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A FREDHOLM OPERATOR APPROACH TO MORITA EQUIVALENCE

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Abstract. Given C^*-algebras A and B and an imprimitivity A-B-bimodule X, we construct an explicit isomorphism $X_*: K_i(A) \to K_i(B)$ where K_i denote the complex K-theory functors for $i = 0, 1$. Our techniques do not require separability nor existence of countable approximate identities. We thus extend, to general C^*-algebras, the result of Brown, Green and Rieffel according to which strongly Morita equivalent C^*-algebras have isomorphic K-groups. The method employed includes a study of Fredholm operators on Hilbert modules.

Key words. Strong Morita equivalence, Hilbert modules, Fredholm operators, sigma-unital C^*-algebras.

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1. Introduction

Strong Morita equivalence for C^*-algebras was introduced by Rieffel ([**13**], [**14**], [**15**]), generalizing the concept of Morita equivalence for rings (see for example, [**16**], Chapter 4). In [**3**], Brown, Green and Rieffel proved that, if the strongly Morita equivalent C^*-algebras A and B are σ-unital (ie., possess countable approximate identities), then they are stably isomorphic in the sense that $\mathcal{K} \otimes A$ is isomorphic to $\mathcal{K} \otimes B$ ($\mathcal{K}$ denoting the algebra of compact operators on a separable Hilbert space). This result has, among many important consequences, the Corollary that strongly Morita equivalent σ-unital C^*-algebras have isomorphic K-theory groups.

In the absence of the σ-unital hypothesis there are examples of strongly Morita equivalent C^*-algebras which are not stably isomorphic, even if one allows tensoring by the compact operators on a non-separable Hilbert space (see [**3**], Theorem 2.7). Thus, without assuming countable approximate identities, the question of whether the K-groups of strongly Morita equivalent C^*-algebras are isomorphic, remained open.

Even though the argument can be made that the majority of C^*-algebras of interest are σ-unital, there is a more serious obstacle in applying the BGR Theorem to concrete situations. The proof of that result rests largely on the work of Brown on full hereditary subalgebras of C^*-algebras [**2**], where a highly non-constructive method is used. The end result is that, given strongly Morita equivalent, σ-unital C^*-algebras A and B, one knows that $\mathcal{K} \otimes A$ and $\mathcal{K} \otimes B$ are isomorphic to each other but there is generally little hope that one can exhibit a concrete isomorphism, unless, of course, an isomorphism is known to exist independently of the Morita equivalence. Likewise, no expression for the isomorphism of K-groups may be described in general.

The primary goal of the present work is to make explicit the isomorphism between $K_*(A)$ and $K_*(B)$ that is predicted by the BGR Theorem. Recently, Rieffel suggested that, if such an explicit construction was possible, then it would probably work without separability, and in fact it does: in Theorems 5.3 and 5.5 we prove that strongly Morita equivalent C^*-algebras have isomorphic K-groups, irrespective of countable approximate identities.

It should be noted that in a crucial point of our argument (see the proof of Theorem 2.7 below), we have to use Kasparov's stabilization Theorem [**7**] which requires a certain countability condition, but in Lemma 2.6, we manage to conform ourselves to Kasparov's hypothesis, without having to assume any countability beforehand.

Let us now briefly describe the strategy adopted to prove our main result. Given a C^*-algebra A, we begin by studying Fredholm operators on Hilbert A-modules and we define the Fredholm index (3.4 and 3.10) which takes values in $K_0(A)$. We then prove that any element of $K_0(A)$ is the index of some Fredholm operator (3.14), showing the surjectivity of the Fredholm index. Next, we characterize in a somewhat geometric way, the pairs of Fredholm operators having identical index (3.16). This enables us to introduce an equivalence relation on the set of all Fredholm operators, under which two operators are equivalent if and only if they have the same index. The quotient by this relation therefore provides an alternate definition for $K_0(A)$ (3.17).

If one is given C^*-algebras A and B and a Hilbert A-B-bimodule X then, for every Fredholm operator $T\colon M \to N$, where M and N are Hilbert A-modules, one can consider the operator

$$T \otimes I_X \quad : \quad M \otimes_B X \quad \to \quad N \otimes_B X.$$

The crux of the whole matter is to prove that this operator is a Fredholm operator (4.3) if X is left-full (in the sense that the range of the A-valued inner-product generates A, see 4.1). This tensor product construction is then used to define (5.1) a group homomorphism $X_*\colon K_0(A) \to K_0(B)$ satisfying $X_*(\mathrm{ind}(T)) = \mathrm{ind}(T \otimes I_X)$. When X is also right-full we show that X_* is an isomorphism (5.3). The case of K_1 is treated, as usual, using suspensions (5.5).

