h,\alpha')}(\alpha'') =$$

$$= \sigma_{gh}(\alpha')\sigma^*_{\varphi(gh,\alpha')}(\alpha'') = \sigma^*_{gh}(\alpha'\alpha'') = \sigma^*_{gh}(\alpha).$$

That α_g^* is bijective on each E^n then follows[2] from (viii), so α^* is indeed an action of G on E^n.

Finally, let us show that φ^* is a cocycle for σ^* on E^n. For this fix g and h in G and let $\alpha \in E^n$. Then, with $\alpha = \alpha'\alpha''$, as before,

$$\varphi^*(gh,\alpha) = \varphi^*(gh,\alpha'\alpha'') = \varphi^*(\varphi(gh,\alpha'),\alpha'') = \varphi^*\Big(\varphi\big(g,\sigma_h(\alpha')\big)\varphi(h,\alpha'),\alpha''\Big) =$$

$$= \varphi^*\Big(\varphi\big(g,\sigma_h(\alpha')\big),\sigma^*_{\varphi(h,\alpha')}(\alpha'')\Big)\varphi^*\big(\varphi(h,\alpha'),\alpha''\big) =: (\star).$$

On the other hand, focusing on the right-hand-side of (2.1), notice that

$$\varphi^*(g,\sigma_h^*(\alpha))\varphi^*(h,\alpha) = \varphi^*\big(g,\sigma_h^*(\alpha'\alpha'')\big)\varphi^*(h,\alpha'\alpha'') =$$

$$= \varphi^*\big(g,\sigma_h(\alpha')\sigma^*_{\varphi(h,\alpha')}(\alpha'')\big)\varphi^*\big(\varphi(h,\alpha'),\alpha''\big) =$$

$$= \varphi^*\Big(\varphi\big(g,\sigma_h(\alpha')\big),\sigma^*_{\varphi(h,\alpha')}(\alpha'')\Big)\varphi^*\big(\varphi(h,\alpha'),\alpha''\big),$$

which coincides with $(\star)$ above. This concludes the proof. □

The only action of G on E^* to be considered in this paper is σ^* so, from now on, we will adopt the shorthand notation

$$g\alpha = \sigma_g^*(\alpha).$$

Moreover, since φ^* extends φ, we will drop the star decoration and denote φ^* simply as φ. The group law, the cocycle condition, and properties (ii), (v), (vi), (vii), (ix) and (x) of Proposition (2.4) may then be rewritten as follows:

2.5. Equations. *For every g and h in G, for every $x \in E^0$, and for every α and β in E^* such that $d(\alpha) = r(\beta)$, one has that*

(a) $(gh)\alpha = g(h\alpha)$,

(b) $\varphi(gh,\alpha) = \varphi\big(g,h\alpha\big)\varphi(h,\alpha)$,

(ii) $\varphi(g,x) = g$,

(v) $r(g\alpha) = gr(\alpha)$,

(vi) $d(g\alpha) = gd(\alpha)$,

(vii) $\varphi(g,\alpha)x = gx$,

(ix) $g(\alpha\beta) = (g\alpha)\ \varphi(g,\alpha)\beta$,

(x) $\varphi(g,\alpha\beta) = \varphi\big(\varphi(g,\alpha),\beta\big)$.

[2] This is why it is useful to include (viii) as a separate statement, since we may now use it to prove bijectivity.

It might be worth noticing that if $\varphi(g, \alpha) = 1$, then (2.5.ix) reads "$g(\alpha\beta) = (g\alpha)\beta$", which may be viewed as an associativity property. However associativity does not hold in general as φ is not always trivial, and hence parentheses must be used.

On the other hand parentheses are unnecessary in expressions of the form $\alpha g\beta$, when $\alpha, \beta \in E^*$, and $g \in G$, since the only possible interpretation for this expression is the concatenation of α with $g\beta$.

Another useful property of φ is in order.

2.6. Proposition. *For every $g \in G$, and every $\alpha \in E^*$, one has that*

$$\varphi(g^{-1}, \alpha) = \varphi(g, g^{-1}\alpha)^{-1}.$$

Proof. We have

$$1 = \varphi(1, \alpha) = \varphi(gg^{-1}, \alpha) = \varphi(g, g^{-1}\alpha)\varphi(g^{-1}, \alpha),$$

from where the conclusion follows. □

3. The universal C*-algebra $\mathcal{O}_{G,E}$.

As in the above section we fix a graph E, an action of a group G on E, and a one-cocycle φ satisfying (2.3).

It is our next goal to build a C*-algebra from this data but first let us recall the following notion from [**30**]:

3.1. Definition. A *Cuntz-Krieger E-family* consists of a set

$$\{p_x : x \in E^0\}$$

of mutually orthogonal projections and a set

$$\{s_e : e \in E^1\}$$

of partial isometries, all lying in some C*-algebra, and satisfying

(i) $s_e^* s_e = p_{d(e)}$, for every $e \in E^1$,

(ii) $p_x = \displaystyle\sum_{e \in r^{-1}(x)} s_e s_e^*$, for every $x \in E^0$ for which $r^{-1}(x)$ is finite and nonempty.

3.2. Definition. We define $\mathcal{O}_{G,E}$ to be the universal unital C*-algebra generated by a set

$$\{p_x : x \in E^0\} \cup \{s_e : e \in E^1\} \cup \{u_g : g \in G\},$$

subject to the following relations:

(a) $\{p_x : x \in E^0\} \cup \{s_e : e \in E^1\}$ is a Cuntz-Krieger E-family,

(b) the map $u : G \to \mathcal{O}_{G,E}$, defined by the rule $g \mapsto u_g$, is a unitary representation of G,

(c) $u_g s_e = s_{ge} u_{\varphi(g,e)}$, for every $g \in G$, and $e \in E^1$,

(d) $u_g p_x = p_{gx} u_g$, for every $g \in G$, and $x \in E^0$.

Observe that, under our standing assumptions (2.3), for every $x \in E^0$ we have that $r^{-1}(x)$ is finite and nonempty. So (3.1.ii) and (3.2.a) imply that

$$u_g p_x u_g^* = \sum_{r(e)=x} u_g s_e s_e^* u_g^* = \sum_{r(e)=x} s_{ge} u_{\varphi(g,e)} u_{\varphi(g,e)}^* s_{ge}^* =$$

$$= \sum_{r(e)=x} s_{ge} s_{ge}^* = \sum_{r(f)=gx} s_f s_f^* = p_{gx},$$

which says that (3.2.d) follows from the other conditions. We have nevertheless included it in (3.2) in the belief that our theory may be generalized to graphs with sources.

Our construction generalizes some well known constructions in the literature as we would now like to mention.

3.3. Example. Let (G, X) be a self similar group as in [**26**: Definition 2.1]. We may then consider a graph E having only one vertex and such that $E^1 = X$. If we define

$$\varphi(g, x) = g|_x,$$

where, in the terminology of [**26**], $g|_x$ is the restriction (or section) of g at x, then the triple (G, E, φ) satisfies (2.3) and one may easily show that $\mathcal{O}_{G,E}$ is isomorphic to the algebra $\mathcal{O}_{(G,X)}$ introduced by Nekrashevych in [**26**].

3.4. Example. As in [**18**], let us assume we are given two $N \times N$ matrices A and B with integer entries, and such that $A_{i,j} \geq 0$, for all i and j. We may then consider the graph E with vertex set

$$E^0 = \{1, 2, \ldots, N\},$$

and such that, for each pair of vertices $i, j \in E^0$, the set of edges from vertex j to vertex i is a set with $A_{i,j}$ elements, say

$$\{e_{i,j,n} : 0 \leq n < A_{i,j}\}.$$

Assuming moreover that A has no identically zero rows, it is easy to see that E has no sources.

Define an action σ of $\mathbb{Z}$ on E, which is trivial on E^0, and which acts on edges as follows: given $m \in \mathbb{Z}$, and $e_{i,j,n} \in E^1$, let $(\hat{k}, \hat{n})$ be the unique pair of integers such that

$$mB_{i,j} + n = \hat{k} A_{i,j} + \hat{n}, \quad \text{and} \quad 0 \leq \hat{n} < A_{i,j}.$$

That is, $\hat{k}$ is the quotient and $\hat{n}$ is the remainder of the Euclidean division of $mB_{i,j} + n$ by $A_{i,j}$. We then put

$$\sigma_m(e_{i,j,n}) = e_{i,j,\hat{n}}.$$

In other words, σ_m corresponds to the addition of $mB_{i,j}$ to the variable "n" of "$e_{i,j,n}$", taken modulo $A_{i,j}$. In turn, the one-cocycle is defined by

$$\varphi(m, e_{i,j,n}) = \hat{k}.$$

Observe that if $A_{i,j} = 0$, then there are no edges from j to i, so the value $B_{i,j}$ is entirely irrelevant for the above construction. Therefore it makes no difference to assume that

$$A_{i,j} = 0 \quad \Rightarrow \quad B_{i,j} = 0.$$

It may then be proved without much difficulty that $\mathcal{O}_{\mathbb{Z},E}$ is isomorphic to Katsura's [**18**] algebra $\mathcal{O}_{A,B}$, under an isomorphism sending each u_m to the m^{th} power of the unitary

$$u := \sum_{i=1}^{N} u_i$$

in $\mathcal{O}_{A,B}$, and sending $s_{e_{i,j,n}}$ to $s_{i,j,n}$.

When $N = 1$, the relevant graph for Katsura's algebras is the same as the one we used above in the description of Nekrashevych's example. However the former is not a special case of the latter because, contrary to what is required in [**26**], the group action might not be faithful.

3.5. Example. Given any finite graph E, and any action σ of a group G on E, the map $\varphi : G \times E^1 \to G$ defined by

$$\varphi(g, a) = g, \quad \forall\, g \in G, \quad \forall\, a \in E^1$$

is a one-cocycle, and the triple (G, E, φ) satisfies (2.3). By (3.2.c), we have that

$$u_g s_a u_g^* = s_{ga},$$

for any g in G, and every a in E^1. It is therefore easy to see that $\mathcal{O}_{G,E}$ is isomorphic to the crossed product of the graph C*-algebra $C^*(E)$ [**30**] by G, relative to the natural action of G on $C^*(E)$ induced by σ. In particular, if σ is the trivial action, we have that $\mathcal{O}_{G,E}$ is the maximal tensor product of $C^*(E)$ by the full group C*-algebra of G.

3.6. Example. Given any finite graph without sources, and any action σ of a group G on E fixing the vertices, consider the map $\varphi : G \times E^1 \to G$ defined by

$$\varphi(g, a) = 1, \quad \forall\, g \in G, \quad \forall\, a \in E^1.$$

It is easy to see that φ is a one-cocycle, and that the triple (G, E, φ) satisfies (2.3). Since E has no sources we have, for any g in G, that

$$u_g = \sum_{x \in E^0} u_g p_x = \sum_{x \in E^0} \sum_{a \in r^{-1}(x)} u_g s_a s_a^* \overset{(3.2.c)}{=} \sum_{x \in E^0} \sum_{a \in r^{-1}(x)} s_{ga} s_a^*,$$

which therefore lies in the copy of $C^*(E)$ within $\mathcal{O}_{G,E}$. Since the natural representation of $C^*(E)$ in $\mathcal{O}_{G,E}$ is faithful by (11.1), the conclusion is that $\mathcal{O}_{G,E} \cong C^*(E)$.

We now return to the general case of a triple (G, E, φ) satisfying (2.3). We initially recall the usual extension of the notation "s_e" to allow for paths of arbitrary length.

3.7. Definition. *Given a finite path α in E^*, we shall let s_α denote the element of $\mathcal{O}_{G,E}$ given by:*

(i) *when $\alpha = x \in E^0$, we let $s_\alpha = p_x$,*
(ii) *when $\alpha \in E^1$, then s_α is already defined above,*
(iii) *when $\alpha \in E^n$, with $n > 1$, write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{n-1}$, and set $s_\alpha = s_{\alpha'} s_{\alpha''}$, by recurrence.*

Commutation relation (3.2.c) may then be generalized to finite paths as follows:

3.8. Lemma. *Given $\alpha \in E^*$, and $g \in G$, one has that*

$$u_g s_\alpha = s_{g\alpha} u_{\varphi(g,\alpha)}.$$

Proof. Let n be the length of α. When $n = 0, 1$, this follows from (3.2.d&c), respectively. When $n > 1$, write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{n-1}$. Using induction, we then have

$$u_g s_\alpha = u_g s_{\alpha'} s_{\alpha''} = s_{g\alpha'} u_{\varphi(g,\alpha')} s_{\alpha''} = s_{g\alpha'} s_{\varphi(g,\alpha')\alpha''} u_{\varphi(\varphi(g,\alpha'),\alpha'')} =$$
$$= s_{(g\alpha')\varphi(g,\alpha')\alpha''} u_{\varphi(g,\alpha'\alpha'')} = s_{g(\alpha'\alpha'')} u_{\varphi(g,\alpha'\alpha'')} = s_{g\alpha} u_{\varphi(g,\alpha)}. \qquad \square$$

Our next result provides a spanning set for $\mathcal{O}_{G,E}$.

3.9. Proposition. *Let*

$$\mathcal{S} = \big\{ s_\alpha u_g s_\beta^* : \alpha, \beta \in E^*,\ g \in G,\ d(\alpha) = g d(\beta) \big\} \cup \{0\}.$$

Then $\mathcal{S}$ is closed under multiplication and adjoints and its closed linear span coincides with $\mathcal{O}_{G,E}$.

Proof. That $\mathcal{S}$ is closed under adjoints is clear. With respect to closure under multiplication, let $s_\alpha u_g s_\beta^*$ and $s_\gamma u_h s_\delta^*$ be elements of $\mathcal{S}$.

From (3.2.a) we know that $s_\beta^* s_\gamma = 0$, unless either $\gamma = \beta\varepsilon$, or $\beta = \gamma\varepsilon$, for some $\varepsilon \in E^*$. If $\gamma = \beta\varepsilon$, then

$$s_\beta^* s_\gamma = s_\beta^* s_{\beta\varepsilon} = s_\beta^* s_\beta s_\varepsilon = s_\varepsilon,$$

and hence

$$(s_\alpha u_g s_\beta^*)(s_\gamma u_h s_\delta^*) = s_\alpha u_g s_\varepsilon u_h s_\delta^* = s_\alpha s_{g\varepsilon} u_{\varphi(g,\varepsilon)} u_h s_\delta^* = s_{\alpha g\varepsilon} u_{\varphi(g,\varepsilon)h} s_\delta^*. \tag{3.9.1}$$

Moreover, since

$$d(\alpha g\varepsilon) = d(g\varepsilon) = g d(\varepsilon) = \varphi(g,\varepsilon) d(\varepsilon) = \varphi(g,\varepsilon) d(\gamma) = \varphi(g,\varepsilon) h d(\delta),$$

we deduce that the element appearing in the right-hand-side of (3.9.1) indeed belongs to $\mathcal{S}$.

In the second case, namely if $\beta = \gamma\varepsilon$, then the adjoint of the term appearing in the left-hand-side of (3.9.1) is

$$(s_\delta u_{h^{-1}} s_\gamma^*)(s_\beta u_{g^{-1}} s_\alpha^*),$$

and the case already dealt with implies that this belongs to $\mathcal{S}$. The result then follows from the fact that $\mathcal{S}$ is self-adjoint.

In order to prove that $\mathcal{O}_{G,E}$ coincides with the closed linear span of $\mathcal{S}$, let A denote the latter. Given that $\mathcal{S}$ is self-adjoint and closed under multiplication, we see that A is a closed *-subalgebra of $\mathcal{O}_{G,E}$. Since A evidently contains s_α for every α in $E^{\leq 1}$, and since it also contains u_g for every g in G, we deduce that $A = \mathcal{O}_{G,E}$. □

4. The inverse semigroup $\mathcal{S}_{G,E}$.

As before, we keep (2.3) in force.

In this section we will give an abstract description of the set $\mathcal{S}$ appearing in (3.9) as well as its multiplication and adjoint operation. The goal is to construct an inverse semigroup from which we will later recover $\mathcal{O}_{G,E}$.

4.1. Definition. Over the set

$$\mathcal{S}_{G,E} = \big\{(\alpha, g, \beta) \in E^* \times G \times E^* : d(\alpha) = gd(\beta)\big\} \cup \{0\},$$

consider a binary *multiplication* operation defined by

$$(\alpha, g, \beta)(\gamma, h, \delta) = \begin{cases} (\alpha g\varepsilon, \quad \varphi(g,\varepsilon)h, \quad \delta), & \text{if } \gamma = \beta\varepsilon, \\ (\alpha, \ g\varphi(h^{-1},\varepsilon)^{-1}, \ \delta h^{-1}\varepsilon), & \text{if } \beta = \gamma\varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

and a unary *adjoint* operation defined by

$$(\alpha, g, \beta)^* := (\beta, g^{-1}, \alpha).$$

Furthermore, the subset of $\mathcal{S}_{G,E}$ formed by all elements (α, g, β), with $g = 1$, will be denoted by $\mathcal{S}_E$.

It is easy to see that $\mathcal{S}_E$ is closed under the above operations, and that it is isomorphic to the inverse semigroup generated by the canonical partial isometries in the graph C*-algebra of E.

Let us begin with a simple, but useful result:

4.2. Lemma. *Given (α, g, β) and (γ, h, δ) in $\mathcal{S}_{G,E}$, one has*

$$\beta = \gamma \quad \Rightarrow \quad (\alpha, g, \beta)(\gamma, h, \delta) = (\alpha, gh, \delta).$$

Proof. Focusing on the first clause of (4.1), write $\gamma = \beta\varepsilon$, with $\varepsilon = d(\beta)$. Then

$$(\alpha, g, \beta)(\gamma, h, \delta) = (\alpha g d(\beta), \ \varphi\big(g, d(\beta)\big)h, \ \delta) = (\alpha d(\alpha), \ gh, \ \delta) = (\alpha, gh, \delta). \quad □$$

4.3. Proposition. *$\mathcal{S}_{G,E}$ is an inverse semigroup with zero.*

Proof. We leave it for the reader to prove that the above operations are well defined and the multiplication is associative. In order to prove the statement it then suffices [**22**: Theorem 1.1.3] to show that, for all $y, z \in \mathcal{S}_{G,E}$, one has that

(i) $yy^*y = y$, and

(ii) yy^* commutes with zz^*.

Given $y = (\alpha, g, \beta) \in \mathcal{S}_{G,E}$, we have by the above Lemma that

$$yy^*y = (\alpha, g, \beta)(\beta, g^{-1}, \alpha)(\alpha, g, \beta) = (\alpha, 1, \alpha)(\alpha, g, \beta) = (\alpha, g, \beta) = y,$$

proving (i). Notice also that

$$yy^* = (\alpha, 1, \alpha) \tag{4.3.1}$$

is an element of the idempotent semi-lattice of $\mathcal{S}_E$, which is a commutative set because $\mathcal{S}_E$ is an inverse semigroup. Point (ii) above then follows immediately, concluding the proof. $\square$

As seen in (4.3.1), the idempotent semi-lattice of $\mathcal{S}_{G,E}$, henceforth denoted by $\mathcal{E}$, is given by

$$\mathcal{E} = \big\{(\alpha, 1, \alpha) : \alpha \in E^*\big\} \cup \{0\}. \tag{4.4}$$

Evidently $\mathcal{E}$ is also the idempotent semi-lattice of $\mathcal{S}_E$.

For simplicity, from now on we will adopt the short-hand notation

$$f_\alpha = (\alpha, 1, \alpha), \quad \forall\, \alpha \in E^*. \tag{4.5}$$

The following is a standard fact in the theory of graph C*-algebras:

4.6. Proposition. *If $\alpha, \beta \in E^*$, then*

$$f_\alpha f_\beta = \begin{cases} f_\alpha, & \text{if there exists } \gamma \text{ such that } \alpha = \beta\gamma, \\ f_\beta, & \text{if there exists } \gamma \text{ such that } \alpha\gamma = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that if α and β are in E^*, we say that $\alpha \preceq \beta$, if α is a *prefix* of β, i.e. if there exists $\gamma \in E^*$, such that $\alpha\gamma = \beta$. It therefore follows from (4.6) that

$$f_\alpha \leq f_\beta \iff \beta \preceq \alpha. \tag{4.7}$$

Another easy consequence of (4.6) is that, for any two elements $e, f \in \mathcal{E}$, one has that either $e \perp f$, or e and f are comparable. It follows that

$$e \Cap f \quad \Rightarrow \quad e \leq f, \text{ or } \ f \leq e. \tag{4.8}$$

5. Pseudo freeness and E*-unitarity.

Again working under (2.3), suppose we are given g in G and a finite path α such that

$$g\alpha = \alpha, \quad \text{and} \quad \varphi(g, \alpha) = 1. \tag{5.1}$$

Then, given any finite path α' extending α, that is a path of the form $\alpha' = \alpha\beta$, where β is another finite path, we have

$$g\alpha' = g(\alpha\beta) = (g\alpha)\varphi(g, \alpha)\beta = \alpha\beta = \alpha',$$

and

$$\varphi(g, \alpha') = \varphi(g, \alpha\beta) = \varphi\big(\varphi(g, \alpha), \beta\big) = \varphi\big(1, \beta\big) = 1.$$

This says that any path α' extending α also satisfies (5.1) so, in particular, every extension of α is fixed by g.

5.2. Definition. If $g \in G$ and α is a finite path satisfying (5.1), we will say that α is *strongly fixed* by g. In addition, if no proper prefix of α is strongly fixed by g, we will say that α is a *minimal* strongly fixed path for g.

The following result is an easy consequence of the discussion above:

5.3. Proposition. *Given g in G, let M_g be the set of all minimal strongly fixed paths for g. Then the set of all strongly fixed paths for g is given by*

$$\bigsqcup_{\mu \in M_g} \{\mu\gamma : \gamma \in E^*,\ d(\mu) = r(\gamma)\},$$

where the square cup stands for disjoint union.

Let us now introduce terminology to describe situations in which nontrivial strongly fixed paths do not exist.

5.4. Definition. We will say that (G, E, φ) is *pseudo free*[3] if, whenever $(g, e) \in G \times E^1$, is such that $ge = e$, and $\varphi(g, e) = 1$, then $g = 1$.

Notice that pseudo freeness is equivalent to the fact that an edge is never a strongly fixed path for a nontrivial group element. In fact we may boost this up to finite paths as follows:

5.5. Proposition. *Suppose that (G, E, φ) is pseudo free and that a finite path α of nonzero length is strongly fixed for some g in G. Then $g = 1$.*

[3] In a preprint version of this work we have used the term *residually free* to refer to the concept presently being defined, but this apparently conflicts with a well established notion in group theory.

Proof. Arguing by contradiction, assume that there is a counter-example to the statement, meaning that there is a strongly fixed path α for a nontrivial group element g. Then, as already mentioned, α has a minimal strongly fixed prefix, so we may assume without loss of generality that α itself is minimal.

By (2.5.ii), α can't be a vertex, and neither can it be an edge, by hypothesis. So $|\alpha| \geq 2$, and we may then write $\alpha = \beta\gamma$, with $\beta, \gamma \in E^*$, and $|\beta|, |\gamma| < |\alpha|$. Then

$$\beta\gamma = \alpha = g\alpha = g(\beta\gamma) = (g\beta)\varphi(g, \beta)\gamma,$$

whence $\beta = g\beta$, and $\gamma = \varphi(g, \beta)\gamma$, by length considerations. Should $\varphi(g, \beta) = 1$, the pair (g, β) would be a smaller counter-example to the statement, violating the minimality of α. So we have that $\varphi(g, \beta) \neq 1$. In addition,

$$\varphi\big(\varphi(g, \beta), \gamma\big) = \varphi(g, \beta\gamma) = \varphi(g, \alpha) = 1.$$

It follows that $\big(\varphi(g, \beta), \gamma\big)$ is a counter-example to the statement, again violating the minimality of α. This is a contradiction and hence no counter-example exists whatsoever, concluding the proof. $\square$

An apparently stronger version of pseudo freeness is in order.

5.6. Proposition. *Suppose that (G, E, φ) is pseudo free. Then, for all $g_1, g_2 \in G$, and $\alpha \in E^*$, one has that*

$$g_1\alpha = g_2\alpha \ \text{ and } \ \varphi(g_1, \alpha) = \varphi(g_2, \alpha) \quad \Rightarrow \quad g_1 = g_2.$$

Proof. Defining $g = g_2^{-1}g_1$, observe that $g\alpha = \alpha$, and we claim that $\varphi(g, \alpha) = 1$. In fact,

$$\varphi(g, \alpha) = \varphi\big(g_2^{-1}g_1, \alpha\big) = \varphi\big(g_2^{-1}, g_1\alpha\big)\varphi\big(g_1, \alpha\big) \overset{(2.6)}{=}$$

$$= \varphi\big(g_2, g_2^{-1}g_1\alpha\big)^{-1}\varphi(g_1, \alpha) = \varphi(g_2, \alpha)^{-1}\varphi(g_1, \alpha) = 1,$$

so it follows that $g = 1$, which is to say that $g_1 = g_2$. $\square$

We will now determine conditions under which $\mathcal{S}_{G,E}$ is E*-unitary. In order to do so we first need to understand when does an element s of $\mathcal{S}_{G,E}$ dominate a nonzero idempotent e, which in turn must necessarily have the form $e = (\gamma, 1, \gamma)$, as seen in (4.3.1). If s indeed dominates a nonzero idempotent, it is clear that s is itself nonzero, so s must have the form (α, g, β).