Our characterization of $K_0(A)$ in terms of Fredholm operators should be compared to the well known fact that $K_0(A)$ is isomorphic to the Kasparov [**8**] group $KK(\mathbf{C}, A)$ ($\mathbf{C}$ denoting the algebra of complex numbers). However, since we do not assume countable approximate identities, that fact does not seem to follow from the existing machinery of KK-theory. Furthermore, this aspect of our work is, perhaps, an indication that KK-theory may be extended beyond the realm of σ-unital C^*-algebras. In particular, it sounds reasonable to conjecture that strongly Morita equivalent C^*-algebras are KK-equivalent. In fact, our methods seem particularly well suited to prove such a conjecture, if only KK-theory were not so inextricably linked to σ-unital C^*-algebras. For the σ-unital case see also [**9**], 2.18.

Our study of Fredholm operators and its connections with K-theory is very much in the spirit of the first part of Mingo's thesis [**10**] with the notable difference that Mingo deals with the unital case, while we are mainly interested in the case were identities (even countable approximate ones) are nowhere in sight.

To some extent, one should also compare what we do here with the Fredholm modules of Connes [**5**], in the sense that we explore, in more detail than usually found in the existing literature, what happens when one plugs the algebra of complex numbers as one of the variables of Kasparov's KK-functor. Connes' Fredholm modules are literally the ingredients of $KK(\cdot\,, \mathbf{C})$, according to the Fredholm picture of KK-theory (see [**1**]). On the other hand, the Fredholm operators we study here are related to $KK(\mathbf{C}, \cdot)$.

Although Hilbert modules permeate all of our work, we believe the subject is old enough that we do not need to spend much time presenting formal definitions from scratch. The reader will find all of the relevant definitions in [**11**], [**13**], [**1**], [**6**] and [**4**]. Nevertheless let us stress that all of our Hilbert modules are supposed to be right modules except, of course, when they are bimodules. The term *operator*, when referring to a map $T\colon M \to N$ between the Hilbert A-modules M and N, will always mean an element of $\mathcal{L}_A(M, N)$, that is, T should be an adjointable linear map in the sense that there exists $T^*\colon N \to M$ satisfying $\langle T(\mu), \nu\rangle = \langle \mu, T^*(\nu)\rangle$ for $\mu \in M$ and $\nu \in N$. See [**6**], 1.1.7 for more details.

2. Preliminaries on Hilbert Modules

In this section we would like to present a few simple facts about Hilbert modules which we shall need in the sequel. Throughout this section, A will denote a fixed C^*-algebra and M and N will always refer to (right) Hilbert modules over A. As usual, A^n will be viewed as a Hilbert module equipped with the inner-product $\langle (a_i)_i, (b_i)_i \rangle = \sum_{i=1}^n a_i^* b_i$.

Let us observe that A^n will often stand for the set of $n \times 1$ (column) matrices over A. In that way, the above inner-product can be expressed, for $v = (a_i)_i$ and $w = (b_i)_i$, as $\langle v, w \rangle = v^* w$. Note that v^* refers to the conjugate-transpose matrix.

For each n-tuple $\mu = (\mu_i)_i$ in M^n, we denote by Ω_μ the operator in $\mathcal{L}_A(A^n, M)$ defined by

$$\Omega_\mu\big((a_i)_i\big) = \sum_{i=1}^n \mu_i a_i, \quad (a_i)_i \in A^n.$$

It is easy to see that Ω_μ^* is given by

$$\Omega_\mu^*(\xi) = \big(\langle \mu_i, \xi \rangle\big)_i, \quad \xi \in M.$$

If $\nu = (\nu_i)_i$ is an n-tuple of elements of N, then the operator $T = \Omega_\nu \Omega_\mu^*$ is in $\mathcal{L}_A(M, N)$. More explicitly we have

$$T(\xi) = \sum_{i=1}^n \nu_i \langle \mu_i, \xi \rangle, \quad \xi \in M.$$

Maps such as T will be called *A-finite rank* operators and the set of all those will be denoted $\mathcal{F}_A(M, N)$, or just $\mathcal{F}_A(M)$ in case $M = N$. The closure of $\mathcal{F}_A(M, N)$ in $\mathcal{L}_A(M, N)$ is denoted $\mathcal{K}_A(M, N)$ and elements from this set will be referred to as *A-compact* operators. An expository treatment of operators on Hilbert modules may be found in [**6**].

2.1. Proposition. *For each $\mu = (\mu_i)_i$ in M^n one has that Ω_μ is in $\mathcal{K}_A(A^n, M)$ and hence also that Ω_μ^* is in $\mathcal{K}_A(M, A^n)$.*

Proof. It is obviously enough to consider the case $n = 1$. Let $(u_\lambda)_\lambda$ be an approximate identity for A (always assumed to be positive and of norm one). It follows from [**6**], 1.1.4 that $\lim_\lambda \mu u_\lambda = \mu$. Therefore we have for all a in A

$$\Omega_\mu(a) = \mu a = \lim_\lambda \mu u_\lambda a = \lim_\lambda \mu \langle u_\lambda, a \rangle = \lim_\lambda \Omega_\mu \Omega_{u_\lambda}^*(a),$$

the limit being uniform in $\|a\| \leq 1$. □

2.2. Definition. *A Hilbert module M will be said to be an A-finite rank module if the identity operator I_M is in $\mathcal{K}_A(M)$.*

Since $\mathcal{F}_A(M)$ is an ideal in $\mathcal{L}_A(M)$, which is dense in $\mathcal{K}_A(M)$, it is easy to see that I_M must, in fact, be in $\mathcal{F}_A(M)$ whenever M is A-finite rank. Note that this implies that M must be finitely generated. We next give the complete characterization of A-finite rank modules.

2.3. Proposition. *M is A-finite rank if and only if there exists an idempotent matrix p in $M_n(A)$ such that M is isomorphic, as Hilbert modules, to pA^n.*

Proof. Initially we should observe that the use of the term "isomorphic", when referring to Hilbert modules, is in accordance with [**6**], 1.1.18. That is, there should exist a linear bijection, preserving the A-valued inner product.

Assume M to be A-finite rank. Then $I_M = \Omega_\nu \Omega_\mu^*$ where μ and ν are in M^n. Observe that $\Omega_\mu^* \Omega_\nu$ is then an idempotent A-module operator on A^n, which therefore corresponds to left multiplication by the idempotent $n \times n$ matrix $p = \big(\langle \mu_i, \nu_j \rangle\big)_{i,j}$. The operator Ω_μ^* then gives an invertible operator in $\mathcal{L}_A(M, pA^n)$. To make that map an (isometric) isomorphism one uses polar decomposition. The converse statement is trivial. $\square$

Any A-finite rank module M clearly becomes a finitely generated projective module over the unitized C^*-algebra $\tilde{A}$ (the unitized algebra is given a new identity element, even if A already has one).

2.4. Definition. *The K-theory class $[M]_0 \in K_0(\tilde{A})$ of an A-finite rank module M, is obviously an element of $K_0(A)$ and will henceforth be denoted* $\mathrm{rank}(M)$. *If M is not necessarily assumed to be A-finite rank, but if P is an idempotent operator in $\mathcal{K}_A(M)$, then* $\mathrm{Im}(P)$ *is clearly an A-finite rank module. In this case we let* $\mathrm{rank}(P) = \mathrm{rank}(\mathrm{Im}(P))$.

If X, Y, Z and W are Hilbert A-modules and T is in $\mathcal{L}_A(X \oplus Y, Z \oplus W)$, then T can be represented by a matrix

$$T = \begin{pmatrix} T_{ZX} & T_{ZY} \\ T_{WX} & T_{WY} \end{pmatrix},$$

where T_{ZX} is in $\mathcal{L}_A(X, Z)$ and similarly for the other matrix entries. Matrix notation is used to define our next important concept.

2.5. Definition. *The Hilbert modules M and N are said to be quasi-stably-isomorphic if there exists a Hilbert module X and an invertible operator T in $\mathcal{L}_A(M \oplus X, N \oplus X)$ such that $I_X - T_{XX}$ is A-compact.*

Of course the concept just defined is meant to be a generalization of the well known concept of stable isomorphism for finitely generated projective A-modules, at least when A is unital. We shall discuss shortly, the precise sense in which that generalization takes place. Before that we need a preparatory result.