5.7. Proposition. *Let α, β and γ be finite paths in E, and let $g \in G$ be such that $d(\alpha) = gd(\beta)$, so that $s := (\alpha, g, \beta)$ is a general nonzero element of $\mathcal{S}_{G,E}$ and $e := (\gamma, 1, \gamma)$ is a general nonzero idempotent element of $\mathcal{S}_{G,E}$. Then $e \leq s$, if and only*

(i) *$\alpha = \beta$,*

(ii) *$\gamma = \alpha\tau$, for some finite path τ,*

(iii) *τ is strongly fixed by g.*

Proof. In order to prove the "if" part, we have

$$se = (\alpha, g, \beta)(\gamma, 1, \gamma) = (\alpha, g, \alpha)(\alpha\tau, 1, \gamma) = (\alpha g\tau, \varphi(g, \tau), \gamma) =$$

$$= (\alpha\tau, 1, \gamma) = (\gamma, 1, \gamma) = e,$$

proving that $e \leq s$. Conversely, assuming that $e \leq s$, we have $se = e$, so in particular $se \neq 0$, and hence by the definition of the multiplication on $\mathcal{S}_{G,E}$, either γ is a prefix of β or vice versa.

In case β is a prefix of γ, we may write $\gamma = \beta\tau$, for some finite path τ, and then

$$(\gamma, 1, \gamma) = e = se = (\alpha, g, \beta)(\beta\tau, 1, \gamma) = (\alpha g\tau, \varphi(g, \tau), \gamma),$$

so we conclude that

$$\alpha g\tau = \gamma = \beta\tau, \quad \text{and} \quad \varphi(g, \tau) = 1.$$

So $\alpha = \beta$, $g\tau = \tau$ and the statement is proved.

On the other hand, if γ is a prefix of β, we may write $\beta = \gamma\varepsilon$ and, again according to the definition of the multiplication on $\mathcal{S}_{G,E}$, the third coordinate of the product se will be $\gamma\varepsilon$, from where we conclude that $\gamma = \gamma\varepsilon$. So $|\varepsilon| = 0$ and then $\gamma = \beta$, which in particular means that β is a prefix of γ, and the proof follows as above. □

5.8. Proposition. *$\mathcal{S}_{G,E}$ is an E^*-unitary inverse semigroup if and only if (G, E, φ) is pseudo free.*

Proof. Let s be an element of $\mathcal{S}_{G,E}$ which dominates a nonzero idempotent element e. As discussed above, we necessarily have

$$s = (\alpha, g, \beta), \quad \text{and} \quad e = (\gamma, 1, \gamma),$$

where α, β and γ are finite paths in E, and $d(\alpha) = gd(\beta)$. Then, by (5.7) we conclude that $g\tau = \tau$, and $\varphi(g, \tau) = 1$ so, assuming that (G, E, φ) is pseudo free, we have $g = 1$. Moreover by (5.7.i) we see that $\alpha = \beta$, so

$$s = (\alpha, g, \beta) = (\alpha, 1, \alpha),$$

which is idempotent as desired. In order to prove the converse, let $(g, e) \in G \times E^1$, be such that $ge = e$, and $\varphi(g, e) = 1$. Then the element

$$s := \big(d(e), g, d(e)\big)$$

lies in $\mathcal{S}_{G,E}$ because

$$gd(e) = d(ge) = d(e).$$

Moreover observe that s dominates the nonzero idempotent element $(e, 1, e)$, since

$$s(e, 1, e) = \big(d(e), g, d(e)\big)\big(d(e)e, 1, e\big) = (d(e)ge, \varphi(g, e), e) = (e, 1, e).$$

So, under the hypothesis that $\mathcal{S}_{G,E}$ is E*-unitary, we conclude that s is idempotent, which is to say that $g = 1$. This proves that (G, E, φ) is pseudo free. □

6. Tight representations of $\mathcal{S}_{G,E}$.

As before, we keep (2.3) in force.

It is the main goal of this section to show that $\mathcal{O}_{G,E}$ is the universal C*-algebra for tight representations of $\mathcal{S}_{G,E}$.

Recall from (4.6) that $f_\alpha \leq f_{d(\alpha)}$, for every $\alpha \in E^*$, so we see that the set

$$\{f_x : x \in E^0\} \tag{6.1}$$

is a *cover* [**6**: Definition 11.5] for $\mathcal{E}$.

6.2. Proposition. *The map*

$$\pi : \mathcal{S}_{G,E} \to \mathcal{O}_{G,E},$$

defined by $\pi(0) = 0$, *and*

$$\pi(\alpha, g, \beta) = s_\alpha u_g s_\beta^*,$$

is a tight [**6**: Definition 13.1] *representation.*

Proof. We leave it for the reader to show that π is in fact multiplicative and that it preserves adjoints.

In order to prove that π is tight, we shall use the characterization given in [**6**: Proposition 11.8], observing that π satisfies condition (i) of [**6**: Proposition 11.7] because, with respect to the cover (6.1), we have that

$$\bigvee_{x\in E^0} \pi(f_x) = \bigvee_{x\in E^0} \pi(x,1,x) = \bigvee_{x\in E^0} p_x = \sum_{x\in E^0} p_x = 1,$$

by (3.2.a). So we assume that $\{f_{\alpha^1}, \ldots, f_{\alpha^n}\}$ is a cover for a given f_β, where $\alpha^1, \ldots \alpha^n, \beta \in E^*$, and we need to show that

$$\bigvee_{i=1}^{n} \pi(f_{\alpha^i}) \geq \pi(f_\beta). \tag{6.2.1}$$

In particular, for each i, we have that $f_{\alpha^i} \leq f_\beta$, so by (4.7) there exists $\gamma^i \in E^*$ such that $\alpha^i = \beta\gamma^i$.

We shall prove (6.2.1) by induction on the variable

$$L = \min_{1\leq i\leq n} |\gamma^i|.$$

If $L = 0$, we may pick i such that $|\gamma^i| = 0$, and then necessarily $\gamma^i = d(\beta)$, in which case $\alpha^i = \beta$, and (6.2.1) is trivially true.

Assuming that $L \geq 1$, let $x := d(\beta)$. Observe that x is not a source either because this is part of our standing assumptions (2.3), or simply because x is the range of every γ^i. In any case let us write

$$r^{-1}(x) = \{e_1, \ldots, e_k\},$$

and observe that

$$\pi(f_\beta) = s_\beta s_\beta^* = s_\beta p_x s_\beta^* \overset{(3.2.a)}{=} \sum_{j=1}^{k} s_\beta s_{e_j} s_{e_j}^* s_\beta^* = \sum_{j=1}^{k} \pi(f_{\beta e_j}).$$

In order to prove (6.2.1) it is therefore enough to show that

$$\bigvee_{i=1}^{n} \pi(f_{\alpha^i}) \geq \pi(f_{\beta e_j}), \tag{6.2.2}$$

for all $j = 1, \ldots, k$. Fixing j we claim that $f_{\beta e_j}$ is covered by the set

$$Z = \big\{ f_{\alpha^i} : 1 \leq i \leq n,\ f_{\alpha^i} \leq f_{\beta e_j} \big\}.$$

In order to see this let y be a nonzero element in $\mathcal{E}$ such that $y \leq f_{\beta e_j}$. Then $y \leq f_\beta$, and so $y \Cap f_{\alpha^i}$ for some i. Thus, to prove the claim it is enough to check that f_{α^i} lies in Z. Observe that

$$y f_{\beta e_j} f_{\alpha^i} = y f_{\alpha^i} \neq 0,$$

which implies that $f_{\beta e_j} \Cap f_{\alpha^i}$.

By (4.8) we have that $f_{\beta e_j}$ and f_{α^i} are comparable, so either $\beta e_j \preceq \alpha^i$ or $\alpha^i \preceq \beta e_j$, by (4.7). Since we are under the hypothesis that $L \geq 1$, and hence that

$$|\alpha^i| = |\beta^i| + |\gamma^i| \geq |\beta| + 1 = |\beta e_j|,$$

we must have that $\beta e_j \preceq \alpha^i$, from where we deduce that $f_{\alpha^i} \leq f_{\beta e_j}$, proving our claim.

Employing the induction hypothesis we then deduce that

$$\bigvee_{z \in Z} \pi(z) \geq \pi(f_{\beta e_j}),$$

verifying (6.2.2), and thus concluding the proof. ☐

We would now like to prove that the representation π above is in fact the *universal* tight representation of $\mathcal{S}_{G,E}$.

6.3. Theorem. *Let A be a unital C^*-algebra and let $\rho : \mathcal{S}_{G,E} \to A$ be a tight representation. Then there exists a unique unital *-homomorphism $\psi : \mathcal{O}_{G,E} \to A$, such that the diagram*

$$\begin{tikzcd} \mathcal{S}_{G,E} \arrow[r, "\pi"] \arrow[dr, "\rho"'] & \mathcal{O}_{G,E} \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

commutes.

Proof. We will initially prove that the elements

$$\tilde{p}_x := \rho(x,1,x), \qquad \forall x \in E^0,$$

$$\tilde{s}_e := \rho\big(e,1,d(e)\big), \qquad \forall e \in E^1,$$

$$\tilde{u}_g := \sum_{x\in E^0} \rho(x,g,g^{-1}x), \quad \forall g \in G,$$

satisfy relations (3.2.a–d). Since the f_x defined in (4.5) are mutually orthogonal idempotents in $\mathcal{S}_{G,E}$, it is clear that the $\tilde{p}_x$ are mutually orthogonal projections. Evidently the $\tilde{s}_e$ are partial isometries so, in order to check (3.2.a), we must only verify (3.1.i) and (3.1.ii). With respect to the former, let $e \in E^1$. Then

$$\tilde{s}_e^*\tilde{s}_e = \rho\big((d(e),1,e)(e,1,d(e)\big) = \rho\big(d(e),1,d(e)\big) = \tilde{p}_{d(e)},$$

proving (3.1.i). In order to prove (3.1.ii), let x be a vertex such that $r^{-1}(x)$ is nonempty and write

$$r^{-1}(x) = \big\{e_1,\ldots,e_n\big\}.$$

Putting $q_i = (e_i,1,e_i)$, we then claim that the set

$$\big\{q_1,\ldots,q_n\big\}$$

is a cover for $q := (x,1,x)$. In order to prove this we must show that, if the nonzero idempotent f is dominated by q, then $f \Cap q_i$ for some i.

Let $f = (\alpha,1,\alpha)$ by (4.4) and notice that

$$0 \neq f = fq = (\alpha,1,\alpha)(x,1,x).$$

So α and x are comparable, and this can only happen when $x = r(\alpha)$. If $|\alpha| = 0$ then necessarily $\alpha = x$, so $f = q$, and it is clear that $f \Cap q_i$ for all i. On the other hand, if $|\alpha| \geq 1$, we write

$$\alpha = \alpha'\alpha'',$$

with $\alpha' \in E^1$, so that $r(\alpha') = r(\alpha) = x$, and hence $\alpha' = e_i$, for some i. Therefore

$$fq_i = (\alpha,1,\alpha)(e_i,1,e_i) = (\alpha,1,\alpha)(\alpha',1,\alpha') = (\alpha,1,\alpha) \neq 0,$$

so $f \Cap q_i$, proving the claim. Since ρ is a tight representation, we deduce that

$$\rho(q) = \bigvee_{i=1}^{n} \rho(q_i),$$

but since the q_i are easily seen to be pairwise orthogonal, their supremum coincides with their sum, whence

$$\tilde{p}_x = \rho(q) = \sum_{i=1}^{n} \rho(q_i) = \sum_{i=1}^{n} \rho(e_i,1,e_i) =$$

$$= \sum_{i=1}^{n} \rho\big((e_i, 1, d(e_i))\ (d(e_i), 1, e_i)\big) = \sum_{i=1}^{n} \tilde{s}_{e_i} \tilde{s}_{e_i}^*,$$

thus verifying (3.1.ii), and hence proving (3.2.a).

With respect to (3.2.b), let us first prove that $\tilde{u}_1 = 1$. Considering the subsets of $\mathcal{E}$ given by

$$X = \varnothing, \quad Y = \varnothing, \quad \text{and} \quad Z = \big\{(x, 1, x) : x \in E^0\big\},$$

notice that, according to [**6**: Definition 11.4], one has that

$$\mathcal{E}^{X,Y} = \mathcal{E},$$

and that Z is a cover for $\mathcal{E}^{X,Y}$, as seen in (6.1). By the tightness condition [**6**: Definition 11.6] we have

$$\bigvee_{z \in Z} \rho(z) \geq \bigwedge_{x \in X} \rho(x) \wedge \bigwedge_{y \in Y} \neg\rho(y).$$

As explained in the discussion following [**6**: Definition 11.6], the right-hand-side above must be interpreted as 1 because X and Y are empty. On the other hand, since the $\rho(z)$ are pairwise orthogonal, the supremum in the left-hand-side above becomes a sum, so

$$1 = \sum_{z \in Z} \rho(z) = \sum_{x \in E^0} \rho(x, 1, x) = \tilde{u}_1.$$

In order to prove that $\tilde{u}$ is multiplicative, let g and h be in G. Then

$$\tilde{u}_g \tilde{u}_h = \sum_{x, y \in E^0} \rho\big((x, g, g^{-1}x)(y, h, h^{-1}y)\big) =$$

$$= \sum_{x \in E^0} \rho\big((x, g, g^{-1}x)(g^{-1}x, h, h^{-1}g^{-1}x)\big) = \sum_{x \in E^0} \rho\big(x, gh, (gh)^{-1}x\big) = \tilde{u}_{gh}.$$

We next claim that $\tilde{u}_g^* = \tilde{u}_{g^{-1}}$, for all g in G. To prove it we compute

$$\tilde{u}_g^* = \sum_{x \in E^0} \rho(x, g, g^{-1}x)^* = \sum_{x \in E^0} \rho(g^{-1}x, g^{-1}, x) = \cdots$$

which, upon the change of variables $y = g^{-1}x$, becomes

$$\cdots = \sum_{y \in E^0} \rho(y, g^{-1}, gy) = \tilde{u}_{g^{-1}}.$$

This shows that $\tilde{u}$ is a unitary representation, verifying (3.2.b). Turning now our attention to (3.2.c), let $g \in G$ and $e \in E^1$. Then

$$\tilde{u}_g \tilde{s}_e = \sum_{x \in E^0} \rho(x, g, g^{-1}x)\, \rho\big(e, 1, d(e)\big) = \rho\big(gr(e), g, r(e)\big)\, \rho\big(e, 1, d(e)\big) =$$

$$= \rho\big(r(ge)ge, \varphi(g,e), d(e)\big) = \rho\big(ge, \varphi(g,e), d(e)\big) = (\star).$$

On the other hand

$$\tilde{s}_{ge}\tilde{u}_{\varphi(g,e)} = \rho\big(ge, 1, d(ge)\big) \sum_{x \in E^0} \rho\big(x, \varphi(g,e), \varphi(g,e)^{-1}x\big) =$$

$$= \rho\big(ge, 1, d(ge)\big)\, \rho\big(d(ge), \varphi(g,e), g^{-1}d(ge)\big) =$$

$$= \rho\big(ge, 1, d(ge)\big)\, \rho\big(d(ge), \varphi(g,e), d(e)\big) = \rho\big(ge, \varphi(g,e), d(e)\big),$$

which coincides with $(\star)$ and hence proves (3.2.c). We leave the proof of (3.2.d) to the reader after which the universal property of $\mathcal{O}_{G,E}$ intervenes to provide us with a *-homomorphism

$$\psi : \mathcal{O}_{G,E} \to A$$

sending

$$p_x \mapsto \tilde{p}_x, \quad s_e \mapsto \tilde{s}_e, \quad \text{and} \quad u_g \mapsto \tilde{u}_g.$$

Now we must show that

$$\psi\big(\pi(\gamma)\big) = \rho(\gamma), \quad \forall\, \gamma \in \mathcal{S}_{G,E}. \tag{6.3.1}$$

We will first do so for the following special cases:

(i) $\gamma = (x, 1, x)$, for $x \in E^0$,

(ii) $\gamma = \big(e, 1, d(e)\big)$, for $e \in E^1$,

(iii) $\gamma = (x, g, g^{-1}x)$, for $x \in E^0$, and $g \in G$.

In case (i) we have

$$\psi(\pi(\gamma)) = \psi(\pi(x, 1, x)) = \psi(p_x) = \tilde{p}_x = \rho(x, 1, x) = \rho(\gamma).$$

As for (ii),

$$\psi(\pi(\gamma)) = \psi\big(\pi\big(e, 1, d(e)\big)\big) = \psi(s_e) = \tilde{s}_e = \rho\big(e, 1, d(e)\big) = \rho(\gamma).$$

Under (iii),

$$\psi(\pi(\gamma)) = \psi\big(\pi(x, g, g^{-1}x)\big) = \psi(p_x u_g p_{g^{-1}x}) = \psi(p_x u_g) = \tilde{p}_x \tilde{u}_g =$$

$$= \rho(x, 1, x) \sum_{y \in E^0} \rho(y, g, g^{-1}y) = \sum_{y \in E^0} \rho\big((x, 1, x)(y, g, g^{-1}y)\big) = \rho(x, g, g^{-1}x) = \rho(\gamma).$$

In order to prove (6.3.1), it is now clearly enough to check that the *-sub-semigroup of $\mathcal{S}_{G,E}$ generated by the elements mentioned in (i–iii), above, coincides with $\mathcal{S}_{G,E}$.

Denoting this *-sub-semigroup by $\mathcal{T}$, we will first show that $\big(\alpha, 1, d(\alpha)\big)$ is in $\mathcal{T}$, for every $\alpha \in E^*$. This is evident for $|\alpha| \le 1$, so we suppose that $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $r(\alpha'') = d(\alpha')$. We then have by induction that

$$\mathcal{T} \ni \big(\alpha', 1, d(\alpha')\big)\big(\alpha'', 1, d(\alpha'')\big) = \big(\alpha'\alpha'', 1, d(\alpha'')\big) = \big(\alpha, 1, d(\alpha)\big).$$

Considering a general element $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, let $x = d(\alpha)$, so that $g^{-1}x = d(\beta)$, and notice that

$$\mathcal{T} \ni \big(\alpha, 1, d(\alpha)\big)(x, g, g^{-1}x)\big(\beta, 1, d(\beta)\big)^* =$$
$$= \big(\alpha, 1, d(\alpha)\big)\big(d(\alpha), g, d(\beta)\big)\big(d(\beta), 1, \beta\big) = (\alpha, g, \beta),$$

which proves that $\mathcal{T} = \mathcal{S}_{G,E}$, and hence that (6.3.1) holds.

To conclude we observe that the uniqueness of ψ follows from the fact that $\mathcal{O}_{G,E}$ is generated by the p_x, the s_e, and the u_g. □

Given an inverse semigroup $\mathcal{S}$ with zero, recall from [**6**: Theorem 13.3] that $\mathcal{G}_{\text{tight}}(\mathcal{S})$ (denoted simply as $\mathcal{G}_{\text{tight}}$ in [**6**]) is the groupoid of germs for the natural action of $\mathcal{S}$ on the space of tight filters over its idempotent semi-lattice. Moreover the C*-algebra of $\mathcal{G}_{\text{tight}}(\mathcal{S})$ is universal for tight representations of $\mathcal{S}$.

6.4. Corollary. *Under the assumptions of (2.3) one has that $\mathcal{O}_{G,E}$ is isomorphic to the C*-algebra of the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.*

Proof. Follows from [**6**: Theorem 13.3] and the uniqueness of universal C*-algebras. □

We should notice that our requirement that G be countable in (2.3) is only used in the above proof, since the application of [**6**: Theorem 13.3] depends on the countability of $\mathcal{S}_{G,E}$.

7. The Lag Group.

It is our next goal to give a concrete description of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, similar to the description given of the groupoid associated to a row-finite graph in [**20**: Definition 2.3]. The crucial ingredient there is the notion of *tail equivalence with lag*. In this section we will construct a group where our generalized *lag* function will take values.

Let G be a group. Within the infinite cartesian product[4]

$$G^\infty = \prod_{n\in\mathbb{N}} G$$

consider the infinite direct sum

$$G^{(\infty)} = \bigoplus_{n\in\mathbb{N}} G$$

formed by the elements $g = (g_n)_{n\in\mathbb{N}} \in G^\infty$ which are eventually trivial, that is, for which there exists n_0 such that $g_n = 1$, for all $n \ge n_0$. It is clear that $G^{(\infty)}$ is a normal subgroup of G^∞.

[4] For the purpose of this cartesian product we adopt the convention that $\mathbb{N} = \{1, 2, 3, \ldots\}$.

7.1. Definition. Given a group G, the *corona* of G is the quotient group

$$\breve{G} = G^\infty / G^{(\infty)}.$$

Consider the *left* and *right shift* endomorphisms of G^∞

$$\lambda, \rho : G^\infty \to G^\infty$$

given for every $\mathrm{g} = (\mathrm{g}_n)_{n\in\mathbb{N}} \in G^\infty$, by

$$\lambda(\mathrm{g})_n = \mathrm{g}_{n+1}, \quad \forall\, n \in \mathbb{N},$$

and

$$\rho(\mathrm{g})_n = \begin{cases} 1, & \text{if } n = 0, \\ \mathrm{g}_{n-1}, & \text{if } n \geq 1. \end{cases}$$

It is readily seen that $G^{(\infty)}$ is invariant under both λ and ρ, so these pass to the quotient providing endomorphisms

$$\breve{\lambda}, \breve{\rho} : \breve{G} \to \breve{G}. \tag{7.2}$$

For every $\mathrm{g} = (\mathrm{g}_n)_{n\in\mathbb{N}} \in G^\infty$, we have that

$$\lambda(\rho(\mathrm{g})) = \mathrm{g}, \quad \text{and} \qquad \rho(\lambda(\mathrm{g})) = (1, \mathrm{g}_2, \mathrm{g}_3, \ldots) \equiv \mathrm{g}, \tag{7.3}$$

where we use "$\equiv$" to refer to the equivalence relation determined by the normal subgroup $G^{(\infty)}$. Therefore both $\breve{\lambda}\breve{\rho}$ and $\breve{\rho}\breve{\lambda}$ coincide with the identity, and hence $\breve{\lambda}$ and $\breve{\rho}$ are each other's inverse. In particular, they are both automorphisms of $\breve{G}$.

Iterating $\breve{\rho}$ therefore gives an action of $\mathbb{Z}$ on $\breve{G}$.

7.4. Definition. Given any countable discrete group G, the *lag group* associated to G is the semi-direct product group

$$\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}.$$

The reason we call this the "lag group" is that it will play a very important role in the next section, as the co-domain for our *lag* function.

8. The tight groupoid of $\mathcal{S}_{G,E}$.

We would now like to give a detailed description of the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. As already mentioned this is the groupoid of germs for the natural action of $\mathcal{S}_{G,E}$ on the space of tight filters over the idempotent semi-lattice $\mathcal{E}$ of $\mathcal{S}_{G,E}$. See [**6**: Section 4] for more details.

By an *infinite path* in E we shall mean any infinite sequence of the form

$$\xi = \xi_1\xi_2\ldots,$$

where $\xi_i \in E^1$, and $d(\xi_i) = r(\xi_{i+1})$, for all i. The set of all infinite paths will be denoted by E^∞. Given an infinite path

$$\xi = \xi_1\xi_2\ldots \in E^\infty,$$

and an integer $n \geq 0$, denote by $\xi|_n$ the finite path of length n given by

$$\xi|_n = \begin{cases} \xi_1\xi_2\ldots\xi_n, & \text{if } n \geq 1, \\ \\ r(\xi_1), & \text{if } n = 0. \end{cases}$$

8.1. Proposition. *There is a unique action*

$$(g, \xi) \in G \times E^\infty \mapsto g\xi \in E^\infty$$

of G on E^∞ such that,

$$(g\xi)|_n = g(\xi|_n),$$

for every $g \in G$, $\xi \in E^\infty$, and $n \in \mathbb{N}$.