2.6. Lemma. *Assume M and N are A-finite rank modules. If M and N are quasi-stably-isomorphic then the module X referred to in 2.5 can be taken to be countably generated.*

Proof. Let $T = \begin{pmatrix} T_{NM} & T_{NX} \\ T_{XM} & T_{XX} \end{pmatrix}$ be an invertible operator in $\mathcal{L}_A(M \oplus X, N \oplus X)$ with $I_X - T_{XX}$ A-compact, as in 2.5, and let $S = \begin{pmatrix} S_{MN} & S_{MX} \\ S_{XN} & S_{XX} \end{pmatrix}$ be the inverse of T. Choose a countable set $\Xi_0 = \{\xi_i\}_{i \in \mathbf{N}}$ of elements in X such that

(i) The images of T_{XM}, S_{XN}, T^*_{NX}, and S^*_{MX} are contained in the submodule of X generated by Ξ_0.

(ii) $I_X - T_{XX}$ can be approximated by A-finite rank operators of the form $\Omega_\nu\Omega^*_\mu$, where the components of $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$ belong to Ξ_0.

Define, inductively, $\Xi_{n+1} = \Xi_n \cup T_{XX}(\Xi_n) \cup S_{XX}(\Xi_n) \cup T^*_{XX}(\Xi_n) \cup S^*_{XX}(\Xi_n)$. The set $\Xi = \bigcup_{n\in\mathbf{N}} \Xi_n$ is then obviously countable, satisfies (i) and (ii) above and, in addition,

(iii) Ξ is invariant under T_{XX}, S_{XX}, T^*_{XX} and S^*_{XX}.

Let X_0 be the Hilbert submodule of X generated by Ξ. Because of (i) and (iii) we see that $T(M \oplus X_0) \subseteq N \oplus X_0$ and $T^*(N \oplus X_0) \subseteq M \oplus X_0$. The restriction of T then gives an operator T' in $\mathcal{L}_A(M \oplus X_0, N \oplus X_0)$. The same reasoning applies to S providing S' in $\mathcal{L}_A(N \oplus X_0, M \oplus X_0)$ which is obviously the inverse of T'. In virtue of (ii) it is clear that T' satisfies the conditions of definition 2.5. □

2.7. Theorem. *If M and N are quasi-stably-isomorphic A-finite rank modules, then* $\operatorname{rank}(M) = \operatorname{rank}(N)$.

Proof. According to 2.6 let X be a countably generated Hilbert module and T be an invertible operator in $\mathcal{L}_A(M \oplus X, N \oplus X)$ such that $I_X - T_{XX}$ is in $\mathcal{K}_A(X)$. By Kasparov's stabilization Theorem [**7**], 3.2 (see also [**6**], 1.1.24), $X \oplus H_A$ is isomorphic to H_A, where H_A is the completion of $\bigoplus_1^\infty A$ (see [**6**], 1.1.6 for a more precise definition). This said, we may assume, without loss of generality, that $X = H_A$. Since M is finitely generated as an A-module, we conclude, again by Kasparov's Theorem, that $M \oplus X$ is isomorphic to H_A. Choose, once and for all, an isomorphism $\varphi\colon H_A \to M \oplus X$ and consider the operators F and G on H_A given by the compositions

$$F \ : \ H_A \xrightarrow{\varphi} M \oplus X \xrightarrow{T} N \oplus X \longrightarrow X \longrightarrow M \oplus X \xrightarrow{\varphi^{-1}} H_A$$

and

$$G \ : \ H_A \xrightarrow{\varphi} M \oplus X \longrightarrow X \longrightarrow N \oplus X \xrightarrow{T^{-1}} M \oplus X \xrightarrow{\varphi^{-1}} H_A,$$

where the unmarked arrows denote either the canonical inclusion or the canonical projection. It can be easily seen that both $I_{H_A} - GF$ and $I_{H_A} - FG$ belong to $\mathcal{K}_A(H_A)$. Since one also has that $FGF = F$ and $GFG = G$, the operators $I_{H_A} - GF$ and $I_{H_A} - FG$ are seen to be idempotents.