Proof. Left to the reader. □

Recall from (4.5) that, for any finite path $\alpha \in E^*$, we denote by f_α the idempotent element $(\alpha, 1, \alpha)$ in $\mathcal{E}$. Thus, given an infinite path $\xi \in E^\infty$, we may look at the subset

$$\mathcal{F}_\xi = \{f_{\xi|_n} : n \in \mathbb{N}\} \subseteq \mathcal{E},$$

which turns out to be an ultra-filter [**6**: Definition] over $\mathcal{E}$. Denoting the set of all ultra-filters over $\mathcal{E}$ by $\widehat{\mathcal{E}}_\infty$, as in [**6**: Definition 12.8], one may also show [**6**: Proposition 19.11] that the correspondence

$$\xi \in E^\infty \mapsto \mathcal{F}_\xi \in \widehat{\mathcal{E}}_\infty$$

is bijective, and we will use it to identify E^∞ and $\widehat{\mathcal{E}}_\infty$. Furthermore, this correspondence may be proven to be a homeomorphism if E^∞ is equipped with the product topology.

Since E is finite, E^∞ is compact by Tychonov's Theorem, and consequently so is $\widehat{\mathcal{E}}_\infty$. Being the closure of $\widehat{\mathcal{E}}_\infty$ within $\widehat{\mathcal{E}}$ [**6**: Theorem 12.9], the space $\widehat{\mathcal{E}}_{\text{tight}}$ formed by the tight filters therefore necessarily coincides with $\widehat{\mathcal{E}}_\infty$.

Identifying $\widehat{\mathcal{E}}_{\text{tight}}$ with E^∞, as above, we may transfer the canonical action of $\mathcal{S}_{G,E}$ from the former to the latter resulting in the following: to each element $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, we associate the partial homeomorphism of E^∞ whose domain is the *cylinder*

$$Z(\beta) := \{\eta \in E^\infty : \eta = \beta\xi, \text{ for some } \xi \in E^\infty\}, \tag{8.2}$$

and which sends each $\eta = \beta\xi \in Z(\beta)$ to $\alpha g\xi$, where the meaning of "$g\xi$" is as in (8.1).

As before we will not use any special symbol to indicate this action, using module notation instead:

$$(\alpha, g, \beta)\eta = \alpha g\xi, \quad \forall\, (\alpha, g, \beta) \in \mathcal{S}_{G,E}, \quad \forall\, \eta = \beta\xi \in Z(\beta). \tag{8.3}$$

Before we proceed let us at least check that $\alpha g\xi$ is in fact an element of E^∞, which is to say that $d(\alpha) = r(g\xi)$. Firstly, for every element $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, we have that $d(\alpha) = gd(\beta)$. Secondly, if $\eta = \beta\xi \in E^\infty$, then $d(\beta) = r(\xi)$. Therefore

$$r(g\xi) = gr(\xi) = gd(\beta) = d(\alpha).$$

This leads to a first, more or less concrete description of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.

8.4. Proposition. *Under (2.3), one has that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is isomorphic to the groupoid of germs for the above action of $\mathcal{S}_{G,E}$ on E^∞.*

Our aim is nevertheless a much more precise description of this groupoid. Recall from [**6**: Definition 4.6] that the germ of an element $s \in \mathcal{S}_{G,E}$ at a point ξ in the domain of s is denoted by $[s, \xi]$. If $s = (\alpha, g, \beta)$, this would lead to the somewhat awkward notation $[(\alpha, g, \beta), \xi]$, which from now on will instead be written as

$$\big[\alpha, g, \beta;\, \xi\big].$$

Thus the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, consisting of all germs for the action of $\mathcal{S}_{G,E}$ on E^∞, is given by

$$\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) = \Big\{ \big[\alpha, g, \beta;\, \xi\big] : (\alpha, g, \beta) \in \mathcal{S}_{G,E},\ \xi \in Z(\beta) \Big\}. \tag{8.5}$$

Let us now prove a useful criterion for equality of germs.

8.6. Proposition. *Suppose that (G, E, φ) is pseudo free and let us be given elements (α_1, g_1, β_1) and (α_2, g_2, β_2) in $\mathcal{S}_{G,E}$, with $|\beta_1| \leq |\beta_2|$, as well as infinite paths η_1 in $Z(\beta_1)$, and η_2 in $Z(\beta_2)$. Then*

$$\big[\alpha_1, g_1, \beta_1;\, \eta_1\big] = \big[\alpha_2, g_2, \beta_2;\, \eta_2\big]$$

if and only if there is a finite path γ and an infinite path ξ, such that

(i) $\alpha_2 = \alpha_1 g_1 \gamma$,

(ii) $g_2 = \varphi(g_1, \gamma)$,

(iii) $\beta_2 = \beta_1 \gamma$,

(iv) $\eta_1 = \eta_2 = \beta_1 \gamma \xi$.

Proof. Assuming that the germs are equal, we have by [**6**: Definition 4.6] that

$$\eta_1 = \eta_2 =: \eta,$$

and there is an idempotent $(\delta, 1, \delta) \in \mathcal{E}$, such that $\eta \in Z(\delta)$, and

$$(\alpha_1, g_1, \beta_1)(\delta, 1, \delta) = (\alpha_2, g_2, \beta_2)(\delta, 1, \delta). \tag{8.6.1}$$

It follows that $\eta = \delta\zeta$, for some $\zeta \in E^\infty$. Upon replacing δ by a longer prefix of η, we may assume that $|\delta|$ is as large as we want. Furthermore the element of $\mathcal{S}_{G,E}$ represented by the two sides of (8.6.1) is evidently nonzero because the partial homeomorphism associated to it under our action has η in its domain. So, focusing on (4.1), we see that β_1 and δ are comparable, and so are β_2 and δ.

Assuming that $|\delta|$ exceeds both $|\beta_1|$ and $|\beta_2|$, we may then write $\delta = \beta_1\varepsilon_1 = \beta_2\varepsilon_2$, for suitable ε_1 and ε_2 in E^*. But since $|\beta_1| \leq |\beta_2|$, this in turn implies that $\beta_2 = \beta_1\gamma$, for some $\gamma \in E^*$, hence proving (iii). Therefore $\delta = \beta_1\gamma\varepsilon_2$, so

$$\eta = \delta\zeta = \beta_1\gamma\varepsilon_2\zeta,$$

and (iv) follows once we choose $\xi = \varepsilon_2\zeta$. Moreover, equation (8.6.1) reads

$$(\alpha_1, g_1, \beta_1)(\beta_1\gamma\varepsilon_2, 1, \beta_1\gamma\varepsilon_2) = (\alpha_2, g_2, \beta_1\gamma)(\beta_1\gamma\varepsilon_2, 1, \beta_1\gamma\varepsilon_2).$$

Computing the products according to (4.1), we get

$$\big(\alpha_1 g_1(\gamma\varepsilon_2),\ \varphi(g_1, \gamma\varepsilon_2),\ \beta_1\gamma\varepsilon_2\big) = \big(\alpha_2 g_2\varepsilon_2,\ \varphi(g_2, \varepsilon_2),\ \beta_1\gamma\varepsilon_2\big),$$

from where we obtain

$$\alpha_2 g_2\varepsilon_2 = \alpha_1 g_1(\gamma\varepsilon_2) = \alpha_1(g_1\gamma)\varphi(g_1, \gamma)\varepsilon_2, \tag{8.6.2}$$

and

$$\varphi(g_2, \varepsilon_2) = \varphi(g_1, \gamma\varepsilon_2) = \varphi\big(\varphi(g_1, \gamma), \varepsilon_2\big). \tag{8.6.3}$$

Since $|g_2\varepsilon_2| = |\varepsilon_2| = |\varphi(g_1, \gamma)\varepsilon_2|$, we deduce from (8.6.2) that

$$g_2\varepsilon_2 = \varphi(g_1, \gamma)\varepsilon_2, \tag{8.6.4}$$

and hence also that

$$\alpha_2 = \alpha_1 g_1\gamma,$$

proving (i). In view of (8.6.3) and (8.6.4), point (ii) follows from (5.6).

Conversely, assume (i–iv) and let us prove equality of the above germs. Setting $\delta = \beta_1\gamma$, we have by (iv) that

$$\eta := \eta_1 = \eta_2 \in Z(\delta),$$

so it suffices to verify (8.6.1), which the reader could do without any difficulty. □

Proposition (8.6) then says that the typical situation in which an equality of germs takes place is

$$\big[\alpha, g, \beta;\ \beta\gamma\xi\big] = \big[\alpha g\gamma, \varphi(g, \gamma), \beta\gamma;\ \beta\gamma\xi\big].$$

Our next two results are designed to offer convenient representatives of germs.

8.7. Proposition. *Given any germ u, there exists an integer n_0, such that for every $n \geq n_0$,*

(i) *there is a representation of u of the form $u = \big[\alpha_1, g_1, \beta_1;\ \beta_1\xi_1\big]$, with $|\alpha_1| = n$.*

(ii) *there is a representation of u of the form $u = \big[\alpha_2, g_2, \beta_2;\ \beta_2\xi_2\big]$, with $|\beta_2| = n$.*

Proof. Write $u = \big[\alpha, g, \beta;\ \eta\big]$, and choose any $n_0 \geq \max\{|\alpha|, |\beta|\}$. Then, for every $n \geq n_0$ we may write $\eta = \beta\gamma\xi$, with $\gamma \in E^*$, $\xi \in E^\infty$, and such that $|\gamma| = n - |\alpha|$ (resp. $|\gamma| = n - |\beta|$). Therefore

$$u = \big[\alpha, g, \beta;\ \beta\gamma\xi\big] = \big[\alpha g\gamma, \varphi(g, \gamma), \beta\gamma;\ \beta\gamma\xi\big],$$

and we have $|\alpha g\gamma| = |\alpha| + |g\gamma| = |\alpha| + |\gamma| = n$ (resp. $|\beta\gamma| = |\beta| + |\gamma| = n$). □

8.8. Corollary. *Given u_1 and u_2 in $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, such that $(u_1, u_2) \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})^{(2)}$, that is, such that the multiplication u_1u_2 is allowed or, equivalently, such that $d(u_1) = r(u_2)$, then there are representations of u_1 and u_2 of the form*

$$u_1 = \big[\alpha_1, g_1, \alpha_2;\ \alpha_2 g_2 \xi\big], \quad \text{and} \quad u_2 = \big[\alpha_2, g_2, \beta;\ \beta\xi\big],$$

and in this case

$$u_1u_2 = \big[\alpha_1, g_1g_2, \beta;\ \beta\xi\big].$$

Proof. Using (8.7), write

$$u_i = \big[\alpha_i, g_i, \beta_i;\ \beta_i\xi_i\big],$$

with $|\beta_1| = |\alpha_2|$. By virtue of (u_1, u_2) lying in $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})^{(2)}$, we have that

$$\beta_1\xi_1 = (\alpha_2, g_2, \beta_2)(\beta_2\xi_2) = \alpha_2 g_2 \xi_2,$$

so in fact $\beta_1 = \alpha_2$, and $\xi_1 = g_2\xi_2$. Then

$$u_1 = \big[\alpha_1, g_1, \beta_1;\ \beta_1\xi_1\big] = \big[\alpha_1, g_1, \alpha_2;\ \alpha_2 g_2 \xi_2\big],$$

and it suffices to put $\xi = \xi_2$, and $\beta = \beta_2$.

With respect to the last assertion we have that $u_1u_2 = [s; \beta\xi]$, where s is the element of $\mathcal{S}_{G,E}$ given by

$$s = (\alpha_1, g_1, \alpha_2)(\alpha_2, g_2, \beta) \overset{(4.2)}{=} (\alpha_1, g_1g_2, \beta),$$

concluding the proof. □

Having extended the action of G to the set of infinite paths in Proposition (8.1), one may ask whether it is possible to do the same for the cocycle φ. The following is an attempt at this which however produces a map taking values in the infinite product G^∞, rather than in G.

8.9. Definition. We will denote by Φ, the map

$$\Phi : G \times E^\infty \to G^\infty$$

defined by the rule

$$\Phi(g, \xi)_n = \varphi(g, \xi|_{n-1}),$$

for $g \in G$, $\xi \in E^\infty$, and $n \geq 1$.

Recall that we are indexing the elements of G^∞ on the set $\{1, 2, 3, \ldots\}$, so the first coordinate of $\Phi(g, \xi)$ is

$$\Phi(g, \xi)_1 = \varphi(g, \xi|_0) = \varphi\big(g, r(\xi)\big) \overset{(2.5.\text{ii})}{=} g.$$

We wish to view Φ as some sort of cocycle but, unfortunately, property (2.5.x) does not hold quite as stated. On the fortunate side, a suitable modification of this relation, involving the left shift endomorphism λ of G^∞, is satisfied:

8.10. Proposition. *Let α be a finite path and let ξ be an infinite path such that $d(\alpha) = r(\xi)$. Then, for every g in G, one has that*

$$\Phi\big(\varphi(g,\alpha),\xi\big) = \lambda^{|\alpha|}\big(\Phi\big(g,\alpha\xi\big)\big).$$

Proof. For all $n \geq 1$, we have

$$\Phi\big(\varphi(g,\alpha),\xi\big)_n = \varphi\big(\varphi(g,\alpha),\xi|_{n-1}\big) = \varphi\big(g,\alpha(\xi|_{n-1})\big) =$$

$$= \varphi\big(g,(\alpha\xi)|_{n-1+|\alpha|}\big) = \lambda^{|\alpha|}\big(\Phi(g,\alpha\xi)\big)_n. \qquad \square$$

Another reason to think of Φ as a cocycle is as follows:

8.11. Proposition. *For every $\xi \in E^\infty$, and every $g,h \in G$, we have that*

$$\Phi(gh,\xi) = \Phi\big(g,h\xi\big)\Phi(h,\xi).$$

Proof. We have for all $n \in \mathbb{N}$, that

$$\Phi(gh,\xi)_n = \varphi(gh,\xi|_{n-1}) \overset{(2.5.b)}{=} \varphi\big(g,h(\xi|_{n-1})\big)\varphi(h,\xi|_{n-1}) \overset{(8.1)}{=}$$

$$= \varphi\big(g,(h\xi)|_{n-1}\big)\varphi(h,\xi|_{n-1}) = \Phi\big(g,h\xi\big)_n\Phi(h,\xi)_n. \qquad \square$$

The following elementary fact might perhaps justify the choice of "$n-1$" in the definition of Φ.

8.12. Proposition. *Given $g \in G$, and $\xi \in E^\infty$, one has that*

$$(g\xi)_n = \Phi(g,\xi)_n\,\xi_n.$$

Proof. By (8.1) we have that $(g\xi)|_n = g(\xi|_n)$, so the n^{th} coordinate of $g\xi$ is also the n^{th} coordinate of $g(\xi|_n)$. In addition we have that

$$g(\xi|_n) = g(\xi|_{n-1}\xi_n) \overset{(2.5.ix)}{=} g(\xi|_{n-1})\varphi(g,\xi|_{n-1})\xi_n,$$

so

$$(g\xi)_n = \varphi(g,\xi|_{n-1})\xi_n = \Phi(g,\xi)_n\,\xi_n. \qquad \square$$

We now wish to define a homomorphism (sometimes also called a one-cocycle) from $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ to the lag group $\breve{G} \rtimes_\rho \mathbb{Z}$, by means of the rule

$$\big[\alpha,g,\beta;\,\beta\xi\big] \mapsto \Big(\rho^{|\alpha|}\big(\Phi(g,\xi)\big),|\alpha|-|\beta|\Big).$$

As it is often the case for maps defined on groupoid of germs, the above tentative definition uses a representative of the germ, so some work is necessary to prove that the definition does not depend on the choice of representative. The technical part of this task is the content of our next result.

8.13. Lemma. *Suppose that (G, E, φ) is pseudo free. For each $i = 1, 2$, let us be given (α_i, g_i, β_i) in $\mathcal{S}_{G,E}$, as well as $\eta_i = \beta_i\xi_i \in Z(\beta_i)$. If*

$$\big[\alpha_1, g_1, \beta_1;\, \eta_1\big] = \big[\alpha_2, g_2, \beta_2;\, \eta_2\big],$$

then

$$\rho^{|\alpha_1|}\big(\Phi(g_1, \xi_1)\big) \equiv \rho^{|\alpha_2|}\big(\Phi(g_2, \xi_2)\big)$$

modulo $G^{(\infty)}$.

Proof. Assuming without loss of generality that $|\beta_1| \leq |\beta_2|$, we may use (8.6) to write

$$\alpha_2 = \alpha_1 g_1 \gamma, \quad g_2 = \varphi(g_1, \gamma), \quad \beta_2 = \beta_1\gamma, \quad \text{and} \quad \eta_1 = \eta_2 = \beta_1\gamma\xi,$$

for suitable $\gamma \in E^*$ and $\xi \in E^\infty$. Then necessarily $\xi_1 = \gamma\xi$, and $\xi_2 = \xi$, and

$$\rho^{|\alpha_2|}\big(\Phi(g_2, \xi_2)\big) = \rho^{|\alpha_1|+|\gamma|}\Big(\Phi\big(\varphi(g_1, \gamma), \xi\big)\Big) \overset{(8.10)}{=}$$

$$= \rho^{|\alpha_1|}\rho^{|\gamma|}\lambda^{|\gamma|}\big(\Phi\big(g_1, \gamma\xi\big)\big) \overset{(7.3)}{\equiv} \rho^{|\alpha_1|}\big(\Phi\big(g_1, \xi_1\big)\big). \qquad \square$$

► Due to our reliance on Proposition (8.6) and Lemma (8.13), from now on and until the end of this section we will assume, in addition to (2.3), that (G, E, φ) is pseudo free.

If g is in G^∞, we will denote by $\breve{g}$ its class in the quotient group $\breve{G}$. Likewise we will denote by $\breve{\Phi}$ the composition of Φ with the quotient map from G^∞ to $\breve{G}$.

$$\begin{tikzcd} G \times E^\infty \arrow[r, "\Phi"] \arrow[rr, bend right, "\breve{\Phi}"'] & G^\infty \arrow[r] & \breve{G} \end{tikzcd}$$

It then follows from (8.13) that the correspondence

$$\big[\alpha, g, \beta;\, \beta\xi\big] \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) \mapsto \breve{\rho}^{|\alpha|}\big(\breve{\Phi}(g, \xi)\big) \in \breve{G}$$

is a well defined map. This is an important part of the one-cocycle we are about to introduce.

8.14. Proposition. *The correspondence*

$$\ell : \big[\alpha, g, \beta;\, \beta\xi\big] \mapsto \Big(\breve{\rho}^{|\alpha|}\big(\breve{\Phi}(g, \xi)\big), |\alpha| - |\beta|\Big)$$

gives a well defined map

$$\ell : \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) \to \breve{G} \rtimes_\rho \mathbb{Z},$$

which is moreover a one-cocycle. From now on ℓ will be called the lag function.

Proof. By the discussion above we have that the first coordinate of the above pair is well defined. On the other hand, in the context of Proposition (8.6) one easily sees that $|\alpha_1| - |\beta_1| = |\alpha_2| - |\beta_2|$, so the second coordinate is also well defined.

In order to show that ℓ is multiplicative, pick $(u_1, u_2) \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})^{(2)}$. We may then use (8.8) to write

$$u_1 = \big[\alpha_1, g_1, \alpha_2;\ \alpha_2 g_2 \xi\big], \quad \text{and} \quad u_2 = \big[\alpha_2, g_2, \beta;\ \beta\xi\big].$$

So

$$\ell(u_1)\ell(u_2) = \Big(\rho^{|\alpha_1|}\big(\Phi(g_1, g_2\xi)\big),\ |\alpha_1| - |\alpha_2|\Big)\Big(\rho^{|\alpha_2|}\big(\Phi(g_2, \xi)\big),\ |\alpha_2| - |\beta|\Big) =$$

$$= \Big(\rho^{|\alpha_1|}\big(\Phi(g_1, g_2\xi)\big)\ \rho^{|\alpha_1|}\big(\Phi(g_2, \xi)\big),\ \ |\alpha_1| - |\alpha_2| + |\alpha_2| - |\beta|\Big) =$$

$$= \Big(\rho^{|\alpha_1|}\Big(\Phi(g_1, g_2\xi)\Phi(g_2, \xi)\Big),\ |\alpha_1| - |\beta|\Big) \overset{(8.11)}{=} \Big(\rho^{|\alpha_1|}\big(\Phi(g_1 g_2, \xi)\big),\ |\alpha_1| - |\beta|\Big) =$$

$$= \ell\big(\big[\alpha_1, g_1 g_2, \beta;\ \beta\xi\big]\big) \overset{(8.8)}{=} \ell(u_1 u_2). \qquad \square$$

The main relevance of this one-cocycle is that, together with the domain and range maps, it uniquely describes the elements of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, as we will now show.

8.15. Proposition. *Given $u_1, u_2 \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, one has that*

$$\left.\begin{array}{l} d(u_1) = d(u_2) \\ r(u_1) = r(u_2) \\ \ell(u_1) = \ell(u_2) \end{array}\right\} \quad \Rightarrow \quad u_1 = u_2.$$

Proof. Using (8.7), write $u_i = \big[\alpha_i, g_i, \beta_i;\ \beta_i\xi_i\big]$, for $i = 1, 2$, with $|\beta_1| = |\beta_2|$. Since

$$\beta_1\xi_1 = d(u_1) = d(u_2) = \beta_2\xi_2,$$

we conclude that $\beta_1 = \beta_2$, and

$$\xi_1 = \xi_2 =: \xi.$$

By focusing on the second coordinate of $\ell(u_i)$, we see that $|\alpha_1| - |\beta_1| = |\alpha_2| - |\beta_2|$, and hence $|\alpha_1| = |\alpha_2|$. Moreover, since

$$\alpha_1 g_1 \xi = \alpha_1 g_1 \xi_1 = r(u_1) = r(u_2) = \alpha_2 g_2 \xi_2 = \alpha_2 g_2 \xi,$$

we see that $\alpha_1 = \alpha_2$, and

$$g_1\xi = g_2\xi. \tag{8.15.1}$$

The fact that $\ell(u_1) = \ell(u_2)$ also implies that

$$\breve\rho^{|\alpha_1|}\big(\breve\Phi(g_1, \xi)\big) = \breve\rho^{|\alpha_2|}\big(\breve\Phi(g_2, \xi)\big),$$

and since $\alpha_1 = \alpha_2$, we conclude that $\breve\Phi(g_1, \xi) = \breve\Phi(g_2, \xi)$, and hence that there exists an integer n_0 such that

$$\varphi(g_1, \xi|_n) = \varphi(g_2, \xi|_n), \quad \forall\, n \geq n_0.$$

By (8.15.1) we also have that $g_1(\xi|_n) = g_2(\xi|_n)$, so (5.6) gives $g_1 = g_2$, whence $u_1 = u_2$. $\square$

As a consequence of the above result we see that the map

$$F : \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) \to E^\infty \times (\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}) \times E^\infty$$

defined by the rule

$$F(u) = \big(r(u), \ell(u), d(u)\big), \tag{8.16}$$

is one-to-one.

Observe that the co-domain of F has a natural groupoid structure, being the cartesian product of the lag group $\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}$ by the graph of the transitive equivalence relation on E^∞.

Putting together (8.14) and (8.15) we may now easily prove:

8.17. Corollary. *F is a groupoid homomorphism (functor), hence establishing an isomorphism from $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ to the range of F.*

The range of F is then the concrete model of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ we are after. But, before giving a detailed description of it, let us make a remark concerning notation: since the co-domain of F is a mixture of cartesian and semi-direct products, the standard notation for its elements would be something like $\big(\eta, (u,p), \zeta\big)$, for $\eta, \zeta \in E^\infty$, $u \in \breve{G}$, and $p \in \mathbb{Z}$. As part of our effort to avoid heavy notation we will instead denote such an element by

$$\big(\eta; u, p; \zeta\big).$$

8.18. Proposition. *The range of F is precisely the subset of $E^\infty \times (\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}) \times E^\infty$, formed by the elements $(\eta; \breve{\mathrm{g}}, p-q; \zeta)$, where $\eta, \zeta \in E^\infty$, $\mathrm{g} \in G^\infty$, and $p, q \in \mathbb{N}$, are such that, for all $n \geq 1$,*

(i) $\mathrm{g}_{n+p+1} = \varphi(\mathrm{g}_{n+p}, \zeta_{n+q})$,

(ii) $\eta_{n+p} = \mathrm{g}_{n+p}\zeta_{n+q}$.