Consider the exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}_A(H_A) \longrightarrow \mathcal{L}_A(H_A) \longrightarrow \mathcal{L}_A(H_A)/\mathcal{K}_A(H_A) \longrightarrow 0.$$

Denoting by π the quotient map, one sees from the discussion above, that $\pi(F)$ and $\pi(G)$ are each others inverse. Two facts need now be stressed. The first one is that the K-theory index map

$$\partial \ : \ K_1\big(\mathcal{L}_A(H_A)/\mathcal{K}_A(H_A)\big) \to K_0\big(\mathcal{K}_A(H_A)\big)$$

assigns to the class of $\pi(F)$, the element $\operatorname{rank}(N) - \operatorname{rank}(M)$, once $\mathcal{K}_A(H_A)$ is identified with $\mathcal{K} \otimes A$ (according to [**7**], 2.4) and $K_0(\mathcal{K} \otimes A)$ is identified with $K_0(A)$ as usual in K-theory. Too see this, one could employ 8.3.1 in [**1**], where the element w, used there, can be taken to be $w = \begin{pmatrix} F & I_{H_A} - FG \\ I_{H_A} - GF & G \end{pmatrix}$.

The second fact to be pointed out is that F is an A-compact perturbation of the identity. In fact, ignoring both φ and φ^{-1} in the definition of F (that is, conjugating F by φ^{-1}) we get the operator

$$F' \ : \ M \oplus X \xrightarrow{T} N \oplus X \longrightarrow X \longrightarrow M \oplus X$$

whose matrix representation is $\begin{pmatrix} 0 & 0 \\ T_{XM} & T_{XX} \end{pmatrix}$. If we now observe that T_{XX} is an A-compact perturbation of the identity and that M and N are A-finite rank modules (and so any operator having either M or N as domain or codomain must be A-compact), one concludes that F differ from the identity by a compact operator as claimed. It follows that $\pi(F) = 1$ and hence that $\partial(\pi(F)) = 0$. This concludes the proof. □

3. Fredholm Operators

We shall now study Fredholm operators between Hilbert modules. As before, A will denote a fixed C^*-algebra and M and N, with or without subscripts, will denote Hilbert A-modules.

3.1. Definition. *Let T be in $\mathcal{L}_A(M, N)$. Suppose there is S in $\mathcal{L}_A(N, M)$ such that $I_M - ST$ is in $\mathcal{K}_A(M)$ and $I_N - TS$ is in $\mathcal{K}_A(N)$. Then T is said to be an A-Fredholm operator. In case the algebra A is understood, we shall just say that T is a Fredholm operator (but we should keep in mind that this notion does not coincide with the classical notion of Fredholm operators).*

As in the classical theory of Fredholm operators (see, for example, [**12**], 3.3.11), it can be proved that whenever T is A-Fredholm, one can find S in $\mathcal{L}_A(N, M)$ such that $I_M - ST$ is in $\mathcal{F}_A(M)$ and $I_N - TS$ is in $\mathcal{F}_A(N)$. In all of our uses of the A-Fredholm hypothesis, below, we shall adopt that characterization.

In the initial part of the present section we shall concentrate on a special class of operators which we will call regular operators. This concept is the natural extension, to Hilbert modules, of the notion of operators on Hilbert spaces having closed image. Contrary to the Hilbert space case, not all A-Fredholm operators have a closed image: take, for example, $A = C([0, 1])$, and let $M = A$ with its natural Hilbert module structure (see the first paragraph of section 2). Let $T\colon M \to M$ be defined by $T(f)(x) = xf(x)$. Since A is unital, any operator on M is A-compact, hence any such operator is A-Fredholm as well. Nevertheless the range of T is not closed. In fact, the function $g(x) = \sqrt{x}$ can be easily seen to be adherent to the image of T but not in that image.

3.2. Definition. *An operator T in $\mathcal{L}_A(M,N)$ is said to be regular if there is S in $\mathcal{L}_A(N,M)$ such that $TST=T$ and $STS=S$. Any operator S having these properties will be called a pseudo-inverse of T.*

It is easy to see that for any pseudo-inverse S of T one has that $I_M - ST$ is the projection onto $\mathrm{Ker}(T)$ and that TS is the projection onto $\mathrm{Im}(T)$. If T is assumed to be regular and Fredholm, there are, according to the above definitions, operators S and S' such that $I_M - S'T$ and $I_N - TS'$ are A-finite rank and, on the other hand, $TST=T$ and $STS=S$.

Observe that, because $I_M - ST = (I_M - S'T)(I_M - ST)$, any pseudo inverse S, must be such that $I_M - ST$ is A-finite rank and similarly for $I_N - TS$. An immediate consequence of the present discussion is the following:

3.3. Proposition. *Let $T \in \mathcal{L}_A(M,N)$ be a regular A-Fredholm operator. Then both* $\mathrm{Ker}(T)$ *and* $\mathrm{Ker}(T^*)$ *are A-finite rank modules.*

Let us now define the Fredholm index for regular A-Fredholm operators. Shortly we shall extend that concept to general A-Fredholm operators.

3.4. Definition. *If T is a regular A-Fredholm operator, then the Fredholm index of T is defined to be the element of $K_0(A)$ given by*

$$\mathrm{ind}(T) = \mathrm{rank}(\mathrm{Ker}(T)) - \mathrm{rank}(\mathrm{Ker}(T^*)).$$

We collect, in our next proposition, some of the elementary properties of the Fredholm index.

3.5. Proposition. *If $T \in \mathcal{L}_A(M,N)$ is a regular Fredholm operator, then*

(i) $\mathrm{ind}(T^*) = -\mathrm{ind}(T)$.

(ii) Any pseudo-inverse S of T is regular Fredholm and we have that $\mathrm{rank}(\mathrm{Ker}(T^*)) = \mathrm{rank}(\mathrm{Ker}(S))$ *and* $\mathrm{ind}(S) = -\mathrm{ind}(T)$.

(iii) If X and Y are Hilbert A-modules and if $U \in \mathcal{L}_A(X,M)$ and $V \in \mathcal{L}_A(N,Y)$ are invertible, then $\mathrm{ind}(VTU) = \mathrm{ind}(T)$.

(iv) If $T_1 \in \mathcal{L}_A(M_1,N_1)$ is regular and Fredholm, then $\mathrm{ind}(T \oplus T_1) = \mathrm{ind}(T) + \mathrm{ind}(T_1)$.

Proof. Left to the reader. □

The first fact about classical Fredholm operators whose generalization to Hilbert modules requires some work is the invariance under compact perturbations which we now prove.

3.6. Theorem. *If T is a regular Fredholm operator in $\mathcal{L}_A(M)$ and if $I_M - T$ is A-compact, then* $\mathrm{ind}(T) = 0$.

Proof. Let S be a pseudo-inverse for T and denote by X the image of the idempotent ST or, equivalently, $X = \mathrm{Im}(S)$. Consider the transformation

$$U\colon \mathrm{Ker}(T) \oplus X \to \mathrm{Ker}(S) \oplus X$$

given by $U(\xi, \eta) = \big((I_M - TS)(\xi + \eta), S(\xi + \eta)\big)$. It is easy to see that U is invertible with inverse given by $V(\xi, \eta) = \big((I_M - ST)(\xi + T(\eta)), ST(\xi + T(\eta))\big)$. The operator U_{XX} (occurring in the matrix representation of U) coincides with S which is easily seen to be an A-compact perturbation of the identity. This shows that the A-finite rank modules $\mathrm{Ker}(T)$ and $\mathrm{Ker}(S)$ are quasi-stably-isomorphic. By 2.7 we conclude that $\mathrm{rank}(\mathrm{Ker}(T)) = \mathrm{rank}(\mathrm{Ker}(S))$ and hence that $\mathrm{ind}(T) = 0$. □

3.7. Corollary. *If $T_1, T_2 \in \mathcal{L}_A(M, N)$ are regular Fredholm operators such that $T_1 - T_2$ is in $\mathcal{K}_A(M, N)$, then* $\mathrm{ind}(T_1) = \mathrm{ind}(T_2)$.

Proof. Let S_1 and S_2 be pseudo inverses for T_1 and T_2. Define operators U and R in $\mathcal{L}_A(M \oplus N)$ by

$$U = \begin{pmatrix} I_M - S_1T_1 & S_1 \\ T_1 & I_N - T_1S_1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & S_1 \\ T_2 & 0 \end{pmatrix}.$$

Observe that $U^2 = I$ so, in particular, U is invertible. Therefore, by 3.5, we have that $\mathrm{ind}(UR) = \mathrm{ind}(R) = \mathrm{ind}(T_2) - \mathrm{ind}(T_1)$. But, since

$$UR = \begin{pmatrix} S_1T_2 & 0 \\ T_2 - T_1S_1T_2 & T_1S_1 \end{pmatrix}$$

is an A-compact perturbation of the identity, it follows that $\mathrm{ind}(UR) = 0$. □

We now start to treat general A-Fredholm operators. The crucial fact which allows us to proceed, is that any Fredholm operator is "regularizable" over a unital algebra.