Proof. Pick a general element $\big[\alpha, g, \beta;\ \beta\xi\big] \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ and, recalling that

$$F(\big[\alpha, g, \beta;\ \beta\xi\big]) = \Big(\alpha g\xi;\ \breve{\rho}^{|\alpha|}\big(\breve{\Phi}(g,\xi)\big),\ |\alpha| - |\beta|;\ \beta\xi\Big), \tag{8.18.1}$$

let $\eta = \alpha g \xi$, $\mathrm{g} = \rho^{|\alpha|}\big(\Phi(g,\xi)\big)$, $p = |\alpha|$, $q = |\beta|$, and $\zeta = \beta\xi$, so that the element depicted in (8.18.1) becomes $(\eta; \breve{\mathrm{g}}, p-q; \zeta)$, and we must now verify (i) and (ii). For all $n \geq 1$, one has that

$$\mathrm{g}_{n+|\alpha|} = \Phi(g,\xi)_n = \varphi(g, \xi|_{n-1}),$$

so

$$\eta_{n+p} = (\alpha g \xi)_{n+|\alpha|} = (g\xi)_n \overset{(8.12)}{=} \varphi(g, \xi|_{n-1})\xi_n = \mathrm{g}_{n+|\alpha|}(\beta\xi)_{n+|\beta|} = \mathrm{g}_{n+p}\zeta_{n+q},$$

proving (ii). Also,

$$\mathrm{g}_{n+p+1} = \mathrm{g}_{n+|\alpha|+1} = \varphi\big(g, \xi|_n\big) = \varphi(g, \xi|_{n-1}\xi_n) = \varphi\big(\varphi(g, \xi|_{n-1}), \xi_n\big) =$$

$$= \varphi\big(\mathrm{g}_{n+|\alpha|}, (\beta\xi)_{n+|\beta|}\big) = \varphi\big(\mathrm{g}_{n+p}, \zeta_{n+q}\big),$$

proving (i) and hence showing that the range of F is a subset of the set described in the statement.

Conversely, pick $\eta, \zeta \in E^\infty$, $\mathrm{g} \in G^\infty$, and $p, q \in \mathbb{N}$ satisfying (i) and (ii), and let us show that the element $(\eta; \breve{\mathrm{g}}, p-q; \zeta)$ lies in the range of F. Let

$$g = \mathrm{g}_{p+1}, \quad \alpha = \eta|_p, \quad \text{and} \quad \beta = \zeta|_q,$$

so $\zeta = \beta\xi$ for a unique $\xi \in E^\infty$. We then claim that $[\alpha, g, \beta;\, \beta\xi]$ lies in $\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E})$. In order to see this notice that

$$gd(\beta) = gd(\zeta_q) = gr(\zeta_{q+1}) = r(\mathrm{g}_{p+1}\zeta_{q+1}) \overset{\text{(ii)}}{=} r(\eta_{p+1}) = d(\eta_p) = d(\alpha),$$

so $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, and therefore $[\alpha, g, \beta;\, \beta\xi]$ is indeed a member of $\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E})$. The proof will then be concluded once we show that

$$F([\alpha, g, \beta;\, \beta\xi]) = (\eta; \breve{\mathrm{g}}, p-q; \zeta),$$

which in turn is equivalent to showing that

(a) $\alpha g\xi = \eta$,
(b) $\breve{\rho}^{|\alpha|}\big(\breve{\Phi}(g,\xi)\big) = \breve{\mathrm{g}}$,
(c) $|\alpha| - |\beta| = p - q$,
(d) $\beta\xi = \zeta$.

Before proving these points we will show that

$$\varphi(\mathrm{g}_{p+1}, \xi|_n) = \mathrm{g}_{n+p+1}, \quad \forall\, n \geq 0. \tag{†}$$

This is obvious for $n = 0$. Assuming that $n \geq 1$ and using induction, we have

$$\varphi(\mathrm{g}_{p+1}, \xi|_n) = \varphi(\mathrm{g}_{p+1}, \xi|_{n-1}\xi_n) = \varphi\big(\varphi(\mathrm{g}_{p+1}, \xi|_{n-1}), \xi_n\big) =$$

$$= \varphi(\mathrm{g}_{n+p}, \zeta_{n+q}) \overset{\text{(i)}}{=} \mathrm{g}_{n+p+1},$$

verifying (†).

Addressing (a) we have to prove that $(\alpha g\xi)_k = \eta_k$, for all $k \geq 1$, but given that α is defined to be $\eta|_p$, this is trivially true for $k \leq p$. On the other hand, for $k = n + p$, with $n \geq 1$, we have

$$(\alpha g\xi)_k = (\alpha g\xi)_{n+p} = (g\xi)_n \overset{(8.12)}{=} \varphi(g, \xi|_{n-1})\xi_n =$$

$$= \varphi(\mathrm{g}_{p+1}, \xi|_{n-1})\xi_n \overset{(†)}{=} \mathrm{g}_{n+p}\zeta_{n+q} \overset{\text{(ii)}}{=} \eta_{n+p} = \eta_k,$$

proving (a). Focusing on (b) we have for all $n \geq 1$ that

$$\rho^{|\alpha|}\big(\Phi(g,\xi)\big)_{p+n} = \Phi(g,\xi)_n = \varphi(\mathrm{g}_{p+1}, \xi|_{n-1}) \overset{(†)}{=} \mathrm{g}_{n+p},$$

proving that $\rho^{|\alpha|}\big(\Phi(g,\xi)\big) \equiv \mathrm{g}$, modulo $G^{(\infty)}$, hence taking care of (b). The last two points, namely (c) and (d) are trivial and so the proof is concluded. $\square$

As an immediate consequence we get a very precise description of the algebraic structure of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$:

8.19. Theorem. *Suppose that (G, E, φ) satisfies the conditions of (2.3) and is moreover pseudo free. Then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is isomorphic to the sub-groupoid of $E^\infty \times (\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}) \times E^\infty$ given by*

$$\mathcal{G}_{G,E} = \left\{ \begin{array}{rcl} (\eta; \breve{\mathrm{g}}, p-q; \zeta) & \in & E^\infty \times (\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}) \times E^\infty : \\ & & \mathrm{g} \in G^\infty,\ p, q \in \mathbb{N}, \\ & & \mathrm{g}_{n+p+1} = \varphi(\mathrm{g}_{n+p}, \zeta_{n+q}), \\ & & \eta_{n+p} = \mathrm{g}_{n+p}\zeta_{n+q},\ \textit{for all } n \geq 1 \end{array} \right\}.$$

Recall from [**20**] that the C*-algebra of every graph is a groupoid C*-algebra for a certain groupoid constructed from the graph, and informally called the groupoid for the *tail equivalence with lag*.

Viewed through the above perspective, our groupoid may also deserve such a denomination, except that the lag is not just an integer as in [**20**], but an element of the lag group $\breve{G} \rtimes_\rho \mathbb{Z}$ precisely described by the lag function ℓ introduced in Proposition (8.14).

9. The topology of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.

It is now time we look at the topological aspects of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. In fact what we will do is simply transfer the topology of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ over to $\mathcal{G}_{G,E}$ via F. Not surprisingly F will turn out to be an isomorphism of topological groupoids.

Recall from [**6**: Proposition 4.14] that, if $\mathcal{S}$ is an inverse semigroup acting on a locally compact Hausdorff topological space X, then the corresponding groupoid of germs, say $\mathcal{G}$, is topologized by means of the basis consisting of sets of the form

$$\Theta(s, U),$$

where $s \in \mathcal{S}$, and U is an open subset of X, contained in the domain of the partial homeomorphism attached to s by the given action. Each $\Theta(s, U)$ is in turn defined by

$$\Theta(s, U) = \Big\{ [s, x] \in \mathcal{G} : x \in U \Big\}. \tag{9.1}$$

See [**6**: 4.12] for more details.

If we restrict the choice of the U's above to a predefined basis of open sets of X, e.g. the collection of all cylinders in E^∞ in the present case, we evidently get the same topology on the groupoid of germs. Therefore, referring to the model of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ presented in (8.5), we see that a basis for its topology consists of the sets of the form

$$\Theta(\alpha, g, \beta; \gamma) := \Big\{ \big[\alpha, g, \beta;\ \xi\big] \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) : \xi \in Z(\gamma) \Big\}, \tag{9.2}$$

where $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, and $\gamma \in E^*$. We may clearly suppose that $|\gamma| \geq |\beta|$ and, since $\Theta(\alpha, g, \beta; \gamma) = \varnothing$, unless β is a prefix of γ, we may also assume that $\gamma = \beta\varepsilon$, for some $\varepsilon \in E^*$.

In this case, given any $\big[\alpha, g, \beta;\ \xi\big] \in \Theta(\alpha, g, \beta; \gamma)$, notice that $\xi \in Z(\gamma)$, and

$$(\alpha, g, \beta)(\gamma, 1, \gamma) = \big(\alpha g \varepsilon, \varphi(g, \varepsilon), \gamma\big),$$

from where one concludes that

$$\big[\alpha, g, \beta;\ \xi\big] = \big[\alpha g \varepsilon, \varphi(g, \varepsilon), \gamma;\ \xi\big],$$

for all $\xi \in Z(\gamma)$, and hence also that

$$\Theta(\alpha, g, \beta; \gamma) = \Theta\big(\alpha g \varepsilon, \varphi(g, \varepsilon), \gamma; \gamma\big).$$

This shows that any set of the form (9.2) coincides with another such set for which $\beta = \gamma$. We may therefore do away with this repetition and redefine

$$\Theta(\alpha, g, \beta) := \Big\{ \big[\alpha, g, \beta;\ \xi\big] \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) : \xi \in Z(\beta) \Big\}. \tag{9.3}$$

We have therefore shown:

9.4. Proposition. *The collection of all sets of the form $\Theta(\alpha, g, \beta)$, where (α, g, β) range in $\mathcal{S}_{G,E}$, is a basis for the topology of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.*

We may now give a precise description of the topology of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, once it is viewed from the alternative point of view of Theorem (8.19):

9.5. Proposition. *For each (α, g, β) in $\mathcal{S}_{G,E}$, the image of $\Theta(\alpha, g, \beta)$ under F coincides with the set*

$$\Omega(\alpha, g, \beta) := \left\{ \begin{array}{ll} (\eta; \breve{\mathrm{g}}, k; \zeta) \in \mathcal{G}_{G,E} : & \eta \in Z(\alpha),\ \mathrm{g} \in G^\infty,\ k = |\alpha| - |\beta|,\ \zeta \in Z(\beta), \\ & \mathrm{g}_{1+|\alpha|} = g, \\ & \mathrm{g}_{n+|\alpha|+1} = \varphi(\mathrm{g}_{n+|\alpha|}, \zeta_{n+|\beta|}), \\ & \eta_{n+|\alpha|} = \mathrm{g}_{n+|\alpha|} \zeta_{n+|\beta|},\ \textit{for all } n \geq 1 \end{array} \right\},$$

and hence the collection of all such sets form the basis for a topology on $\mathcal{G}_{G,E}$, with respect to which the latter is isomorphic to $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ as topological groupoids.

Proof. Left for the reader. □

We may now summarize the main results obtained so far:

9.6. Theorem. *Suppose that (G, E, φ) satisfies the conditions of (2.3) and is moreover pseudo free. Then $\mathcal{O}_{G,E}$ is *-isomorphic to the C^*-algebra of the groupoid $\mathcal{G}_{G,E}$ described in (8.19), once the latter is equipped with the topology generated by the basis of open sets $\Omega(\alpha, g, \beta)$ described in (9.3), for all (α, g, β) in $\mathcal{S}_{G,E}$.*

10. $\mathcal{O}_{G,E}$ as a Cuntz-Pimsner algebra.

Inspired by Nekrashevych's paper [**24**], we will now give a description of $\mathcal{O}_{G,E}$ as a Cuntz-Pimsner algebra [**29**]. With this we will also be able to prove that $\mathcal{O}_{G,E}$ is nuclear and that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is amenable when G is an amenable group. As before, we will work under the conditions of (2.3).

We begin by introducing the algebra of coefficients over which the relevant Hilbert bimodule, also known as a correspondence, will later be constructed.

Since the action of G on E preserves length by (2.4.iv), we see that the set of vertices of E is G-invariant, so we get an action of G on E^0 by restriction. By dualization G acts on the algebra $C(E^0)$ of complex valued functions[5] on E^0. We may therefore form the crossed-product C*-algebra

$$A = C(E^0) \rtimes G.$$

Since $C(E^0)$ is a unital algebra, there is a canonical unitary representation of G in the crossed product, which we will denote by $\{v_g\}_{g\in G}$.

On the other hand, $C(E^0)$ is also canonically isomorphic to a subalgebra of A and we will therefore identify these two algebras without further warnings.

For each x in E^0, we will denote the characteristic function of the singleton $\{x\}$ by q_x, so that $\{q_x : x \in E^0\}$ is the canonical basis of $C(E^0)$, and thus A coincides with the closed linear span of the set

$$\big\{q_x v_g : x \in E^0,\ g \in G\big\}. \tag{10.1}$$

For later reference, notice that the covariance condition in the crossed product reads

$$v_g q_x = q_{gx} v_g, \quad \forall\, x \in E^0, \quad \forall\, g \in G. \tag{10.2}$$

Our next step is to construct a correspondence over A. In preparation for this we denote by A^e the right ideal of A generated by $q_{d(e)}$, for each $e \in E^1$. In technical terms

$$A^e = q_{d(e)} A.$$

With the obvious right A-module structure, and the inner product defined by

$$\langle y, z\rangle = y^* z, \quad \forall\, y, z \in A^e,$$

one has that A^e is a right Hilbert A-module. Notice that this is not necessarily a full Hilbert module since $\langle A^e, A^e\rangle$ is the two-sided ideal of A generated by $q_{d(e)}$, which might be a proper ideal in some cases.

As already seen in (10.1), A is spanned by the elements of the form $q_x v_g$. Therefore A^e is spanned by the elements of the form $q_{d(e)} q_x v_g$, but, since the q's are mutually orthogonal, this is either zero or equal to $q_{d(e)} v_g$. Therefore we see that

$$A^e = \overline{\text{span}}\{q_{d(e)} v_g : g \in G\}.$$

[5] Notice that, since E^0 is a finite set, $C(E^0)$ is nothing but $\mathbb{C}^{|E^0|}$.

Introducing the right Hilbert A-module which will later be given the structure of a correspondence over A, we define

$$M = \bigoplus_{e\in E^1} A^e.$$

Observe that if x is a vertex which is the source of many edges, say

$$d^{-1}(x) = \{e_1, e_2, \ldots, e_n\},$$

then

$$A^{e_i} = q_{d(e_i)}A = q_x A,$$

for all i, so that $q_x A$ appears many times as a direct summand of M. However these copies of $q_x A$ should be suitably distinguished, according to which edge e_i is being considered.

On the other hand, notice that if $d^{-1}(x) = \varnothing$, then $q_x A$ does not appear among the summands of M, at all.

Addressing the fullness of M, observe that

$$\langle M, M\rangle = \sum_{\substack{x\in E^0\\ d^{-1}(x)\neq\varnothing}} A q_x A,$$

so, when E has no *sinks*, that is, when $d^{-1}(x)$ is nonempty for every x, one has that M is full.

Given $e \in E^1$, the element $q_{d(e)}$, when viewed as an element of $A^e \subseteq M$, will play a very special role in what follows, so we will give it a special notation, namely

$$t_e := q_{d(e)}. \tag{10.3}$$

There is a small risk of confusion here in the sense that, if $e_1, e_2 \in E^1$ are such that

$$x := d(e_1) = d(e_2),$$

then (10.3) assigns q_x to both t_{e_1} and t_{e_2}. However the coordinate in which q_x appears in t_{e_i} is determined by the corresponding e_i, so if $e_1 \neq e_2$, then $t_{e_1} \neq t_{e_2}$.

In order to completely dispel any confusion, here is the technical definition:

$$t_e = (m_f)_{f\in E^1},$$

where

$$m_f = \begin{cases} q_{d(e)}, & \text{if } f = e,\\ 0, & \text{otherwise.}\end{cases}$$

We should notice that

$$t_e q_{d(e)} = t_e, \tag{10.4}$$

and that any element $y \in M$ may be written uniquely as

$$y = \sum_{e\in E^1} t_e y_e, \tag{10.5}$$

where each $y_e \in A^e$.

As the next step in constructing a correspondence over A, we would now like to define a certain *-homomorphism from A to the algebra $\mathcal{L}(M)$ of adjointable linear operators on M. Since A is a crossed product algebra, this will be accomplished once we produce a covariant representation (ψ, V) of the C*-dynamical system $\big(C(E^0), G\big)$ on M. We begin with the group representation V.

10.6. Definition. For each $g \in G$, let V_g be the linear operator on M given by

$$V_g\Big(\sum_{e\in E^1} t_e y_e\Big) = \sum_{e\in E^1} t_{ge} v_{\varphi(g,e)} y_e,$$

whenever $y_e \in A^e$, for each e in E^1.

By the uniqueness in (10.5), it is clear that V_g is well defined.

10.7. Proposition. *Each V_g is a unitary operator on M. Moreover, the correspondence $g \mapsto V_g$ is a unitary representation of G.*

Proof. Pick g in G. We begin by claiming that the two sides in the expression defining V_g, above, coincide whenever the y_e are in A, even if y_e does not belong to A^e. Since V_g is clearly additive, we only need to check that

$$V_g(t_e y) = t_{ge} v_{\varphi(g,e)} y, \quad \forall\, y \in A.$$

Observing that $t_e = t_e q_{d(e)}$, we have

$$V_g(t_e y) = V_g(t_e q_{d(e)} y) = t_{ge} v_{\varphi(g,e)} q_{d(e)} y =$$

$$= t_{ge} q_{\varphi(g,e)d(e)} v_{\varphi(g,e)} y \overset{(2.5.\text{vii})}{=} t_{ge} q_{d(ge)} v_{\varphi(g,e)} y = t_{ge} v_{\varphi(g,e)} y,$$

proving the claim. One therefore concludes that V_g is right-A-linear.

We next claim that, for all $e, f \in E^1$, one has

$$\langle V_g(t_e), t_f\rangle = \langle t_e, V_{g^{-1}}(t_f)\rangle. \tag{10.7.1}$$

We have

$$\langle V_g(t_e), t_f\rangle = \langle t_{ge} v_{\varphi(g,e)}, t_f\rangle = v^*_{\varphi(g,e)}\langle t_{ge}, t_f\rangle = [ge{=}f]\, v^{-1}_{\varphi(g,e)} q_{d(ge)} =$$

$$= [ge{=}f]\, q_{\varphi(g,e)^{-1}d(ge)} v^{-1}_{\varphi(g,e)} \overset{(2.5.\text{vii})}{=} [ge{=}f]\, q_{d(e)} v^{-1}_{\varphi(g,e)} = (\star).$$

Starting from the right-hand-side of (10.7.1), we have

$$\langle t_e, V_{g^{-1}}(t_f)\rangle = \langle t_e, t_{g^{-1}f} v_{\varphi(g^{-1},f)}\rangle = [e{=}g^{-1}f]\, q_{d(e)} v_{\varphi(g^{-1},f)} =$$

$$= [ge{=}f]\, q_{d(e)} v_{\varphi(g,g^{-1}f)^{-1}} = [ge{=}f]\, q_{d(e)} v^{-1}_{\varphi(g,e)},$$

which agrees with $(\star)$ above, and hence proves claim (10.7.1). If $y, z \in A$, we then have that

$$\langle V_g(t_e y), t_f z\rangle = y^*\langle V_g(t_e), t_f\rangle z = y^*\langle t_e, V_{g^{-1}}(t_f)\rangle z = \langle t_e y, V_{g^{-1}}(t_f z)\rangle,$$

from where one sees that $\langle V_g(\xi), \eta\rangle = \langle \xi, V_{g^{-1}}(\eta)\rangle$, for all $\xi, \eta \in M$, hence proving that V_g is an adjointable operator with $V_g^* = V_{g^{-1}}$.

Let us next prove that

$$V_g V_h = V_{gh}, \quad \forall\, g, h \in G.$$

By A-linearity it is enough to prove that these operators coincide on the set formed by the t_e's, which is a generating set for M. We thus compute

$$V_g\big(V_h(t_e)\big) = V_g\big(t_{he} v_{\varphi(h,e)}\big) = t_{ghe} v_{\varphi(g,he)} v_{\varphi(h,e)} = t_{ghe} v_{\varphi(gh,e)} = V_{gh}(t_e).$$

Since it is evident that V_1 is the identity operator on M we obtain, as a consequence, that $V_g^{-1} = V_{g^{-1}} = V_g^*$, so each V_g is unitary and the proof is concluded. $\square$

In order to complete our covariant pair we must now construct a *-homomorphism from $C(E^0)$ to $\mathcal{L}(M)$. With this in mind we give the following:

10.8. Definition. For every x in E^0, let

$$M_x = \bigoplus_{e\in r^{-1}(x)} A^e,$$

which we view as a complemented sub-module of M. In addition, we let Q_x be the orthogonal projection from M to M_x, so that

$$Q_x(t_e y) = [r(e)=x]\, t_e y, \quad \forall\, e \in E^1, \quad \forall\, y \in A. \tag{10.8.1}$$

Observe that the Q_x are pairwise orthogonal projections and that $\sum_{x\in E^0} Q_x = 1$.

10.9. Definition. Let $\psi : C(E^0) \to \mathcal{L}(M)$ be the unique unital *-homomorphism such that

$$\psi(q_x) = Q_x, \quad \forall\, x \in E^0.$$

From our working hypothesis that E has no sources, we see that for every x in E^0, there is some $e \in E^1$ such that $r(e) = x$. So

$$Q_x(t_e) = t_e,$$

whence $Q_x \neq 0$. Consequently ψ is injective.

10.10. Proposition. *The pair (ψ, V) is a covariant representation of the C*-dynamical system $\big(C(E^0), G\big)$ in $\mathcal{L}(M)$.*

Proof. All we must do is check the covariance condition

$$V_g\psi(y) = \psi\big(\sigma_g(y)\big)V_g, \quad \forall\, g \in G, \quad \forall\, y \in C(E^0),$$

where σ is the name we temporarily give to the action of G on $C(E^0)$. Since $C(E^0)$ is spanned by the q_x, it suffices to consider $y = q_x$, in which case the above identity becomes

$$V_g Q_x = Q_{gx} V_g. \tag{10.10.1}$$

Furthermore M is generated, as an A-module, by the t_e, for $e \in E^1$, so we only need to verify this on the t_e. We have

$$V_g\big(Q_x(t_e)\big) = [r(e)=x]\, V_g(t_e) = [r(e)=x]\, t_{ge} v_{\varphi(g,e)},$$

while

$$Q_{gx}\big(V_g(t_e)\big) = Q_{gx}\big(t_{ge} v_{\varphi(g,e)}\big) = [r(ge)=gx]\, t_{ge} v_{\varphi(g,e)},$$

verifying (10.10.1) and concluding the proof. ☐

It follows from [**28**: Proposition 7.6.4 and Theorem 7.6.6] that there exists a *-homomorphism

$$\Psi : C(E^0) \rtimes G \to \mathcal{L}(M),$$

such that

$$\Psi(q_x) = Q_x, \quad \forall\, x \in E^0,$$

and

$$\Psi(v_g) = V_g, \quad \forall\, g \in G.$$

Equipped with the left-A-module structure provided by Ψ, we then have that M is a correspondence over A.

For later reference we record here a few useful calculations involving the left-module structure of M.

10.11. Proposition. *Let $g \in G$, $e \in E^1$, and $x \in E^0$. Then*

(i) $v_g t_e = t_{ge} v_{\varphi(g,e)}$,

(ii) $q_x v_g t_e = [r(ge)=x]\, t_{ge} v_{\varphi(g,e)}$.

Proof. We have

$$v_g t_e = \Psi(v_g) t_e = V_g(t_e) = t_{ge} v_{\varphi(g,e)},$$

proving (a). Also

$$q_x v_g t_e = \Psi(q_x)(v_g t_e) = Q_x(t_{ge} v_{\varphi(g,e)}) = [r(ge)=x]\, t_{ge} v_{\varphi(g,e)}. \qquad \square$$

It is our next goal to prove that $\mathcal{O}_{G,E}$ is naturally isomorphic to the Cuntz-Pimsner algebra associated to the correspondence M, which we denote by $\mathcal{O}_M$. As a first step, we identify a certain Cuntz-Krieger E-family.

10.12. Proposition. *The following relations hold within $\mathcal{O}_M$.*

(a) *For every $x \in E^0$, one has that $\sum_{e\in r^{-1}(x)} t_e t_e^* = q_x$.*

(b) $\sum_{e\in E^1} t_e t_e^* = 1$.

(c) *The set $\{q_x : x \in E^0\} \cup \{t_e : e \in E^1\}$ is a Cuntz-Krieger E-family.*

Proof. We first claim that, for every $x \in E^0$, and every $m \in M$, one has that

$$\sum_{e\in r^{-1}(x)} t_e t_e^* m = q_x m.$$

To prove it, it is enough to consider the case in which $m = t_f$, for $f \in E^1$, since these generate M. In this case we have

$$\sum_{e\in r^{-1}(x)} t_e t_e^* t_f = [r(f)=x]\, t_f t_f^* t_f = [r(f)=x]\, t_f \overset{(10.8.1)}{=} Q_x(t_f) = q_x t_f,$$

proving the claim. This says that the pair $\big(q_x, \sum_{e\in r^{-1}(x)} t_e t_e^*\big)$ is a redundancy or, adopting the terminology of [**29**], that the generalized compact operator

$$\sum_{e\in r^{-1}(x)} \Omega_{t_e,t_e}$$

is mapped to $\Psi(q_x)$ via $\Psi^{(1)}$. Therefore

$$q_x = \sum_{e\in r^{-1}(x)} t_e t_e^*,$$

in $\mathcal{O}_M$, proving (a). Point (b) then follows from the fact that $\sum_{x\in E^0} q_x = 1$.

Focusing now on (c), it is evident that $\{q_x : x \in E^0\}$ is a family of mutually orthogonal projections. Moreover, for each $e \in E^1$, we have

$$t_e^* t_e = \langle t_e, t_e\rangle = q_{d(e)},$$

proving (3.1.i) and also that t_e is a partial isometry. Property (3.1.ii) also holds in view of (a), so the proof is concluded. □

10.13. Proposition. *There exists a unique surjective *-homomorphism*

$$\Lambda : \mathcal{O}_{G,E} \to \mathcal{O}_M$$

such that $\Lambda(p_x) = q_x$, $\Lambda(s_e) = t_e$, *and* $\Lambda(u_g) = v_g$.

Proof. By the universal property of $\mathcal{O}_{G,E}$, in order to prove the existence of Λ it is enough to check that the q_x, t_e, and v_g satisfy the conditions of Definition (3.2).

Condition (3.2.a) has already been proved above while (3.2.b) is evidently true since v is a representation of G in $C(E^0) \rtimes G \subseteq \mathcal{O}_M$. Condition (3.2.c) is precisely (10.11.i), while (3.2.d) was taken care of in (10.2).

Since A is spanned by the q_x and the v_g by (10.1), and since M is generated over A by the t_e, we see that $\mathcal{O}_M$ is spanned by the set

$$\{q_x, t_e, v_g : x \in E^0,\ e \in E^1,\ g \in G\},$$

so Λ is surjective. □

Let us now prove that Λ is invertible by providing an inverse to it. Since A is the crossed product C*-algebra $C(E^0) \rtimes G$, one sees that (3.2.a&d) guarantees the existence of a *-homomorphism

$$\theta_A : A \to \mathcal{O}_{G,E},$$

sending the q_x to the p_x, and the v_g to the u_g. For each e in E^1, consider the linear mapping

$$\theta_M : M \to \mathcal{O}_{G,E},$$

given, for every $m = (m_e)_{e\in E^1} \in M$, by

$$\theta_M(m) = \sum_{e\in E^1} s_e\theta_A(m_e) \in \mathcal{O}_{G,E}.$$

Notice that $\theta_M(t_e) = s_e$, for all $e \in E^1$, because

$$\theta_M(t_e) = s_e\theta_A(q_{d(e)}) = s_e p_{d(e)} = s_e.$$

10.14. Lemma. *The pair (θ_A, θ_M) is a representation of the correspondence M in the sense of* [**29**: Theorem 3.4], *meaning that for all $y \in A$ and all $\xi, \xi' \in M$,*

(i) $\theta_M(\xi)\theta_A(y) = \theta_M(\xi y)$,

(ii) $\theta_A(y)\theta_M(\xi) = \theta_M(y\xi)$,

(iii) $\theta_M(\xi)^*\theta_M(\xi') = \theta_A(\langle\xi, \xi'\rangle)$.

Proof. Considering the various spanning sets at our disposal, we may assume that $y = q_x v_g$, that $\xi = t_e z$, and $\xi' = t_{e'} z'$, with $x \in E^0$, $g \in G$, $e, e' \in E^1$, $z \in q_{d(e)}A$, and $z' \in q_{d(e')}A$. We then have

$$\theta_M(\xi)\theta_A(y) = \theta_M(t_e z)\theta_A(y) = s_e\theta_A(z)\theta_A(y) = s_e\theta_A(zy) = \theta_M(t_e zy) = \theta_M(\xi y),$$

proving (i). As for (ii), we have

$$\theta_A(y)\theta_M(\xi) = \theta_A(q_x v_g)\theta_M(t_e z) = p_x u_g s_e\theta_A(z) = p_x s_{ge} u_{\varphi(g,e)}\theta_A(z) =$$

$$= [r(ge)=x]\, s_{ge}\theta_A(v_{\varphi(g,e)}z) = [r(ge)=x]\, \theta_M(t_{ge}v_{\varphi(g,e)}z) \overset{(10.11.ii)}{=} \theta_M(q_x v_g t_e z) = \theta_M(y\xi),$$

proving (ii). Focusing now on (iii), we have

$$\theta_M(\xi)^*\theta_M(\xi') = \big(s_e\theta_A(z)\big)^* s_{e'}\theta_A(z') = [e=e']\, \theta_A(z)^* p_{d(e)}\theta_A(z') =$$

$$= [e=e']\, \theta_A(z^* q_{d(e)} z') = \theta_A(\langle\xi, \xi'\rangle). \qquad \square$$

It is well known [**29**: Theorem 3.4] that the Toeplitz algebra for the correspondence M, usually denoted $\mathcal{T}_M$, is universal for representations of M, so there exists a *-homomorphism

$$\Theta_0 : \mathcal{T}_M \to \mathcal{O}_{G,E},$$

coinciding with θ_A on A and with θ_M on M.

10.15. Theorem. *The map Θ_0, defined above, factors through $\mathcal{O}_M$, providing a *-isomorphism*

$$\Theta : \mathcal{O}_M \to \mathcal{O}_{G,E},$$

such that $\Theta(q_x) = p_x$, $\Theta(t_e) = s_e$, and $\Theta(v_g) = u_g$, for all $x \in E^0$, $e \in E^1$, and $g \in G$.

Proof. The factorization property follows immediately from (10.12.b) and an easy modification of [**14**: Proposition 7.1] to Cuntz-Pimsner algebras.

In order to prove that Θ is an isomorphism, observe that $\Theta \circ \Lambda$ coincides with the identity map on the generators of $\mathcal{O}_{G,E}$, by (10.13), and hence $\Theta \circ \Lambda = id$. The result then follows from the fact that Λ is surjective. □

10.16. Corollary. *If G is amenable then $\mathcal{O}_{G,E}$ is nuclear.*

Proof. The amenability of G ensures that $C(E^0) \rtimes G$ is nuclear. The result then follows from Theorem (10.15), the fact that Toeplitz-Pimsner algebras over nuclear coefficient algebras is nuclear [**5**: Theorem 4.6.25], and so are quotients of nuclear algebras [**5**: Theorem 9.4.4]. □

10.17. Remark. Since E^0 is finite, the nuclearity of $C(E^0) \rtimes G$ is equivalent to the amenability of G. However, if the present construction is generalized for infinite graphs, one could produce examples of non amenable groups acting amenably on E^0, in which case $C(E^0) \rtimes G$ would be nuclear. The proof of Corollary (10.16) could then be adapted to prove that $\mathcal{O}_{G,E}$ is nuclear.

10.18. Corollary. *If G is amenable, then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is an amenable groupoid. If moreover (G, E, φ) is pseudo free, then its sibling $\mathcal{G}_{G,E}$ is an amenable groupoid.*

Proof. For $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, it follows from (10.16), (6.4) and [**5**: Theorem 5.6.18]. For $\mathcal{G}_{G,E}$, it follows from (10.16), (9.6) and [**5**: Theorem 5.6.18]. □

Nekrashevych has proven in [**26**: Theorem 5.6], that a certain groupoid of germs, denoted $\mathcal{D}_G$, constructed in the context of self-similar groups, is amenable under the hypothesis that the group is *contracting* and *self-replicating*. Even though there are numerous differences between $\mathcal{D}_G$ and $\mathcal{G}_{G,E}$, including a different notion of *germs* and Nekrashevych's requirement that group actions be *faithful*, we believe it should be interesting to try to generalize Nekrashevych's result to our context.

11. Representing $C^*(E)$ and G into $\mathcal{O}_{G,E}$.

In this section we will study natural representations of the graph C*-algebra $C^*(E)$ and of the group G in $\mathcal{O}_{G,E}$. As before, we keep (2.3) in force.

Given that

$$\{p_x : x \in E^0\} \cup \{s_e : e \in E^1\}$$

is a Cuntz-Krieger E-family, the universal property of the graph C*-algebra $C^*(E)$ [**30**] provides for the existence of a *-homomorphism

$$\iota : C^*(E) \to \mathcal{O}_{G,E},$$

sending the canonical Cuntz-Krieger E-family of $C^*(E)$ to the corresponding one within $\mathcal{O}_{G,E}$.

11.1. Proposition. *The *-homomorphism ι above is injective.*

Proof. Using the universal property of $\mathcal{O}_{G,E}$, it is easy to see that, for each complex number z, with $|z| = 1$, there is a *-homomorphism

$$\gamma_z : \mathcal{O}_{G,E} \to \mathcal{O}_{G,E},$$

satisfying

$$\gamma_z(p_x) = p_x, \quad \gamma_z(s_e) = zs_e, \quad \text{and} \quad \gamma_z(u_g) = u_g,$$

for all $x \in E^0$, $e \in E^1$ and $g \in G$. It is also easy to see that the correspondence $z \to \gamma_z$ defines an action of the circle group on $\mathcal{O}_{G,E}$, and moreover that ι is covariant relative to this action on $\mathcal{O}_{G,E}$, on the one hand, and the standard gauge action on $C^*(E)$, on the other. In order to prove the injectivity of ι we may then apply the gauge invariant uniqueness Theorem [**30**: Theorem 2.2], which requires, in addition, that we verify that the p_x are nonzero.

To prove this we observe that, in the groupoid model of $\mathcal{O}_{G,E}$ given by (9.6), for each x in E^0, the element p_x is the characteristic function of the cylinder $Z(x)$, seen as a subset of E^∞, which in turn is the unit space of the groupoid $\mathcal{G}_{G,E}$. Since E has no sources, we have that $Z(x)$ is nonempty, whence p_x is nonzero, as required. This concludes the proof. □

In particular, (11.1) implies that $\mathcal{S}_E$ is $*$-isomorphic to the inverse semigroup of $\mathcal{O}_{G,E}$ generated by $\{s_a : a \in E^1\}$.

With respect to the injectivity of the representation of G into $\mathcal{O}_{G,E}$, we have to work a bit more to obtain a result in the line of (11.1).

11.2. Lemma. *Let $\pi : \mathcal{S}_{G,E} \to \mathcal{O}_{G,E}$ and $u : G \to \mathcal{O}_{G,E}$ be the natural maps. If π is injective, the so is u.*

Proof. Let $g \in G$ such that $u_g = u_1$. For any $x \in E^0$ we have $\pi(x, g, g^{-1}x) = p_x u_g$ and $\pi(x, 1, x) = p_x u_1 = p_x$. Since $u_g = u_1$, we get $(x, g, g^{-1}x) = (x, 1, x) \in \mathcal{S}_{G,E}$, whence $g = 1$. □

We need to recall some extra definitions. Let $\mathcal{G}$ be an étale groupoid, i.e. a topological groupoid whose unit space $\mathcal{G}^{(0)}$ is locally compact and Hausdorff in the relative topology, and such that the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism (and then so is the source map $d : \mathcal{G} \to \mathcal{G}^{(0)}$). An open set $U \subset \mathcal{G}$ is a slice if the restrictions of r and d to U are injective (see e.g. [**27**]). In particular, $\mathcal{G}^{(0)}$ is a slice [**6**: Proposition 3.4], and the collection of all slices forms a basis for the topology of $\mathcal{G}$ [**6**: Proposition 3.5].

11.3. Definition. We denote by $\mathcal{S}\ell(G, E)$ the set of all compact slices. It is well known (see e.g. [**27**: Proposition 2.2.4]) that $\mathcal{S}\ell(G, E)$ forms a $*$-inverse semigroup with the operations

$$UV = \{uv : u \in U, v \in V, (u, v) \in \mathcal{G}^{(2)}\}, \text{ and } U^* = \{u^{-1} : u \in U\}.$$

Moreover, if $\mathcal{U}_{G,E} = \{1_U : U \text{ is a compact slice}\} \subseteq C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ is the semigroup formed by their characteristic functions, then

$$\mathcal{U}_{G,E} \cong \mathcal{S}\ell(G, E). \tag{11.3.1}$$

Fix the canonical action θ of $\mathcal{S}_{G,E}$ on E^∞. Given any $(\alpha,g,\beta)\in\mathcal{S}_{G,E}$, notice that the domain $\mathrm{Dom}(\theta_{(\alpha,g,\beta)})$ of the partial homeomorphism of E^∞ given by the action of (α,g,β) is $Z(\beta)$. Now, given $(\alpha,g,\beta)\in\mathcal{S}_{G,E}$ and any open set $U\subseteq Z(\beta)$, set (see Section 9)

$$\Theta((\alpha,g,\beta),U)=\{[\alpha,g,\beta;\eta]:\eta\in U\}.$$

According to [**6**: Proposition 4.18], for every $(\alpha,g,\beta)\in\mathcal{S}_{G,E}$ and every open set $U\subseteq Z(\beta)$, $\Theta((\alpha,g,\beta),U)$ is a slice (in fact, they form a basis for the topology of $\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E})$). Then, we have the following result

11.4. Lemma. $\mathcal{S}\ell(G,E)=\langle\Theta((\alpha,g,\beta),Z(\beta)):(\alpha,g,\beta)\in\mathcal{S}_{G,E}\rangle$.

Proof. By [**33**: Proposition 5.13(7)], $C^*(\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E}))$ is generated by

$$\{1_{\Theta((\alpha,g,\beta),Z(\beta))}:(\alpha,g,\beta)\in\mathcal{S}_{G,E}\}.$$

Thus, the result holds by (11.3.1). □

The next result is the key point for proving the injectivity of $u:G\to\mathcal{O}_{G,E}$.

11.5. Lemma. *If (G,E,φ) is pseudo free, then the map*

$$\begin{array}{cccc}\rho: & \mathcal{S}_{G,E} & \to & \mathcal{S}\ell(G,E)\\ & (\alpha,g,\beta) & \mapsto & \Theta((\alpha,g,\beta),Z(\beta))\end{array}$$

is a $*$-semigroup isomorphism.

Proof. The surjectivity of ρ derives from (11.4).

Now, let $(\alpha,g,\beta),(\gamma,h,\eta)\in\mathcal{S}_{G,E}$ such that $\Theta((\alpha,g,\beta),Z(\beta))=\Theta((\gamma,h,\eta),Z(\eta))$. Then, for any $\omega\in Z(\beta)$ we have $[\alpha,g,\beta;\omega]=[\gamma,h,\eta;\omega]$. Since (G,E,φ) is pseudo free, by (8.6) there exists $\tau\in E^*$ such that $\gamma=\alpha\cdot g\tau$, $\eta=\beta\tau$ and $h=\varphi(g,\tau)$. If $(\alpha,g,\beta)\neq(\gamma,h,\eta)$, then we can pick $\delta\neq\tau$ in E^* and $\widehat{\omega}=\beta\delta\widetilde{\omega}\in Z(\beta)$. Thus, $\widehat{\omega}\in Z(\eta)$ but $[(\gamma,h,\eta),\widehat{\omega}]$ is not defined, contradicting the hypothesis. Hence, $(\alpha,g,\beta)=(\gamma,h,\eta)$, whence ρ is injective, as desired. □

11.6. Proposition. *There exists a $*$-isomorphism $\phi:\mathcal{O}_{G,E}\to C^*(\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E}))$ such that $\phi(p_x)=1_{\Theta((x,1,x),Z(x))}$ for every $x\in E^0$, $\phi(s_a)=1_{\Theta((a,1,d(a)),Z(a))}$ for every $a\in E^1$, and $\phi(u_g)=\sum\limits_{x\in E^0}1_{\Theta((x,g,g^{-1}x),Z(g^{-1}x))}$ for every $g\in G$.*

Proof. Notice that $u_g=\sum\limits_{x\in E^0}u_gp_x$. Then it is direct but tedious to check that

$$\left\{1_{\Theta((x,1,x),Z(x))}:x\in E^0\right\}\cup\left\{1_{\Theta((a,1,d(a)),Z(a))}:a\in E^1\right\}\cup$$
$$\cup\left\{\sum_{x\in E^0}1_{\Theta((x,g,g^{-1}x),Z(g^{-1}x))}:g\in G\right\}$$

satisfy the defining relations for $\mathcal{O}_{G,E}$. Thus, by the Universal Property of $\mathcal{O}_{G,E}$, the map φ is an $*$-homomorphism. Notice that φ is the homomorphism given in [**6**: Theorem 13.3], and so is injective. Surjectivity is due to [**33**: Proposition 5.13(7)]. □

11.7. Corollary. *If (G, E, φ) is pseudo free, then $\pi : \mathcal{S}_{G,E} \to \mathcal{O}_{G,E}$ is injective.*

Proof. The composition map

$$\begin{array}{ccccc} \mathcal{S}_{G,E} & \to & \mathcal{O}_{G,E} & \to & C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) \\ (\alpha, g, \beta) & \mapsto & s_\alpha u_g s_\beta^* & \mapsto & 1_{\Theta((\alpha,g,\beta),Z(\beta))} \end{array}$$

is injective by (11.4) and (11.5). By (11.6),

$$\mathcal{O}_{G,E} \to C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$$

is injective. Thus, π is injective. □

Hence, we conclude

11.8. Proposition. *If (G, E, φ) is pseudo free, then $u : G \to \mathcal{O}_{G,E}$ is injective.*

Proof. By Corollary (11.7), π is injective. Then, so is u by (11.2). □

11.9. Remark. *In particular, (11.8) implies that, if (G, E, φ) is pseudo free, then $\mathcal{S}_{G,E}$ is $*$-isomorphic to the inverse semigroup of $\mathcal{O}_{G,E}$ generated by $\{s_a : a \in E^1\} \cup \{p_x u_g : x \in E^0, g \in G\}$.*

Proposition (11.8) provides the best situation possible, as the next example shows:

11.10. Example. Let E be the graph with only one vertex and one edge, and let G be any noncommutative group. Fix the trivial action of G on E, and let φ be the one-cocycle of G defined by $\varphi(g, a) = 1$ for every $g \in G, a \in E^1$. Then, it is easy to see that $\mathcal{O}_{G,E} \cong C^*(E) \cong C(\mathbb{T})$, which is a commutative C*-algebra, so that it cannot contain any faithful copy of G.

12. The Hausdorff property for $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.

Again considering a triple (G, E, φ) satisfying (2.3), we will now give a characterization of the Hausdorff property for the tight groupoid of $\mathcal{S}_{G,E}$. The first result we may present in this direction is:

12.1. Proposition. *If (G, E, φ) is pseudo free, then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is a Hausdorff groupoid.*

Proof. If (G, E, φ) is pseudo free, then $\mathcal{S}_{G,E}$ is E*-unitary by (5.8), so $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff by [**12**: Corollary 3.17]. This could also be obtained from [**6**: Propositions 6.4 and 6.2]. □

The converse of the above result is not true: as we will see in Example (18.15), there are examples in which (G, E, φ) fails to be pseudo free but still $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff.

This may be interpreted as saying that the above assumption that (G, E, φ) is pseudo free is a much too strong hypothesis which one would therefore like to relax.

On the other hand, recall from (5.5) that the failure of pseudo freeness for (G, E, φ) is equivalent to the existence of strongly fixed paths for nontrivial group elements. The result below consists in allowing a limited amount of minimal strongly fixed paths, and hence a limited number of counter-examples for pseudo freeness, without harming Hausdorffnes.

12.2. Theorem. *Assuming that (G, E, φ) satisfies (2.3), the following are equivalent:*

(a) *for every g in G, there are at most finitely many minimal strongly fixed paths for g,*

(b) *$\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff.*

Proof. We will of course use [**12**: Theorem 3.16]. So, given s in $\mathcal{S}_{G,E}$, we must provide a finite cover for $\mathcal{J}_s$. Since such a cover exists by trivial reasons when $\mathcal{J}_s$ is empty, let us assume that s dominates at least one nonzero idempotent element. By (5.7) we then have that s necessarily has the form

$$s = (\alpha, g, \alpha),$$

and the set of nonzero idempotent elements dominated by s is given by

$$\mathcal{J}_s = \big\{(\alpha\tau, 1, \alpha\tau) : \tau \in E^*,\ d(\alpha) = r(\tau),\ \tau \text{ is strongly fixed by } g\big\}.$$

Using (5.3) we may further describe $\mathcal{J}_s$ as

$$\mathcal{J}_s = \big\{(\alpha\mu\gamma, 1, \alpha\mu\gamma) : \mu \in M_g,\ \gamma \in E^*,\ d(\alpha) = r(\mu),\ d(\mu) = r(\gamma)\big\}, \tag{12.2.1}$$

where M_g is the set of all minimal strongly fixed paths for g.

Assuming (a), we have that M_g is finite and then it is clear that

$$\big\{(\alpha\mu, 1, \alpha\mu) : \mu \in M_g,\ d(\alpha) = r(\mu)\big\}$$

is a finite cover for $\mathcal{J}_s$, whence $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff by [**12**: Theorem 3.16].

Assuming (b), let $g \in G$, and for each vertex x in E^0, denote by M_g^x the set of all minimal strongly fixed paths for g whose range coincides with x. Since E is finite, in order to prove that M_g is finite, it is enough to check that each M_g^x is finite.

If M_g^x is empty, there is nothing to do, so let us assume the contrary. Given any μ in M_g^x, we then have that

$$x = r(\mu) = r(g\mu) = gr(\mu) = gx,$$

so x is fixed by g. Consequently $s := (x, g, x)$ lies in $\mathcal{S}_{G,E}$ and, assuming (b), we have by [**12**: Theorem 3.16] that $\mathcal{J}_s$ admits a finite cover which, in view of (12.2.1), must necessarily be of the form

$$\big\{(\mu_i\gamma_i, 1, \mu_i\gamma_i)\big\}_{i=1}^n,$$

where the $\mu_i \in M_g^x$, and the γ_i are paths with $d(\mu_i) = r(\gamma_i)$. We then claim that the μ_i apearing above exhaust M_g^x, meaning that

$$M_g^x = \{\mu_i : i = 1, \ldots, n\}. \tag{12.2.2}$$

To see this, let $\mu \in M_g^x$, so that $(\mu, 1, \mu) \leq s$, by (5.7), and hence $(\mu, 1, \mu) \in \mathcal{J}_s$. For some i, one would then have that

$$(\mu_i\gamma_i, 1, \mu_i\gamma_i)(\mu, 1, \mu) \neq 0,$$

in which case either μ is a prefix of $\mu_i\gamma_i$, or vice versa. This implies that either μ is a prefix of μ_i, or vice versa, but since both μ and μ_i are minimal, we must have $\mu = \mu_i$, proving (12.2.2), and hence that M_g^x is finite. Consequently M_g, which decomposes as the disjoint union of the M_g^x, is also finite. This verifies (a) and hence concludes the proof. $\square$

13. Minimality for $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.

In this section we will study conditions under which $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is minimal. Some of the results we obtain here are analog to those proved in [**9**] for the case of partial actions of groups.

Given a triple (G, E, φ) satisfying (2.3), there are two relations among vertices in E^0 which are relevant for the question at hand. First of all let us say that

$$x \rightharpoonup y$$

provided there exists a path α in E^* such that $d(\alpha) = x$ and $r(\alpha) = y$. Notice that this relation is reflexive (take α to be x) and transitive (take the concatenation of the relevant paths). However this is neither symmetric nor antisymmetric, hence it is not an equivalence relation nor an order relation.

The other relation we have in mind is simply the orbit relation, defined by

$$x \sim y$$

when there exists g in G such that $gx = y$. Unlike "$\rightharpoonup$", it is well known that "$\sim$" is an equivalence relation.

We may then consider the smallest transitive relation extending both "$\rightharpoonup$" and "$\sim$", by saying that vertices x and y are related when one may find a sequence of vertices $x_0, x_1, \ldots, x_{2n}$ such that

$$x = x_0 \rightharpoonup x_1 \sim x_2 \rightharpoonup x_3 \sim \ldots \sim x_{2n-2} \rightharpoonup x_{2n-1} \sim x_{2n} = y. \tag{13.1}$$

The situation is in fact not so complicated due to the following:

13.2. Proposition. *Let x and y be vertices in E^0. Then the following are equivalent;*

(i) *there exists a vertex u such that $x \rightharpoonup u \sim y$,*

(ii) *there exists a vertex v such that $x \sim v \rightharpoonup y$.*

Proof. The fact that $x \rightharpoonup u \sim y$ means that there exists a path α in E^* such that $d(\alpha) = x$, and $r(\alpha) = u$, and there exists some g in G such that $gu = y$. Considering the path $\beta = g\alpha$, and the vertex $v = gx$, notice that

$$d(\beta) = d(g\alpha) = gd(\alpha) = gx = v,$$

while

$$r(\beta) = r(g\alpha) = gr(\alpha) = gu = y,$$

so $x \sim v \rightharpoonup y$. Conversely, assuming (ii) we have that $gx = v = d(\beta)$ and $r(\beta) = y$, for suitable g in G and β in E^*. Defining $u = g^{-1}y$, and $\alpha = g^{-1}\beta$, we then have that

$$d(\alpha) = g^{-1}d(\beta) = x,$$

and

$$r(\alpha) = g^{-1}r(\beta) = g^{-1}y = u,$$

so $x \rightharpoonup u \sim y$. □

13.3. Definition. Given x and y in E^0, we will say that

$$x \gg y$$

if the equivalent conditions of (13.2) are satisfied.