3.8. Lemma. *Let B be a unital C^*-algebra and M and N be Hilbert B-modules. If $T \in \mathcal{L}_B(M, N)$ is B-Fredholm (but not necessarily regular), then there exists an integer n and a regular B-Fredholm operator $\tilde{T}$ in $\mathcal{L}_B(M \oplus B^n, N \oplus B^n)$ such that $\tilde{T}_{NM} = T$.*

Proof. Let S in $\mathcal{L}_B(N, M)$ be such that both $I_M - ST$ and $I_N - TS$ are B-finite rank. So, let $\nu = (\nu_i)_i$ and $\mu = (\mu_i)_i$ be such that $I_M - ST = \Omega_\nu \Omega_\mu^*$. Define

$$\tilde{T} = \begin{pmatrix} T & 0 \\ \Omega_\mu^* & 0 \end{pmatrix} \quad \text{and} \quad \tilde{S} = \begin{pmatrix} S & \Omega_\nu \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\tilde{T}$ and $\tilde{S}$ are each others pseudo-inverse hence, in particular, $\tilde{T}$ is regular. The hypothesis that B have a unit implies that B^n is B-finite rank and hence that

$$I - \tilde{S}\tilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & I_{B^n} \end{pmatrix}$$

and

$$I - \tilde{T}\tilde{S} = \begin{pmatrix} I_N - TS & -T\Omega_\nu \\ -\Omega_\mu^* S & I_{B^n} - \Omega_\mu^*\Omega_\nu \end{pmatrix}$$

are B-finite rank operators. We conclude that $\tilde{T}$ is Fredholm. □

Since we shall not assume the algebras we work with to be unital (nor σ-unital, as already stressed in the introduction), we will keep all the applications of the above Lemma to the unitized algebra $\tilde{A}$, in the following way. Given an A-Fredholm operator T in $\mathcal{L}_A(M,N)$ consider M and N as Hilbert modules over the unitized algebra $\tilde{A}$, as it is done in [**6**], remark 1.1.5. Obviously T is $\tilde{A}$-Fredholm as well. Let therefore $\tilde{T} \in \mathcal{L}_{\tilde{A}}(M \oplus \tilde{A}^n, N \oplus \tilde{A}^n)$ be the operator constructed as in 3.8. Since $\tilde{T}$ is regular and Fredholm, $\operatorname{ind}(\tilde{T})$ is well defined as an element of $K_0(\tilde{A})$.

The following result will allow us to return to the realm of non-unital (meaning non-necessarily-unital) algebras after our brief encounter with units.

3.9. Proposition. *The Fredholm index of $\tilde{T}$ belongs to $K_0(A)$.*

Proof. Denote by $\varepsilon\colon \tilde{A} \to \mathbf{C}$ the augmentation homomorphism. That is $\varepsilon(a+\lambda) = \lambda$ for λ in $\mathbf{C}$ and a in A. Since $K_0(A)$ is defined to be the kernel of the map $\varepsilon_*\colon K_0(\tilde{A}) \to K_0(\mathbf{C})$, all we need to do is show that $\varepsilon_*(\operatorname{ind}(\tilde{T})) = 0$.

Let S, $\tilde{S}$, μ and ν be as in the proof of 3.8. Note that $I - \tilde{S}\tilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & I_{\tilde{A}^n} \end{pmatrix}$, so $\operatorname{rank}(\operatorname{Ker}(\tilde{T})) = n$. We then need to show that $\varepsilon_*(\operatorname{rank}(\operatorname{Ker}(\tilde{S})))$ is also equal to n. The kernel of $\tilde{S}$ is the image of the idempotent

$$I - \tilde{T}\tilde{S} = \begin{pmatrix} I_N - TS & -T\Omega_\nu \\ -\Omega_\mu^* S & I_{\tilde{A}^n} - \Omega_\mu^*\Omega_\nu \end{pmatrix} \tag{3.9.1}$$

which we shall simply denote by P. Since P is $\tilde{A}$-compact, there are m-tuples $\phi = (\phi_1, \ldots, \phi_m)$ and $\psi = (\psi_1, \ldots, \psi_m)$ of elements of $N \oplus \tilde{A}^n$ such that $P = \Omega_\phi \Omega_\psi^*$. Write each ϕ_i as $\phi_i = (\xi_i, v_i)$ and $\psi_i = (\eta_i, w_i)$.