Observe that "$\gg$" coincides with the relation defined in (13.1), thanks to (13.2), and hence it is clearly transitive. It is also evident that "$\gg$" is reflexive but, again, it is neither symmetric nor antisymmetric. Nevertheless we will view it as a *defective order relation*, in the sense that it satisfies all of the postulates of a (partial) order relation but for antisymmetry.

Anytime we have such a defective order relation, it is possible to turn it into a bona fide partial order by identifying elements whenever antisymmetry fails. By this we mean that two vertices x and y in E^0 will be called equivalent, in symbols

$$x \approx y$$

whenever $x \gg y$ and $y \gg x$. Writing $[x]$ for the equivalence class of each x in E^0, the set of all equivalent classes, namely

$$\frac{E^0}{\approx} = \big\{[x] : x \in E^0\big\}$$

becomes a partially ordered set via the well defined order relation

$$[x] \geq [y] \iff x \gg y.$$

13.4. Definition. Under the assumptions of (2.3), we will say that:

(i) E is *G-transitive* if, for any two vertices x and y in E^0, one has that $x \gg y$,

(ii) E is *weakly G-transitive* if, given any infinite path ξ, and any vertex x in E^0, there is some vertex v along ξ such that $v \gg x$.

The notion of G-transitivity generalizes the well known notion of transitivity in graphs. When it holds, E^0 has a single equivalence class.

On the other hand, weak G-transitivity is inspired by the notion of cofinality introduced in [**20**: Section 3], (see also [**8**: Definition 37.16]). The reader is however warned that the notions of weak G-transitivity and cofinality may only be reconciled upon a reversal of the direction of the edges in E^1, following the new trend in graph algebras started by Katsura (see the penultimate paragraph of the introduction in [**17**]).

It is evident that every G-transitive graph is weakly G-transitive, but these are sometimes equivalent notions as we will now show:

13.5. Proposition. *In addition to the assumptions in (2.3), suppose that E has no sinks, meaning that $d^{-1}(x)$ is nonempty for every x in E^0. Then, if E is weakly G-transitive, it must also be G-transitive.*

Proof. Since E^0 is finite, we may choose a minimal element $[x]$ in $E^0/\approx$. Using that E has no sinks, we may find an infinite sequence of edges

$$\ldots, \alpha_{-i-1}, \alpha_{-i}, \ldots, \alpha_{-2}, \alpha_{-1}, \alpha_0 \in E^1,$$

such that $d(\alpha_0) = x$, and $d(\alpha_{i-1}) = r(\alpha_i)$, for every $i \leq 0$. Since E^1 is also finite, there must be repetitions among the α_i, say $\alpha_m = \alpha_n$, where $m < n \leq 0$. The finite path

$$\gamma = \alpha_m \alpha_{m+1} \ldots \alpha_{n-1},$$

therefore satisfies

$$d(\gamma) = d(\alpha_{n-1}) = r(\alpha_n) = r(\alpha_m) = r(\gamma),$$

and hence γ may be concatenated with itself infinitely many times producing the infinite path

$$\xi = \gamma\gamma\gamma \ldots$$

Given any y in E^0, and assuming weak G-transitivity, there is some vertex v along ξ, such that $v \gg y$. Since ξ is made of repetitions of γ, one has that $v = r(\alpha_k)$, for some k in the integer interval $[m, n]$. We then have

$$x = d(\alpha_0) = d(\alpha_k \ldots \alpha_{-2}\alpha_{-1}\alpha_0) \rightharpoonup r(\alpha_k) = v \gg y,$$

so $x \gg y$, but since $[x]$ is minimal, we deduce that $[x] = [y]$, which is to say that $x \approx y$.

The conclusion is that $E^0/\approx$ is a singleton, from where G-transitivity follows. ☐

Of course the above result has taken advantage of the fact that E is a finite graph in an essential way, so nothing like this is to be expected for infinite graphs.

Regardless of the absence of sinks, we have:

13.6. Theorem. *Given (G, E, φ) satisfying (2.3), one has that the following are equivalent:*

(i) *the standard action of $\mathcal{S}_{G,E}$ on E^∞ defined in (8.3) is irreducible,*

(ii) *$\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is minimal,*

(iii) *E is weakly G-transitive.*

Proof. The equivalence between (i) and (ii) is a consequence of [**12**: Proposition 5.4]. We will next show that the above condition (iii) is equivalent to condition (iii) of [**12**: Theorem 5.5], from where the result will follow. In doing so, it is useful to understand how do idempotents in $\mathcal{E}$ behave under conjugation by elements in $\mathcal{S}_{G,E}$, and we leave it for the reader to verify that, given (α, g, β) in $\mathcal{S}_{G,E}$ and $(\gamma, 1, \gamma) \in \mathcal{E}$, one has that

$$(\alpha, g, \beta)(\gamma, 1, \gamma)(\alpha, g, \beta)^* = \begin{cases} (\alpha g\varepsilon, 1, \alpha g\varepsilon), & \text{if } \gamma = \beta\varepsilon, \\ (\alpha,\ 1,\ \alpha), & \text{if } \gamma\varepsilon = \beta, \\ 0, & \text{otherwise}. \end{cases} \tag{13.6.1}$$

(iii)⇒[**12**: Theorem 5.5.iii]: Given any two nonzero idempotent elements in $\mathcal{E}$, necessarily of the form

$$f_\alpha = (\alpha, 1, \alpha), \quad \text{and} \quad f_\beta = (\beta, 1, \beta),$$

to employ the notation introduced in (4.5), we must find an outer cover of f_α (in the sense of [**12**: Definition 2.9]) formed by a finite number of conjugates of f_β. As a first step, notice that $s := \big(d(\beta), 1, \beta\big)$ lies in $\mathcal{S}_{G,E}$ and

$$sf_\beta s^* = \big(d(\beta), 1, \beta\big)\ (\beta, 1, \beta)\ \big(\beta, 1, d(\beta)\big) = \big(d(\beta), 1, d(\beta)\big) = f_{d(\beta)}.$$

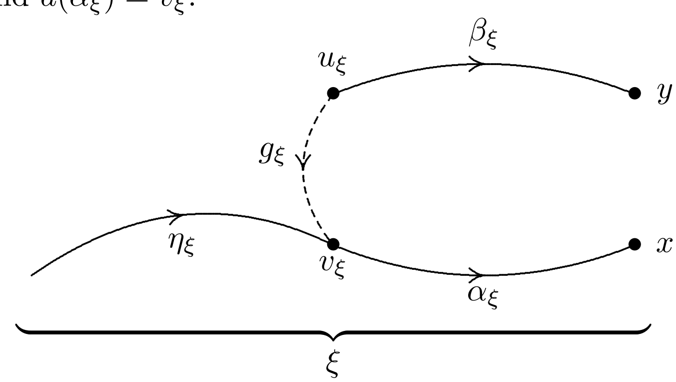
Thus, anything that may be obtained by conjugating $f_{d(\beta)}$ by an element $t \in \mathcal{S}_{G,E}$, may also be obtained by conjugating f_β by ts. It therefore suffices to find an outer cover of f_α formed by conjugates of $f_{d(\beta)}$.

On the other hand, observe that $f_\alpha \leq f_{r(\alpha)}$, so any outer cover of $f_{r(\alpha)}$ is necessarily also an outer cover of f_α. This said we see that we may assume, without loss of generality, that α and β are vertices.

Our task thus gets simplifyed in the sense that we now need to find an outer cover of f_x made of conjugates of f_y, for any given vertices x and y in E^0.

Recall from (8.2) that the set of all infinite paths with a given prefix γ is denoted $Z(\gamma)$. In case we take $\gamma = x$, we then have that $Z(x)$ is the set of all infinite paths with range x.

Thanks to weak G-transitivity, for each ξ in $Z(x)$, we may choose a vertex v_ξ along ξ such that $v_\xi \gg y$. This is to say that we may write $\xi = \alpha_\xi \eta_\xi$, where α_ξ is a finite path, η_ξ is an infinite path and $d(\alpha_\xi) = v_\xi$.



The fact that $v_\xi \gg y$ may be expressed by saying that $v_\xi \sim u_\xi \rightharpoonup y$, for some vertex u_ξ, so there exists g_ξ in G, and a finite path β_ξ, such that $g_\xi u_\xi = v_\xi$, $d(\beta_\xi) = u_\xi$, and $r(\beta_\xi) = y$.

Speaking of the cylinders $Z(\alpha_\xi)$, it is obvious that $\xi \in Z(\alpha_\xi)$, so we see that the collection of cylinders

$$\big\{Z(\alpha_\xi)\big\}_{\xi \in Z(x)}$$

is an open cover (in the topological sense of the word) for $Z(x)$. Since $Z(x)$ is compact, we may extract a finite subcover, say

$$Z(x) \subseteq \bigcup_{\xi \in F} Z(\alpha_\xi), \tag{13.6.2}$$

where F is a finite subset of $Z(x)$. We next claim that $\{f_{\alpha_\xi}\}_{\xi\in F}$ is an outer cover[6] of f_x.

To see this, let e be a nonzero idempotent in $\mathcal{E}$, with $e \leq f_x$. Then e is necessarily given by $e = (\gamma, 1, \gamma)$, for some finite path γ such that $r(\gamma) = x$. Using our standing hypothesis (2.3) according to which E has no sources, we may prolong γ to an infinite path η, which will then share ranges with γ, whence $\eta \in Z(x)$. By (13.6.2) we then have that η lies in $Z(\alpha_\xi)$, for some $\xi \in F$.

This implies that both α_ξ and γ are prefixes of η, from where it is easy to see that α_ξ is a prefix of γ or vice-versa. In particular we conclude that $f_{\alpha_\xi} \Cap f_\gamma$, proving our claim. Incidentally this could also be obtained from [**12**: Proposition 3.8].

We next claim that each f_{α_ξ} is a conjugate of f_y. To see this, observe that, since

$$g_\xi d(\beta_\xi) = g_\xi u_\xi = v_\xi = d(\alpha_\xi),$$

one has that $s_\xi := (\alpha_\xi, g_\xi, \beta_\xi)$ lies in $\mathcal{S}_{G,E}$, and

$$s_\xi f_y s_\xi^* = (\alpha_\xi, g_\xi, \beta_\xi)(y, 1, y)(\beta_\xi, g_\xi, \alpha_\xi) = (\alpha_\xi, 1, \alpha_\xi) = f_{\alpha_\xi}.$$

This concludes the proof of condition (iii) of [**12**: Theorem 5.5].

[**12**: Theorem 5.5.iii]⇒(iii): Given any infinite path ξ, and any vertex y in E^0, we must show that there is some vertex v along ξ such that $v \gg y$. Letting $x = r(\xi)$, let us use the hypothesis regarding the nonzero idempotents

$$f_x = (x, 1, x), \quad \text{and} \quad f_y = (y, 1, y).$$

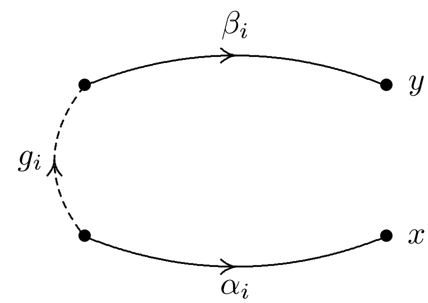
This is to say that there are $s_1, s_2, \ldots, s_n$ in $\mathcal{S}_{G,E}$, such that $\{s_i f_y s_i^*\}_{1\leq i\leq n}$ is an outer cover for f_x. For each i, write $s_i = (\alpha_i, g_i, \beta_i)$, so that

$$s_i f_y s_i^* = (\alpha_i, g_i, \beta_i)(y, 1, y)(\beta_i, g_i, \alpha_i).$$

Observe that, unless $r(\beta_i) = y$, the element displayed above vanishes, so it cannot possibly have any use as a member of a cover. We may then safely discard it, being left only with those β_i such that that $r(\beta_i) = y$. In this case, by the second clause in (13.6.1) we have

$$s_i f_y s_i^* = (\alpha_i, 1, \alpha_i) = f_{\alpha_i}.$$

Unless $r(\alpha_i) = x$, notice that $f_{\alpha_i} \perp f_x$, in which case f_{α_i} may again be discarded as it plays no role in an outer cover for f_x. We may therefore suppose, without loss of generality that $r(\alpha_i) = x$, for all i.



[6] This is in fact a cover but we do not need to worry about this right now.

Given that (α_i, g, β_i) lies in $\mathcal{S}_{G,E}$, we necessarily have that $d(\alpha_i) = g_i d(\beta_i)$. Recalling that the infinite path ξ, chosen at the beginning of the present argument, has range x, we claim that ξ is necessarily of the form

$$\xi = \alpha_i \xi',$$

for some i and some infinite path ξ'. To see this, write

$$\xi = \delta \xi'',$$

where ξ'' is an infinite path and δ is a finite path whose length exceeds the length of all of the α_i. Observing that $r(\delta) = x$, we then have that

$$f_\delta := (\delta, 1, \delta) \leq (x, 1, x) = f_x.$$

So, by the covering property we must have $f_\delta \Cap f_{\alpha_i}$, for some i, which implies that either δ is a prefix of α_i or vice versa. However, due to the fact that $|\delta| > |\alpha_i|$, by construction, the first alternative cannot hold, meaning that α_i is a prefix of δ, and hence also of ξ, proving the claim.

It follows that $d(\alpha_i)$ is a vertex along ξ, and it is clear from the above diagram that $d(\alpha_i) \gg y$. This concludes the proof. □

Combining the above result with (13.5), we immediately deduce:

13.7. Corollary. *If, in addition to the assumptions of (13.6) we have that E has no sinks, then conditions (13.6.i–iii) are also equivalent to:*

(iv) *E is G-transitive.*

It is interesting to observe that G-transitivity, when it holds, is the result of a joint effort by the action of G and the edges, both of which may be seen as pushing vertices around. However, sometimes only one of the players bear the responsibility to do the pushing around:

(1) If G acts transitively on E^0, then E is G-transitive regardless of the graph. Easy examples of this situation may be built on a graph formed by a disjoint union of loops, for instance.

(2) If G fixes all vertices, then E is (weakly) G-transitive if and only if E is (weakly) transitive [**8**: Definition 37.16]. This is the case of Katsura algebras, when seen in the present framework.

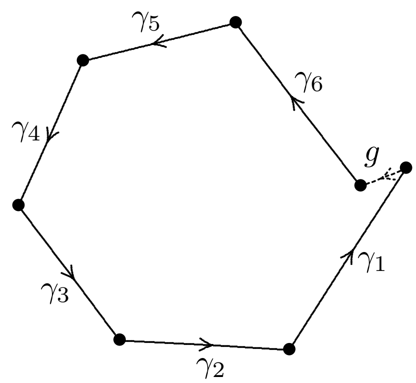
14. Essentially principal groupoids.

In this section we will discuss conditions under which $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is an essentially principal groupoid, a condition which is intimately tied to the action of $\mathcal{S}_{G,E}$ on E^∞ being topologically free. The reader is referred to [**12**: Section 4] for the definition of the notion of topologically free actions of inverse semigroups, as well as some of the main tools we shall use here.

14.1. Definition.

(1) A *circuit*[7] is a finite path $\gamma \in E^*$ of nonzero length such that $d(\gamma) = r(\gamma)$.

(2) A *G-circuit* is a pair (g, γ), where $g \in G$, and $\gamma \in E^*$ is a finite path of nonzero length such that $d(\gamma) = gr(\gamma)$.



A G-circuit.

Thus, a G-circuit needs a little help from the group to close it up. Notice that a (usual) circuit γ may be concatenated with itself infinitely many times producing an infinite path

$$\xi = \gamma\gamma\gamma\ldots$$

Moreover, if

$$s = \big(\gamma, 1, d(\gamma)\big),$$

then, regarding the standard action of $\mathcal{S}_{G,E}$ on E^∞ defined in (8.3), it is easy to see that $s\xi = \xi$, which is to say that ξ is a fixed point for s. It is also possible to create fixed points from G-circuits as follows:

14.2. Proposition. *Given a G-circuit (g, γ), define a sequence $\{\gamma^n\}_{n\geq 1}$ of finite paths, and a sequence $\{g_n\}_{n\geq 1}$ of group elements, recursively by $\gamma^1 = \gamma$, $g_1 = g$, and*

$$\begin{cases} \gamma^{n+1} &= g_n\gamma^n \\ g_{n+1} &= \varphi(g_n, \gamma^n), \end{cases}$$

for all $n \geq 1$. Then

(i) *$d(\gamma^n) = r(\gamma^{n+1})$, for all $n \geq 1$,*

(ii) *the concatenation $\xi = \gamma^1\gamma^2\gamma^3\ldots$ is a well defined infinite path,*

(iii) *for every finite path β such that $d(\beta) = r(\gamma)$, one has that $s := (\beta\gamma, g, \beta)$ lies in $\mathcal{S}_{G,E}$, and $\beta\xi$ is a fixed point for s.*

---

[7] Circuits are also called loops or cycles in the graph C*-algebra literature. Our preference for *circuits* comes from the fact that it is the terminology of choice among graph theorists and also because in the established graph theory terminology the word *loop* refers to a single edge whose source and range coincide.

Proof. In order to prove the case $n = 1$ of (i), we have

$$d(\gamma^1) = d(\gamma) = gr(\gamma) = r(g\gamma) = r(g_1\gamma^1) = r(\gamma^2).$$

For $n \geq 1$, we have

$$d(\gamma^{n+1}) = d(g_n\gamma^n) = g_n d(\gamma^n) \overset{(2.5.\text{vii})}{=} \varphi(g_n, \gamma^n)d(\gamma^n) = g_{n+1}d(\gamma^n) \overset{(\star)}{=}$$

$$= g_{n+1}r(\gamma^{n+1}) = r(g_{n+1}\gamma^{n+1}) = r(\gamma^{n+2}),$$

where we have used induction in the step marked with $(\star)$ above. This proves (i), which in turn immediately implies (ii).

In order to show that the element s defined in (iii) indeed lies in $\mathcal{S}_{G,E}$, it is enough to observe that

$$gd(\beta) = gr(\gamma) = d(\gamma) = d(\beta\gamma).$$

Before proving the last part of (iii), we claim that

$$g_n(\gamma^n\gamma^{n+1}\dots\gamma^{n+k}) = \gamma^{n+1}\gamma^{n+2}\dots\gamma^{n+k+1}, \quad \forall\, n \geq 1, \quad \forall\, k \geq 0.$$

In case $k = 0$, this is true by the recursive definition above, and if $k \geq 1$, we have

$$g_n(\gamma^n\gamma^{n+1}\dots\gamma^{n+k}) = (g_n\gamma^n)\varphi(g_n, \gamma^n)(\gamma^{n+1}\dots\gamma^{n+k}) =$$

$$= \gamma^{n+1}g_{n+1}(\gamma^{n+1}\dots\gamma^{n+k}),$$

and the claim then follows easily by induction. A useful consequence is that

$$g_1(\gamma^1\gamma^2\dots\gamma^n) = \gamma^2\gamma^3\dots\gamma^{n+1}, \quad \forall\, n \geq 1,$$

from where we further deduce that

$$g\xi = g_1(\gamma^1\gamma^2\gamma^3\dots) = \gamma^2\gamma^3\gamma^4\dots \tag{14.2.1}$$

With this we may now tackle the final task:

$$s(\beta\xi) = (\beta\gamma, g, \beta)(\beta\xi) \overset{(8.3)}{=} \beta\gamma g\xi \overset{(14.2.1)}{=} \beta\gamma\gamma^2\gamma^3\dots\gamma^{n+1} = \beta\xi. \qquad \square$$

The above method does not give us all fixed points of every single element s in $\mathcal{S}_{G,E}$, but in certain cases it does:

14.3. Proposition. *Given $s := (\alpha, g, \beta)$ in $\mathcal{S}_{G,E}$, suppose that $|\alpha| > |\beta|$. Then, regarding the standard action of $\mathcal{S}_{G,E}$ on E^∞, one has that:*

(i) *s admits at most one fixed point,*

(ii) *if s admits a fixed point ζ, then there is a G-circuit (g, γ) such that $\alpha = \beta\gamma$, and ζ coincides with the fixed point $\beta\xi$ mentioned in (14.2.iii), constructed from (g, γ).*

Proof. Assuming that ζ is a fixed point for s, it must lie in $Z(\beta)$, so necessarily $\zeta = \beta\xi$, for a suitable infinite path ξ. We then have

$$\beta\xi = \zeta = s\zeta = (\alpha, g, \beta)(\beta\xi) = \alpha g\xi. \tag{14.3.1}$$

This imples that both α and β are prefixes of ζ, so one must be a prefix of the other, but since $|\alpha| > |\beta|$, the only alternative is that β is a prefix of α. We may therefore write

$$\alpha = \beta\gamma,$$

for some finite path γ, which necessarily satisfies

$$gr(\gamma) = gd(\beta) = d(\alpha) = d(\gamma).$$

In other words, (g, γ) is a G-circuit. From (14.3.1) we also deduce that

$$\beta\xi = \alpha g\xi = \beta\gamma g\xi,$$

so $\xi = \gamma g\xi$. Let us now write $\xi = \gamma^1\gamma^2\gamma^3\ldots$, where each γ^i is a finite path with $|\gamma^i| = |\gamma|$. Then

$$\gamma^1\gamma^2\gamma^3\ldots = \xi = \gamma g\xi = \gamma g(\gamma^1\gamma^2\gamma^3\ldots) = \gamma(g_1\gamma^1)(g_2\gamma^2)(g_3\gamma^3)\ldots,$$

where the g_i are recursively defined by $g_1 = g$, and $g_{n+1} = \varphi(g_n, \gamma^n)$. It then follows that $\gamma^1 = \gamma$, and $\gamma^{n+1} = g_n\gamma^n$, for all $n \geq 1$, so we see that the γ^n and the g_n are precisely defined as in (14.2). This concludes the proof. □

As already announced we will eventually be interested in determining conditions under which the standard action of $\mathcal{S}_{G,E}$ on E^∞ is topologically free, so the fixed points that will really interest us are the interior ones.

Under the conditions of the above result, when there is at most one fixed point, the existence of interior fixed points hinges on whether or not the unique fixed point is isolated in E^∞. We will now introduce certain concepts designed to study isolated fixed points.

Recall from (2.3) that our graph E has no sources, meaning that $r^{-1}(x)$ is nonempty for every vertex x.

14.4. Definition.

(1) We shall say that a vertex x in E^0 is a *simple vertex* if $r^{-1}(x)$ is a singleton.

(2) Given a path $\gamma = \gamma_1\gamma_2\ldots\gamma_n$ in E^*, where each γ_i is in E^1, we will say that γ *has no entry* if $d(\gamma_i)$ is a simple vertex for every $i = 1, \ldots, n$.

(3) If the condition above fails, we will say that γ *has an entry*.

Thus, if a path $\gamma = \gamma_1\gamma_2\ldots\gamma_n$ has no entry, then $r^{-1}\big(d(\gamma_i)\big)$ is a singleton for every i, and we may obviously guess which is the edge forming this singleton, namely

$$r^{-1}\big(d(\gamma_i)\big) = \{\gamma_{i+1}\},$$

provided $i < n$. However the same cannot be said when $i = n$, unless (g, γ) is a G-circuit, in which case

$$r^{-1}\big(d(\gamma_n)\big) = \{g\gamma_1\}.$$

The notion of entryless paths will only be useful when applied to G-circuits.

14.5. Proposition. *Under the conditions of (14.3.ii), let (g, γ) be the G-circuit and ζ be the fixed point for s mentioned there. Then the following are equivalent:*

(i) *ζ is an isolated point in E^∞,*

(ii) *γ has no entry.*

Proof. In case γ has no entry, writing $\gamma = \gamma_1\gamma_2 \ldots \gamma_n$, where the γ_i are edges, notice that the only infinite path extending γ_1 is the path ξ referred to in (14.2.ii). The fixed point $\zeta = \beta\xi$ mentioned in (14.3.ii) is therefore the only infinite path extending $\beta\gamma_1$, whence

$$Z(\beta\gamma_1) = \{\zeta\},$$

which implies that ζ is isolated.

Conversely, assuming that ζ is isolated, there exists a sufficiently long prefix ε of ζ, such that

$$Z(\varepsilon) = \{\zeta\}.$$

This means that ζ is the only infinite path extending ε. Writing

$$\zeta = \zeta_1\zeta_2\zeta_3 \ldots,$$

where the ζ_i are edges, one then has that, for sufficiently large i, there is only one edge whose range is $d(\zeta_i)$. Letting $\{\gamma^n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ be the sequences defined in (14.2), we then have that, for sufficiently large n, the G-circuit (g_n, γ^n) has no entry. Since G acts on E by graph automorphisms, we may easily prove by induction that all G-circuits (g_k, γ^k) have no entry, including $(g_1, \gamma^1) = (g, \gamma)$. □

Since we are interested in topologically free actions, we would like to avoid isolated fixed points and hence we will be interested in situations when every G-circuit has an entry. However, given that we are working with finite graphs only, the action of G on E turns out not to be relevant in this respect. In precise terms, what we mean is that:

14.6. Proposition. *Under the conditions of (2.3), the following are equivalent:*

(i) *every G-circuit has an entry,*

(ii) *every circuit has an entry.*

Proof. Since every circuit γ gives rise to the G-circuit $(1, \gamma)$, it is evident that (i) implies (ii). Conversely, assume (ii) and let γ be a G-circuit. Leting $\{\gamma^n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ be as in (14.2), consider the infinite path $\xi = \gamma^1\gamma^2\gamma^3 \ldots$ mentioned in (14.2.iii). Notice that the γ^i are all in the orbit of γ under the action of G, and hence the length of γ^i coincides with that of γ. As E is finite, there is only a finite number of paths of this length, so there must necessarily be repetitions among the γ^i, say $\gamma^i = \gamma^j$, where $i < j$. Then

$$r(\gamma^{i+1}) = d(\gamma^i) = d(\gamma^j),$$

so the path

$$\gamma^{i+1}\gamma^{i+2} \ldots \gamma^j$$

is a circuit, which by hypothesis has an entry. It is now easy to see that some γ^k must have an entry. Finally, since γ^k is in the orbit of γ under the action of G, then γ likewise has an entry, concluding the proof. □

Observe that we have used the finiteness of E in a very strong way above. Thus, should our theory ever be extended to infinite graphs, one might have to distinguish between conditions (14.6.i) and (14.6.ii).

We should point out that a graph in which every circuit has an entry is usually said to satisfy condition (L).

The above results, mainly (14.3) and (14.5), may also be used to study the fixed points for elements $s := (\alpha, g, \beta)$ when $|\alpha| < |\beta|$, since such fixed points are precisely the same as the fixed points of s^*, and s^* clearly satisfies the hypothesis of (14.3). However we still have work to do in order to treat the remaining case $|\alpha| = |\beta|$.

14.7. Proposition. *Let $s := (\alpha, g, \beta) \in \mathcal{S}_{G,E}$, whith $|\alpha| = |\beta|$, and suppose that s admits a fixed point. Then*

(i) $\alpha = \beta$,

(ii) *the fixed points of s in E^∞ are precisely the elements of the form $\zeta = \beta\xi$, where ξ is an infinite path such that $r(\xi) = d(\beta)$, and $g\xi = \xi$.*

Proof. Left for the reader. □

The conclusion of the previous Proposition is that when $|\alpha| = |\beta|$, understanding the fixed points for s requires understanding the fixed points for the action of g on E^∞. One may easily describe such fixed points in terms of the action of G on E and the cocycle φ, but apparently there is no smart way to control each and every one of them. However, since our main interest is in studying topological freeness, we need only focus on large (meaning open) sets of fixed points:

14.8. Proposition. *Suppose that $s := (\alpha, g, \alpha)$ lies in $\mathcal{S}_{G,E}$, and that ζ is an interior fixed point for s. Then there is a finite path γ, such that:*

(i) $g\gamma = \gamma$,

(ii) $d(\alpha) = r(\gamma)$,

(iii) $\zeta \in Z(\alpha\gamma)$,

(iv) *the group element $h := \varphi(g, \gamma)$ pointwise fixes[8] the cylinder $Z(d(\gamma))$.*

Conversely, if γ is any finite path satisfying (i), (ii) and (iv), then every $\zeta \in Z(\alpha\gamma)$ is a (necessarily interior) fixed point for s.

Proof. In particular ζ a fixed point for s so, by (14.7) we have that $\zeta = \alpha\xi$, with $g\xi = \xi$.

Moreover there exists a neighborhood U of ζ consisting of fixed points for ζ. Since the cylinders form a basis for the topology of E^∞, we may assume without loss of generality that $U = Z(\beta)$, for some finite path β, which we may assume is as long as we wish, and our wish in this case is simply that $|\beta| > |\alpha|$.

Since ζ lies in $Z(\beta)$, we have that β is a prefix of ζ, so we may write $\zeta = \beta\eta$, for some infinite path η. We then have

$$\beta\eta = \zeta = \alpha\xi.$$

[8] By this we mean that every point in $Z(d(\gamma))$ is fixed by h.

Given that $|\beta| > |\alpha|$, this implies that α is a prefix of β, so we write $\beta = \alpha\gamma$, for a suitable finite path γ, obviously satisfying (ii). Consequently

$$\zeta = \beta\eta = \alpha\gamma\eta \in Z(\alpha\gamma),$$

proving (iii). Given any infinite path $\mu \in Z(d(\gamma))$, we may form the path $\alpha\gamma\mu$, which necessarily lies in $Z(\alpha\gamma) = Z(\beta)$, and hence is fixed under s. Therefore

$$\alpha\gamma\mu = (\alpha, g, \alpha)(\alpha\gamma\mu) = \alpha g(\gamma\mu) = \alpha\ (g\gamma)\ \big(\varphi(g,\gamma)\mu\big) = \alpha\ (g\gamma)\ (h\mu),$$

whence $\gamma = g\gamma$, proving (i), and $\mu = h\mu$, in turn proving (iv).

In order to prove the last sentence in the statement it is enough to notice that any element in $Z(\alpha\gamma)$ is necessarily of the form $\alpha\gamma\mu$, where $\mu \in Z(d(\gamma))$, and the last calculation displayed above could be used to check that $\alpha\gamma\mu$ is fixed under s. □

Searching for conditions under which the standard action of $\mathcal{S}_{G,E}$ on E^∞ is topologically free, one should probably worry about group elements fixing whole cylinders, as in (14.8.iv). The following notion is designed to pinpoint situations under which whole cylinders of the form $Z(x)$ are in fact fixed.

14.9. Definition. Given $g \in G$, and $x \in E^0$, we shall say that g is *slack* at x, if there is a non-negative integer n such that all finite paths γ with $r(\gamma) = x$, and $|\gamma| \geq n$, are strongly fixed by g, as defined in (5.1).

As already discussed at the begining of section (5), if γ is strongly fixed by g, then g fixes any finite path extending γ, and hence also all infinite paths in $Z(\gamma)$.

If g is slack at x, and if n is as in (14.9), notice that

$$Z(x) = \bigcup_{\substack{r(\gamma)=x\\ |\gamma|=n}} Z(\gamma),$$

and since each γ occuring above is strongly fixed by g, we have that g pointwise fixes $Z(\gamma)$, and hence also the whole cylinder $Z(x)$.

Notice that a path of length zero, namely a vertex x, is never strongly fixed by a nontrivial group element g, because

$$\varphi(g,x) \overset{(2.5.\text{ii})}{=} g \neq 1.$$

The concept of slackness above should therefore be seen as the best replacement for the notion of being strongly fixed in case of a vertex.

We are now ready for a main result:

14.10. Theorem. *Under the conditions of (2.3), the standard action of $\mathcal{S}_{G,E}$ on E^∞ is topologically free if and only if the following two conditions hold:*

(i) *every G-circuit has an entry*[9],

(ii) *given a vertex x, and a group element g fixing every infinite path in $Z(x)$, then necessarily g is slack at x.*

[9] Recall that this is the same as saying that every circuit has an entry by (14.6).

Proof. Suppose (i) and (ii) hold and let ζ be an interior fixed point for some $s = (\alpha, g, \beta)$ in $\mathcal{S}_{G,E}$. In order to prove topological freeness, we need to prove that ζ is a trivial fixed point for s.

CASE 1: Let us first assume that $|\alpha| = |\beta|$. Letting γ and h as in (14.8), we then have that h pointwise fixes the cylinder $Z(d(\gamma))$. By (ii) we then conclude that h is slack at $d(\gamma)$, so there is n such that every finite path of length n and range $d(\gamma)$ is strongly fixed by h.

By (14.8.iii) we have that ζ lies in $Z(\alpha\gamma)$, so we may write $\zeta = \alpha\gamma\xi$, for some infinite path ξ with $d(\gamma) = r(\xi)$. Denoting by ε the path formed by the first n edges of ξ, we then have that

$$r(\varepsilon) = r(\xi) = d(\gamma),$$

so ε is strongly fixed by h, and we may further write

$$\zeta = \alpha\gamma\varepsilon\xi',$$

for a suitable infinite path ξ'. If follows that $\zeta \in Z(\alpha\gamma\varepsilon)$, which is the domain of the idempotent

$$f_{\alpha\gamma\varepsilon} = (\alpha\gamma\varepsilon, 1, \alpha\gamma\varepsilon).$$

In addition

$$sf_{\alpha\gamma\varepsilon} = (\alpha, g, \alpha)(\alpha\gamma\varepsilon, 1, \alpha\gamma\varepsilon) = \big(\alpha g(\gamma\varepsilon), \varphi(g, \gamma\varepsilon), \alpha\gamma\varepsilon\big), \tag{14.10.1}$$

and we claim that the element at the end of the above calculation coincides with $f_{\alpha\gamma\varepsilon}$. To see this notice that

$$g(\gamma\varepsilon) = (g\gamma)\big(\varphi(g, \gamma)\varepsilon\big) \overset{(14.8.i)}{=} \gamma h\varepsilon = \gamma\varepsilon,$$

while

$$\varphi(g, \gamma\varepsilon) \overset{(2.5.x)}{=} \varphi\big(\varphi(g, \gamma), \varepsilon\big) = \varphi\big(h, \varepsilon\big) = 1.$$

Plugging the last two identities at the end of (14.10.1) leads to $sf_{\alpha\gamma\varepsilon} = f_{\alpha\gamma\varepsilon}$, thus proving that ζ is a trivial fixed point, as needed.

CASE 2: Let us now assume that $|\alpha| > |\beta|$. By (14.3) we have that ζ is the only fixed point for s, necessarily given in terms of a G-circuit (g, γ), as in (14.3.iii).

Being the unique fixed point, as well as an interior member of the set of fixed points, we see that ζ is isolated in E^∞. So (g, γ) has no entry by (14.5), contradicting (i). This implies that in fact s has no interior fixed points, so there is nothing to do.

CASE 3: The last remaining alternative, namely when $|\alpha| < |\beta|$, may be treated by simply observing that the fixed points for s are the same as the fixed points for $s^* = (\beta, g^{-1}, \alpha)$, and that s^* fits the previous case studied, so there are no interior fixed points for s^*, either.

This concludes the proof that (i) and (ii) imply topological freeness. In order to prove that topological freeness implies (i), assume the former and suppose by contradiction that a G-circuit (g, γ) exists with no entry. Let $x = r(\gamma)$, and notice that

$$gx = gr(\gamma) = d(\gamma),$$

so the triple $s := (\gamma, g, x)$ is seen to lie in $\mathcal{S}_{G,E}$. We may then use (14.2) to obtain a fixed point ζ for s, and by (14.5) we have that ζ is an isolated point of E^∞, hence also an interior fixed point.

Working under the assumption of topological freeness, we deduce that ζ is a trivial fixed point, which is to say that there is an idempotent e in $\mathcal{E}$, whose domain contains ζ, and such that $se = e$. Observing that e cannot possibly be zero, we deduce that $e = (\varepsilon, 1, \varepsilon)$, for some finite path ε. We then have that

$$(\varepsilon, 1, \varepsilon) = e = se = (\gamma, g, x)(\varepsilon, 1, \varepsilon) = \big(\gamma g\varepsilon, \varphi(g, \varepsilon), \varepsilon\big).$$

In particular this implies that $\varepsilon = \gamma g\varepsilon$, so

$$|\varepsilon| = |\gamma g\varepsilon| = |\gamma| + |g\varepsilon| = |\gamma| + |\varepsilon|,$$

whence $|\gamma| = 0$, contradicting the fact that G-circuits have nonzero length by definition. This shows that there are no G-circuit without an entry, hence proving (i).

We next show that topological freeness implies (ii). So we suppose that some g in G pointwise fixes a whole cylinder $Z(x)$, where x is a vertex. In particuler we have that $gx = x$, so the element

$$s := (x, g, x)$$

belongs to $\mathcal{S}_{G,E}$, and it clearly also fixes every point in $Z(x)$. Each ζ in $Z(x)$ is therefore an interior fixed point for s, hence necessarily a trivial one by hypothesis. This means that there exists an idempotent $e = (\gamma, 1, \gamma) \in \mathcal{E}$, such that ζ lies in the domain of e, also known as $Z(\gamma)$, and moreover $se = e$. Therefore

$$(\gamma, 1, \gamma) = e = se = (x, g, x)(\gamma, 1, \gamma) = \big(xg\gamma, \varphi(g, \gamma), \gamma\big),$$

from where we deduce that $g\gamma = \gamma$, and $\varphi(g, \gamma) = 1$, which is to say that γ is strongly fixed by g.

Given that $\zeta \in Z(\gamma)$, we have that γ is a prefix of ζ, whence $r(\gamma) = r(\xi) = x$, so

$$\zeta \in Z(\gamma) \subseteq Z(x).$$

We then deduce that $Z(x)$ is the union of the $Z(\gamma)$, where γ range in the set of all finite paths strongly fixed by g, with $r(\gamma) = x$. By compactness we may find a finite collection of such finite paths, say $\gamma_1, \gamma_2, \ldots, \gamma_k$, such that

$$Z(x) = \bigcup_{i=1}^{k} Z(\gamma_i). \tag{14.10.2}$$

We next wish to argue that the above γ_i's may be taken so that their length is constant. To see this let n be the length of the longer γ_i, and observe that, for each i, one has that

$$Z(\gamma_i) = \bigcup_{\substack{r(\varepsilon)=d(\gamma_i) \\ |\varepsilon|=n-|\gamma_i|}} Z(\gamma_i\varepsilon).$$

Moreover, each $\gamma_i\varepsilon$ occuring above is also strongly fixed by g, as seen in the discussion near the beginning of section (5). Thus, if we replace each γ_i by the set of all $\gamma_i\varepsilon$, where ε is as above, all of the properties so far mentioned of the original γ_i's will be preserved, and now

$$|\gamma_i\varepsilon| = |\gamma_i| + |\varepsilon| = n.$$

Therefore we may and will assume, from now on, that the γ_i have a constant length, say n. From (14.10.2) it is now easy to conclude that the γ_i exhaust the set of all finite paths with range x and length n. In fact, if α is such a path, we may extend it to an infinite path of the form $\xi = \alpha\eta$. Since $\xi \in Z(x)$, then $\xi \in Z(\gamma_i)$, for some i, whence γ_i is a prefix of ξ and, by considering lengths, we see that $\alpha = \gamma_i$.

The conclusion is that every finite path with length n and range x is strongly fixed by g, which is to say that g is slack at x. □

14.11. Remark. If for any $g \in G \setminus \{1\}$ and for any $x \in E^0$ there exists $\eta \in Z(x)$ such that $g\eta \neq \eta$, then (14.10.ii) holds trivially. This fact will be used in (18.9) and subsequent examples.

14.12. Remark. Regarding [**12**: Theorem 4.10.ii], and letting γ_i be as in (14.10.2), one may show that $\{f_{\gamma_i}\}_i$ is a cover of f_x consisting of idempotents fixed under s (in the sense of [**12**: Definition 4.8.1]).

In case (G, E, φ) is pseudo free, and if g is a nontrivial group element, then g admits no strongly fixed paths by (5.5), so g will never be slack at any vertex. Condition (14.10.ii) can therefore only be satisfied if no nontrivial group element pointwise fixes a cylinder $Z(x)$, and hence we have the following immediate consequence of (14.10):

14.13. Corollary. *In addition to the conditions of (2.3), suppose that (G, E, φ) is pseudo free. Then the standard action of $\mathcal{S}_{G,E}$ on E^∞ is topologically free if and only if the following two conditions hold:*

(i) *every G-circuit has an entry (which is the same as saying that every circuit has an entry by (14.6)),*

(ii) *for every g in G, with $g \neq 1$, and for every x in E^0, there is at least one ζ in $Z(x)$ such that $g\zeta \neq \zeta$.*

An important case for the theory of self-similar groups is when G acts faithfully[10] on E^∞, and E is a graph with a single vertex.

14.14. Corollary. *Under the conditions of (2.3), suppose moreover that:*

(a) *E has a single vertex, and at least two edges,*

(b) *G acts faithfully on E^∞.*

Then the standard action of $\mathcal{S}_{G,E}$ on E^∞ is topologically free.

[10] Meaning that if $g\xi = \xi$, for all ξ in E^∞, then $g = 1$.

Proof. In the present situation the conditions of (14.10) become trivially true because: (i) all path are circuits and all circuits have entries, and (ii) there is only one $Z(x)$ to consider, namely the whole space E^∞, and by faithfulness no nontrivial group element acts trivially on E^∞. □

As the title of the present section suggests, our main interest is in determining conditions for $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ to be an essentially principal groupoid. Having understood topological freeness, an immediate consequence of [**12**: Theorem 4.7] is:

14.15. Corollary. *Under the assumptions of (2.3), one has that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is essentially principal if and only if (14.10.i&ii) hold.*

Two other similar results could be stated giving conditions for $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ to be essentially principal, by combining [**12**: Theorem 4.7] with either (14.13) or (14.14), but we will refrain from doing it here since the reader can easily guess them.

15. Local contractivity for $\mathcal{S}_{G,E}$.

In [**12**: Section 6] local contractivity for groupoids and for actions of inverse semigroups is studied. We will now use these results to characterize local contractivity for the tight groupoid associated to an inverse semigroup $\mathcal{S}$.

15.1. Theorem. *Under the conditions of (2.3), one has that the following are equivalent:*

(i) *$\mathcal{S}_{G,E}$ is a locally contracting inverse semigroup,*

(ii) *the standard action $\theta : \mathcal{S}_{G,E} \curvearrowright E^\infty$ is locally contracting,*

(iii) *$\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is a locally contracting groupoid,*

(iv) *every circuit in E has an entry.*

Proof. As already mentioned in section (8), every tight filter in $\mathcal{E}$ is an ultra-filter, so the equivalence between (i) and (ii) follows from [**12**: Theorem 6.5].

(ii)⇒(iii): Follows immediately from [**12**: Proposition 6.3].

(iii)⇒(iv): We will prove this by contraposition, that is, assuming the existence of a circuit γ without an entry, we will show that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is not locally contracting.

Our task is actually very easy. Given an entryless circuit γ, the path $\xi = \gamma\gamma\gamma\ldots$ is an isolated point, whence $U := \{\xi\}$ is an open subset of E^∞. Viewing the latter as the unit space of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, as usual, and plugging U into [**12**: Definition 6.1], clearly there can be no open set V, and bissection S, as mentioned there, simply because a chain of nonempty subsets

$$S\overline{V}S^{-1} \subsetneq V \subseteq U$$

cannot possibly exist withing a singleton such as U. This shows that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is not locally contracting, as desired.

(iv)⇒(i): Assuming that every circuit has an entry, we will show local contractivity of $\mathcal{S}_{G,E}$ via [**12**: Proposition 6.7]. Given a nonzero idempotent e in $\mathcal{E}$, write $e = (\mu, 1, \mu)$ for some finite path μ. Using that E has no sources, we may find an infinite path $\xi = \xi_1\xi_2\xi_3\ldots$, such that $d(\mu) = r(\xi)$. Since E is a finite graph, there must be repetitions amongst the ξ_i, say $\xi_i = \xi_j$, for some $i < j$. Letting

$$\alpha = \xi_1\xi_2 \ldots \xi_i, \quad \text{and} \quad \gamma = \xi_{i+1}\xi_{i+2} \ldots \xi_j,$$

notice that

$$d(\gamma) = d(\xi_j) = d(\xi_i) = r(\xi_{i+1}) = r(\gamma),$$

so γ is a circuit. It is also clear that $\mu\alpha$ and $\mu\alpha\gamma$ are well defined paths. Noticing that

$$d(\alpha) = r(\gamma) = d(\gamma),$$

we have that $s := (\mu\alpha\gamma, 1, \mu\alpha)$ lies in $\mathcal{S}_{G,E}$. Moreover, setting

$$\beta_1 = \mu\alpha,$$

and using the notation introduced in (4.5), we have

$$sf_{\beta_1}s^* = (\mu\alpha\gamma, 1, \mu\alpha)(\mu\alpha, 1, \mu\alpha)(\mu\alpha\gamma, 1, \mu\alpha)^* \overset{(13.6.1)}{=} (\mu\alpha\gamma, 1, \mu\alpha\gamma) \leq f_{\beta_1},$$

thus verifying [**12**: Proposition 6.7.ii]. By hypothesis γ has an entry, so we may find a path γ', with $r(\gamma') = r(\gamma)$, which is not a prefix of γ, or vice versa. Setting

$$\beta_0 = \mu\alpha\gamma',$$

we then have

$$\begin{aligned} 0 \neq f_{\mu\alpha\gamma'} &\leq f_{\mu\alpha} \qquad \leq f_\mu \quad \Rightarrow \\ 0 \neq \ \ f_{\beta_0} \ \ &\leq f_{\beta_1} = s^*s \leq e, \end{aligned}$$

verifying [**12**: Proposition 6.7.i]. Focusing now on [**12**: Proposition 6.7.iii] notice that

$$f_{\beta_0}s = (\mu\alpha\gamma', 1, \mu\alpha\gamma')(\mu\alpha\gamma, 1, \mu\alpha) = 0,$$

precisely because γ and γ' are not each other's prefix. So evidently $f_{\beta_0}sf_{\beta_1} = 0$, proving the last condition in [**12**: Proposition 6.7], and hence that $\mathcal{S}_{G,E}$ is locally contracting, thus proving (i). ☐

It is worth noticing that many results of [**12**] used in the above proof, such as [**12**: Proposition 6.3], [**12**: Theorem 6.5] and [**12**: Proposition 6.7], comparing local contractivity for groupoids, inverse semigroups, and actions, are either one way implications only, or the converse depends on special conditions. Nevertheless, the situation in which we are working has fortunately allowed for a downright equivalence of the various manifestations of contractivity.

However, this result should be taken with a certain skepticism. First of all it is well known that the above condition on circuits is not sufficient for local contractivity for the groupoid associated to *infinite* graphs [**21**]. Considering that *finite* graphs are special cases of our theory (just take the acting group to be the trivial group), it is not unreasonable to believe that our results admit natural generalizations to infinite graphs, but then a characterization of local contractivity for the corresponding groupoid will certainly not follow from the fact that every circuit has an entry, since this is false for infinite graphs, as mentioned above.

Secondly, observe that the condition on the existence of entries for circuits completely ignores the group G, but, again, a generalization to infinite graphs will probably depend on the action. A hypothesis such as "every vertex connects to a G-circuit with an entry", to paraphrase the main hypothesis of [**21**: Lemma 3.8], is probably more realistic in the conjectured infinite graph scenario.

16. Simplicity and pure infiniteness for $\mathcal{O}_{G,E}$.

In this section we use the results in the previous sections to characterize when $\mathcal{O}_{G,E}$ is simple and purely infinite. The central results are the following:

16.1. Theorem. *Assume that (G, E, φ) satisfies (2.3), that G is amenable, and that for every $g \in G$ there are at most finitely many minimal strongly fixed paths for g. Then $\mathcal{O}_{G,E}$ is simple if and only if the following conditions are satisfied:*

(a) *E is weakly-G-transitive.*

(b) *Every G-circuit has an entry.*

(c) *Given a vertex x, and a group element g fixing $Z(x)$ pointwise, then necessarily g is a slack at x.*

Proof. By (12.2), the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff. Clearly $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is étale with second countable unit space. By (10.18), $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is amenable. Then, by (6.4), $\mathcal{O}_{G,E} \cong C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) = C^*_r(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$. By [**4**: Theorem 5.1], $\mathcal{O}_{G,E}$ is simple if and only if $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is minimal and essentially principal. Since minimality of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is equivalent to (a) by (13.6), and essential principality of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is equivalent to (b&c) by (14.15), the result holds. □

With respect to pure infiniteness, we have:

16.2. Theorem. *Let (G, E, φ) be under (2.3), and let G be an amenable group. If $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is essentially principal, then every hereditary subalgebra of $\mathcal{O}_{G,E}$ contains an infinite projection.*

Proof. By the same argument as in (16.1), $\mathcal{O}_{G,E} \cong C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) = C^*_r(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$. By (14.10.i), every circuit of E has an entry. Thus, $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is locally contracting by (15.1). Hence, by [**1**: Proposition 2.4], every nonzero hereditary sub-C*-algebra of $\mathcal{O}_{G,E}$ contains an infinite projection, as desired. □

As an immediate consequence we have

16.3. Corollary. *If (G, E, φ) satisfies (2.3), the group G is amenable, and $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff then, whenever $\mathcal{O}_{G,E}$ is simple, it is necessarily also purely infinite (simple).*

Proof. By (14.10) and (16.1), $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is essentially principal. Thus, by (16.2), every nonzero hereditary sub-C*-algebra of $\mathcal{O}_{G,E}$ contains an infinite projection. Hence, $\mathcal{O}_{G,E}$ is purely infinite simple, as desired. □

17. Revisiting Nekrashevych algebras.

In this section we will analyze Nekrashevych algebras from our point of view.

The Nekrashevych C*-algebra $\mathcal{O}_{(G,X)}$, associated to a self-similar action of a group G on a finite alphabet X [**26**], is a direct example of our definition (see (3.3)). Here, the graph E is the rose of n petals for $n = |X| \geq 2$, so that the action on vertices is trivial, and the action is faithful. Since $|E^0| = 1$, we have the following facts:

(1) E is G-transitive, whence $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is minimal by (13.6).

(2) $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is essentially principal by (14.14) and [**12**: Theorem 14.7]. In particular, $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is locally contracting by (14.10) and (15.1).

Thus, if $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff, we conclude that $C_r^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ is a purely infinite simple C*-algebra by [**12**: Theorem 6.8]. Hausdorffness of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is equivalent, according to (12.2), to the existence of at most finitely many minimal strongly fixed paths for every $g \in G$.

In this sense, it is interesting to remark that Nekrashevych also gave a presentation of its algebra as a groupoid C*-algebra associated to a groupoid of germs of an inverse semigroup $\mathcal{S}$ [**26**: Section 5]. While $\mathcal{S}$ turns out to be $\mathcal{S}_{G,E}$, the notion of germ that he used is the one adopted by Arzumanian and Renault [**3**], which differs from the one we used, due to Patterson [**27**: Page 140]. Luckily, both definitions coincide when the action of $\mathcal{S}_{G,E}$ on E^∞ is topologically free, which is the case of Nekrashevych triples, as we noticed above. So, Nekrashevych's groupoid and $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ coincide, and the characterization of Hausdorffness we obtained in (12.2) coincide with that given by Nekrashevych [**26**: Lemma 5.4].

In order to obtain a characterization of (pure infinite) simplicity for $\mathcal{O}_{(G,X)}$, we need to keep control of whether $\mathcal{O}_{(G,X)}$ is nuclear. So, it only remains to determine when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is amenable, which implies that $C^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})) = C_r^*(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$. By (10.18), if G is an amenable group, then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is an amenable groupoid. Thus, we obtain

17.1. Proposition. *If (G, X, φ) is a Nekrashevych triple, with G an amenable group and $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ a Hausdorff groupoid, then $\mathcal{O}_{(G,X)}$ is a nuclear, separable, purely infinite simple C*-algebra.*

Here, Nekrashevych's approach differs from ours. In [**26**] he stated a sufficient condition for the amenability of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, which apparently does not require the group G to be amenable. The condition relies on two concepts associated to self-similar groups: self-replication and contractiveness (see [**25**] or [**26**] for definitions of these concepts). Nekrashevych [**26**: Theorem 5.6] proved that if $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdorff and (G, X) is self-replicating and contractive, then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is of polynomial growth [**25**], and thus it its amenable by [**2**: Proposition 3.2.32].

18. Revisiting Katsura algebras.

In this section we will analyze Katsura algebras from our point of view.

We will quickly recall the definition and basic properties of Katsura algebras that will be needed in the sequel. This is borrowed from [**18**].

18.1. Definition. Let $N \in \mathbb{N} \cup \{\infty\}$, let $A \in M_N(\mathbb{Z}^+)$ and $B \in M_N(\mathbb{Z})$ be row-finite matrices. Define a set Ω_A by

$$\Omega_A := \{(i,j) \in \{1,2,\ldots,N\} \times \{1,2,\ldots,N\} : A_{i,j} \geq 1\}.$$

For each $i \in \{1,2,\ldots,N\}$, define a set $\Omega_A(i) \subset \{1,2,\ldots,N\}$ by

$$\Omega_A(i) := \{j \in \{1,2,\ldots,N\} : (i,j) \in \Omega_A\}.$$

Notice that, by definition, $\Omega_A(i)$ is finite for all i. Finally, fix the following relation:

(0) $\Omega_A(i) \neq \emptyset$ for all i, and $B_{i,j} = 0$ for $(i,j) \notin \Omega_A$.

With these data we can define Katsura algebras

18.2. Definition. Define $\mathcal{O}_{A,B}$ to be the universal C*-algebra generated by mutually orthogonal projections $\{q_i\}_{i=1}^N$, partial unitaries $\{u_i\}_{i=1}^N$ with $u_iu_i^* = u_i^*u_i = q_i$, and partial isometries $\{s_{i,j,n}\}_{(i,j)\in\Omega_A, n\in\mathbb{Z}}$ satisfying the relations:

(i) $s_{i,j,n}u_j = s_{i,j,n+A_{i,j}}$ and $u_is_{i,j,n} = s_{i,j,n+B_{i,j}}$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$.

(ii) $s_{i,j,n}^*s_{i,j,n} = q_j$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$.

(iii) $q_i = \sum_{j\in\Omega_A(i)} \sum_{n=1}^{A_{i,j}} s_{i,j,n}s_{i,j,n}^*$ for all i.

18.3. Remark. Now, the following facts holds:

(1) The C*-algebra $\mathcal{O}_{A,B}$ is separable, nuclear, in the UCT class [**18**: Proposition 2.9].

(2) If the matrices A, B satisfy the following additional properties:

   (a) A is irreducible, and

   (b) $|A_{ii}| \geq 2$ and $B_{i,i} = 1$ for every $1 \leq i \leq N$,

   then the C*-algebra $\mathcal{O}_{A,B}$ is purely infinite simple, and hence a Kirchberg algebra [**18**: Proposition 2.10].

(3) The K-groups of $\mathcal{O}_{A,B}$ are [**18**: Proposition 2.6]:

   (a) $K_0(\mathcal{O}_{A,B}) \cong \operatorname{coker}(I - A) \oplus \ker(I - B)$, and

   (b) $K_1(\mathcal{O}_{A,B}) \cong \operatorname{coker}(I - B) \oplus \ker(I - A)$.

(4) Every Kirchberg algebra can be represented, up to isomorphism, by an algebra $\mathcal{O}_{A,B}$ for matrices A, B satisfying the conditions in (18.3.2) [**19**: Proposition 4.5].

As we have seen in (3.4), unital Katsura algebras are natural examples of our construction. So, it is easy to use our results in order to characterize some properties, like simplicity or pure infinite simplicity, in terms of matrices A and B. This work has been previously done in [**10**], but the approach we chose there was fairly more direct and computational, so that the conditions appearing there were less elegant and clear than the ones we will present here.

Across this section, we will say that a triple $(\mathbb{Z}, E, \varphi)$ is a Katsura triple if there exist finite matrices A, B satisfying (18.1) such that the triple associated to the algebra $\mathcal{O}_{A,B}$ is

$(\mathbb{Z}, E, \varphi)$; in particular, E is the graph whose adjacency matrix is A. Also, we will fix the following agreement: let ξ be either in E^* or in E^∞, i.e.

$$\xi = e_{i_1,i_2,n_1} e_{i_2,i_3,n_2} \cdots e_{i_k,i_{k+1},n_k} \text{ or } \xi = e_{i_1,i_2,n_1} e_{i_2,i_3,n_2} \cdots e_{i_k,i_{k+1},n_k} \cdots,$$

then, for any $r \in \mathbb{N}$ we define

$$B_{\xi_{|r}} := \prod_{t=1}^{r} B_{i_t,i_{t+1}} \text{ and } A_{\xi_{|r}} := \prod_{t=1}^{r} A_{i_t,i_{t+1}}.$$

The first step to work out the corresponding results to the ones we obtained for the general setting is to determine when a finite path $\alpha \in E^*$ is fixed under the action of an element $l \in \mathbb{Z}$.

18.4. Lemma. *Let $(\mathbb{Z}, E, \varphi)$ be a Katsura triple. Given an element α of E^* of length r and an integer $l \in \mathbb{Z}$, the following are equivalent:*

(1) *α is fixed under the action of l.*

(2) *For every $1 \leq j \leq r$ the element $K_j := l \dfrac{B_{\alpha_{|j}}}{A_{\alpha_{|j}}}$ belongs to $\mathbb{Z}$.*

Proof. Set $\alpha = e_{i_1,i_2,n_1} e_{i_2,i_3,n_2} \cdots e_{i_r,i_{r+1},n_r}$. By definition of $(\mathbb{Z}, E, \varphi)$, $\alpha = l\alpha$ if and only if there exists a sequence $(K_j)_{j\geq 0} \subseteq \mathbb{Z}$ such that:

(i) $K_0 = l$.

(ii) For every $1 \leq j \leq r$, $n_{j-1} + K_{j-1} B_{i_j,i_{j+1}} = n_{j-1} + K_j A_{i_j,i_{j+1}}$.

Notice that (ii) is equivalent to ask $K_{j-1} B_{i_j,i_{j+1}} = K_j A_{i_j,i_{j+1}}$ for every $j \geq 1$.

Now, for $j = 1$ we have $K_0 B_{i_1,i_2} = l B_{i_1,i_2} = K_1 A_{i_1,i_2}$, so that $K_1 = l \dfrac{B_{i_1,i_2}}{A_{i_1,i_2}}$. Now, suppose that for $1 \leq t \leq j-1$ we have proved that $K_t := l \dfrac{B_{\alpha_{|t}}}{A_{\alpha_{|t}}}$. Hence

$$K_j A_{i_j,i_{j+1}} = K_{j-1} B_{i_j,i_{j+1}} = l \frac{B_{\alpha_{|j-1}}}{A_{\alpha_{|j-1}}} \cdot B_{i_j,i_{j+1}},$$

so that $K_j = l \dfrac{B_{\alpha_{|j}}}{A_{\alpha_{|j}}}$. This completes the proof. $\square$

Now, we are ready to characterize pseudo freeness for a Katsura triple $(\mathbb{Z}, E, \varphi)$.

18.5. Lemma. *Let $(\mathbb{Z}, E, \varphi)$ be a Katsura triple. Then, the following are equivalent:*

(1) *$(\mathbb{Z}, E, \varphi)$ is pseudo free.*

(2) *$B_{i,j} = 0$ if and only if $(i, j) \notin \Omega_A$.*

Proof. Let $\alpha = e_{i_1,i_2,n_1}e_{i_2,i_3,n_2}\cdots e_{i_r,i_{r+1},n_r}$ of E^*, and let $l \in Z$. By (18.4), $l\alpha = \alpha$ exactly when the elements $K_j := l\dfrac{B_{\alpha_{|j}}}{A_{\alpha_{|j}}}$ belongs to $\mathbb{Z}$ for every $1 \leq j \leq r$. Since $\varphi(l, \alpha_{|j}) = K_j$, $\varphi(l,\alpha) = 0$ exactly when $K_j = 0$ for some $j \leq r$. Thus, the situation reduces to $K_{j-1}e_{i_j,i_{j+1},n_l} = e_{i_j,i_{j+1},n_l}$ and $K_j = 0$ for some $1 \leq j \leq r$, which corresponds to the equation $n_j + K_{j-1}B_{i_j,i_{j+1}} = n_j$. And this occurs exactly when $B_{i_j,i_{j+1}} = 0$, so we are done. □

Which these results in mind, we are ready to characterize when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff

18.6. Theorem. *Let $(\mathbb{Z}, E, \varphi)$ be a Katsura triple. Then, the following are equivalent:*

(1) *$\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff.*

(2) *Whenever $(i,j) \in \Omega_A$ with $B_{i,j} = 0$, then for any $l \in \mathbb{Z}$ there exist finitely many finite paths $\alpha \in E^*$ with $d(\alpha) = i$ such that $l\dfrac{B_{\alpha_{|t}}}{A_{\alpha_{|t}}} \in \mathbb{Z}$ for every $1 \leq t \leq r-1$.*

Proof. The result holds by (12.2) and (18.5). □

The next step is to determine the minimality of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$.

18.7. Theorem. *Let $(\mathbb{Z}, E, \varphi)$ be a Katsura triple. Then, the following are equivalent:*

(1) *$\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is minimal.*

(2) *The adjacency matrix A of E is irreducible.*

Proof. First notice that E has no sinks by (18.1.(0)). Moreover, the action of $\mathbb{Z}$ on E fixes all the vertices. Then, by (13.6), $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is minimal if and only if E is transitive, which is equivalent to the matrix A being irreducible, so we are done. □

Now, we will give a characterization of when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is essentially principal.

18.8. Theorem. *Let $(\mathbb{Z}, E, \varphi)$ be a Katsura triple. Then, the following are equivalent:*

(1) *$\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is essentially principal.*

(2) (a) *Every circuit in E has an entry.*

(b) *If $1 \leq i \leq N$, $l \in \mathbb{Z}$, and for any $\xi \in Z(i)$ the elements $l\dfrac{B_{\xi_{|n}}}{A_{\xi_{|n}}} \in \mathbb{Z}$ for all $n \in \mathbb{N}$, then there exists $m \in \mathbb{N}$ such that $B_{\xi_{|m}} = 0$ for all $\xi \in Z(i)$.*

Proof. Since the action of $\mathbb{Z}$ fixes all the vertices of E, (2a) is (14.10.i). On the other side, (2b) is exactly (14.10.ii) because of (18.4) and (18.5). Thus, the result is consequence of (14.15). □

We can obtain an easy sufficient condition for $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ being essentially principal.

18.9. Corollary. *Let $(\mathbb{Z}, E, \varphi)$ be a Katsura triple. If*

(1) *Every circuit of E has an entry, and*

(2) *For every $1 \leq i \leq N$ and every $l \in \mathbb{Z}$ there exists $\eta \in Z(i)$ such that $\lim\limits_{n\to\infty} l\dfrac{B_{\eta_{|n}}}{A_{\eta_{|n}}} = 0$,*

then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is essentially principal.

Proof. By (18.4), condition (2) implies that $l\eta \neq \eta$ for any $l \in \mathbb{Z}$, whence the triple (Z, E, φ) trivially satisfies (14.10.ii), as remarked in (14.11). $\square$

Corollary (18.9) applies when we have a pair of finite matrices A, B under (18.1), such that for every $1 \leq i \leq N$ we have $|A_{ii}| \geq 2$ and $B_{ii} < |A_{ii}|$. In particular, $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is essentialy principal for Katsura systems $(\mathbb{Z}, E, \varphi)$ satisfying (18.3.2).

Also, it is immediate to characterize when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is locally contracting.

18.10. Theorem. *Let $(\mathbb{Z}, E, \varphi)$ be a Katsura triple. Then, the following are equivalent:*

(1) *$\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is locally contracting.*

(2) *Every circuit of E has an entry.*

Proof. This is (15.1). $\square$

Finally, we have the following fact

18.11. Proposition. *If $(\mathbb{Z}, E, \varphi)$ is a Katsura triple, then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is an amenable groupoid.*

Proof. Since $\mathbb{Z}$ is an amenable group, (10.18) applies. $\square$

Now, we are ready to characterize simplicity of the algebra $\mathcal{O}_{A,B}$, as follows

18.12. Theorem. *Let $(\mathbb{Z}, E, \varphi)$ be a Katsura triple such that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff (see (18.6)). Then, the following are equivalent:*

(1) (a) *The matrix A is irreducible.*

(b) *Every circuit of E has an entry.*

(c) *If $1 \leq i \leq N$, $l \in \mathbb{Z}$, and for any $\xi \in Z(i)$ the elements $l\dfrac{B_{\xi_{|n}}}{A_{\xi_{|n}}} \in \mathbb{Z}$ for all $n \in \mathbb{N}$, then there exists $m \in \mathbb{N}$ such that $B_{\xi_{|m}} = 0$.*

(2) *$\mathcal{O}_{A,B}$ is simple.*

Proof. This is exactly (16.1) for the Katsura triple $(\mathbb{Z}, E, \varphi)$, because of (18.7), (18.8) and (18.11). $\square$

In particular, when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff and $\mathcal{O}_{A,B}$ is simple, the $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is locally contracting by (18.10) and (18.12.1b). Hence, we have

18.13. Corollary. *If $(\mathbb{Z}, E, \varphi)$ is a Katsura triple such that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff and $\mathcal{O}_{A,B}$ is simple, then $\mathcal{O}_{A,B}$ is purely infinite simple.*

Proof. This is by (16.3). $\square$

18.14. Remark. Notice that, because of (18.9), Katsura's condition (18.3.2) for $\mathcal{O}_{A,B}$ being a purely infinite simple C*-algebra derive directly from (18.12) and (18.13) when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff. Moreover, when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff, (18.12) provides a characterization of simplicity for $\mathcal{O}_{A,B}$, improving Katsura's results on that direction, where only sufficient conditions are given [**18**].

We close this section by presenting a couple of examples. The first one illustrates the difference between (G, E, φ) being pseudo free and $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ being Hausdorff, and also the difference between the action of $\mathcal{S}_{G,E}$ on E^∞ being topologically free and the action of G on E^∞ being topologically free.

18.15. Example. Set $N = 2$, and consider the matrices $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $(\mathbb{Z}, E, \varphi)$ be the associated Katsura triple. Then, we have the following:

(1) Since $\mathbb{Z}$ is amenable, then so is $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$.

(2) Since A is irreducible, $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is minimal.

(3) Every circuit in E has an entry.

(4) Since $A_{1,2} \neq 0$ and $B_{1,2} = 0$, $(\mathbb{Z}, E, \varphi)$ is not pseudo free by (18.5).

(5) Notice that the only possible quotient values $\dfrac{B_{i,j}}{A_{i,j}}$ are $\dfrac{B_{1,1}}{A_{1,1}} = \dfrac{B_{2,2}}{A_{2,2}} = \dfrac{1}{2}$ and $\dfrac{B_{1,2}}{A_{1,2}} = \dfrac{B_{2,1}}{A_{2,1}} = \dfrac{0}{1} = 0$. Then, for any $l \in \mathbb{Z}$, it is clear that there exists only finitely many minimal strongly fixed paths for l. Thus, $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff by (18.6).

(6) Moreover, by the argument in point (5), the only infinite paths fixed by the action of $\mathbb{Z}$ are the ones associated to minimal strongly fixed paths, and thus trivial. Hence, the action of $\mathcal{S}_{\mathbb{Z},E}$ on E^∞ is topologically free. But the action of $\mathbb{Z}$ is not topologically free, since every element of $\mathbb{Z}$ fix the cylinders $Z(e_{1,2,1})$ and $Z(e_{2,1,1})$.

Notice that $\mathcal{O}_{A,B}$ is purely infinite simple by (18.12) and (18.13).

The second example shows that (G, E, φ) being pseudo free do not imply that the action of $\mathcal{S}_{G,E}$ is topologically free.

18.16. Example. Set $N = 1$, and set $A = B = (n)$ for any $n \geq 2$. Let $(\mathbb{Z}, E, \varphi)$ be the associated Katsura triple. Then, we have the following:

(1) Since $\mathbb{Z}$ is amenable, then so is $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$.

(2) Since A is irreducible, $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is minimal.

(3) Every circuit in E has an entry.

(4) Since $A = B$, $(\mathbb{Z}, E, \varphi)$ is pseudo free, whence in particular $\mathcal{G}_{\text{tight}}(\mathcal{S}_{\mathbb{Z},E})$ is Hausdorff.

(5) Since $A = B$, the action of $\mathbb{Z}$ on E is trivial. Thus, the action of $\mathcal{S}_{\mathbb{Z},E}$ on E^∞ (and of $\mathbb{Z}$) cannot be topologically free, because there are no slacks.

References

[1] C. Anantharaman-Delaroche, "Purely infinite C^*-algebras arising form dynamical systems", *Bull. Soc. Math. France,* **125** (1997), no. 2, 199–225.

[2] C. Anantharaman-Delaroche and J. Renault, "Amenable groupoids", Monogr. Enseign. Math. 36, Université de Genève, 2000.

[3] V. Arzumanian, J. Renault, "Examples of pseudogroups and their C^*-algebras", Operator algebras and quantum field theory (Rome, 1996), 93-104, Int. Press, Cambridge, MA, 1997.

[4] J. Brown, L. O. Clark, C. Farthing and A. Sims, "Simplicity of algebras associated to étale groupoids", *Semigroup Forum,* **88** (2014), 433–452.

[5] N. P. Brown and N. Ozawa, "C*-algebras and finite-dimensional approximations", Graduate Studies in Mathematics, 88, American Mathematical Society, 2008.

[6] R. Exel, "Inverse semigroups and combinatorial C*-algebras", *Bull. Braz. Math. Soc.,* **39** (2008), no. 2, 191–313.

[7] R. Exel, "Non-Hausdorff étale groupoids", *Proc. Amer. Math. Soc.,* **139** (2011), no. 3, 897–907.

[8] R. Exel, "Partial Dynamical Systems, Fell Bundles and Applications", Licensed under a Creative Commons Attribution-ShareAlike 4.0 International License, 351pp, 2014. Available online from mtm.ufsc.br/∼exel/publications. PDF file md5sum: bc4cbce3debdb584ca226176b9b76924.

[9] R. Exel and M. Laca, "Cuntz-Krieger algebras for infinite matrices", *J. reine angew. Math.,* **512** (1999), 119–172.

[10] R. Exel and E. Pardo, "Representing Kirchberg algebras as inverse semigroup crossed products", arXiv:1303.6268 [math.OA], 2013.

[11] R. Exel and E. Pardo, "Graphs, groups and self-similarity", arXiv:1307.1120 [math.OA], 2013.

[12] R. Exel and E. Pardo, "The tight groupoid of an inverse semigroup", arXiv:1408.5278 [math.OA], 2014.

[13] R. Exel and C. Starling, "Self-similar graph C*-algebras and partial crossed products", arXiv:1406.1086 [math.OA], 2014.

[14] R. Exel and A. Vershik, "C*-algebras of irreversible dynamical systems", *Canadian Mathematical Journal,* **58** (2006), 39–63.

[15] R. I. Grigorchuk, "On Burnside's problem on periodic groups", *Funct. Anal. Appl.,* **14** (1980), 41–43.

[16] N. D. Gupta and S. N. Sidki, "On the Burnside problem for periodic groups", *Math. Z.,* **182** (1983), 385–388.

[17] T. Katsura, "A class of C*-algebras generalizing both graph algebras and homeomorphism C*-algebras. I. Fundamental results", *Trans. Amer. Math. Soc.,* **356** (2004), no. 11, 4287–4322.

[18] T. Katsura, "A construction of actions on Kirchberg algebras which induce given actions on their K-groups", *J. reine angew. Math.,* **617** (2008), 27–65.

[19] T. Katsura, "A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras IV, pure infiniteness", *J. Funct. Anal.,* **254** (2008), 1161–1187.

[20] A. Kumjian, D. Pask, I. Raeburn and J. Renault, "Graphs, groupoids, and Cuntz-Krieger algebras", *J. Funct. Anal.,* **144** (1997), 505–541.

[21] A. Kumjian, D. Pask and I. Raeburn, "Cuntz-Krieger algebras of directed graphs", *Pacific J. Math.,* **184** (1998), no. 1, 161–174.

[22] M. V. Lawson, "Inverse semigroups, the theory of partial symmetries", World Scientific, 1998.

[23] M. V. Lawson, "Compactable semilattices", *Semigroup Forum,* **81** (2010), no. 1, 187–199.

[24] V. Nekrashevych, "Cuntz-Pimsner algebras of group actions", *J. Operator Theory,* **52** (2004), 223–249.

[25] V. Nekrashevych, "Self-similar groups", Mathematical Surveys and Monographs, **117,** Amer. Math. Soc., Providence, RI, 2005.

[26] V. Nekrashevych, "C*-algebras and self-similar groups", *J. reine angew. Math.,* **630** (2009), 59–123.

[27] A. L. T. Paterson, "Groupoids, inverse semigroups, and their operator algebras", Birkhäuser, 1999.

[28] G. K. Pedersen, "C*-algebras and Their Automorphism Groups", Academic Press, 1979.

[29] M. V. Pimsner, "A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$", *Fields Inst. Commun.,* **12** (1997), 189–212.

[30] I. Raeburn, "Graph algebras", CBMS Regional Conference Series in Mathematics, **103** (2005), pp. vi+113.

[31] J. Renault, "Cartan subalgebras in C^*-algebras", *Irish Math. Soc. Bull.,* **61** (2008), 29–63.

[32] C. Starling, "Boundary quotients of C*-algebras of right LCM semigroups", arXiv:1409.1549 [math.OA].

[33] B. Steinberg, "A groupoid approach to discrete inverse semigroup algebras", *Adv. Math.*, **223** (2010), 689–727.

Departamento de Matemática; Universidade Federal de Santa Catarina; 88010-970 Florianópolis SC; Brazil
(ruyexel@gmail.com)

Departamento de Matemáticas, Facultad de Ciencias; Universidad de Cádiz, Campus de Puerto Real; 11510 Puerto Real (Cádiz); Spain
(enrique.pardo@uca.es)