Replacing, if necessary, ϕ_i by $P(\phi_i)$, we can assume that $P\Omega_\phi = \Omega_\phi$ and therefore that $Q = \Omega_\psi^*\Omega_\phi$ is an idempotent operator on $\tilde{A}^n$ whose image is isomorphic, as $\tilde{A}$-modules, to the image of P. As an $m \times m$ matrix, Q is given by $Q = \big(\langle \psi_i, \phi_j \rangle\big)_{i,j}$. Our goal is then to show that the trace of the complex idempotent matrix $\varepsilon(Q) = \big(\varepsilon\big(\langle \psi_i, \phi_j \rangle\big)\big)_{i,j}$ equals n. That trace is given by

$$\sum_{i=1}^m \varepsilon\big(\langle \psi_i, \phi_i \rangle\big) = \sum_{i=1}^m \varepsilon\big(\langle w_i, v_i \rangle\big) = \varepsilon\bigg(\sum_{i=1}^m \sum_{r=1}^n \langle w_i, e_r \rangle \langle e_r, v_i \rangle\bigg),$$

where $\{e_i, \ldots, e_n\}$ is the canonical basis of $\tilde{A}^n$. The above then equals

$$\varepsilon\bigg(\sum_{i=1}^m \sum_{r=1}^n \langle e_r, v_i \langle w_i, e_r \rangle\rangle\bigg) = \varepsilon\bigg(\sum_{r=1}^n \langle (0, e_r), P(0, e_r)\rangle\bigg).$$

Using the definition of P in 3.9.1, the term $\sum_{r=1}^n \langle (0, e_r), P(0, e_r)\rangle$ can be expressed as

$$\sum_{r=1}^n \langle e_r, (I - \Omega_\mu^*\Omega_\nu) e_r \rangle = n - \sum_{r=1}^n \langle \mu_r, \nu_r \rangle.$$

The last term above clearly maps to n under ε so the proof is complete. ∎

The statement of 3.9 is meant to refer to the specific construction of $\tilde{T}$ obtained in 3.8. But note that any regular Fredholm operator in $\mathcal{L}_{\tilde{A}}(M \oplus \tilde{A}^n, N \oplus \tilde{A}^n)$, which has T in the upper left corner, will differ from the $\tilde{T}$ above, by an $\tilde{A}$-compact operator. Therefore its index will coincide with that of $\tilde{T}$ by 3.7, and so will be in $K_0(A)$ as well.

3.10. Definition. *If T is an A-Fredholm operator in $\mathcal{L}_A(M, N)$, then the Fredholm index of T, denoted* $\operatorname{ind}(T)$, *is defined to be the index of the regular Fredholm operator $\tilde{T}$ constructed in 3.8.*

Clearly, if T is already regular, we can take $n = 0$ in 3.8 so that the above definition extends the one given in 3.4. Elementary properties of the Fredholm index are collected in the next proposition.

3.11. Proposition. *If $T \in \mathcal{L}_A(M, N)$ is a Fredholm operator, then*

(i) $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$

(ii) If U in $\mathcal{L}_A(X, M)$ and V in $\mathcal{L}_A(Y, N)$ are invertible, then $\operatorname{ind}(VTU) = \operatorname{ind}(T)$.

(iii) If $T' \in \mathcal{L}_A(M, N)$ is such that $T' - T$ is in $\mathcal{K}_A(M, N)$, then T' is also Fredholm and $\operatorname{ind}(T') = \operatorname{ind}(T)$.

(iv) If $S \in \mathcal{L}_A(N, M)$ is such that $I_M - ST$ is in $\mathcal{K}_A(M)$, then $\operatorname{ind}(S) = -\operatorname{ind}(T)$.

(v) If $T_1 \in \mathcal{L}_A(M_1, N_1)$ is Fredholm, then $\operatorname{ind}(T \oplus T_1) = \operatorname{ind}(T) + \operatorname{ind}(T_1)$.

Proof. Left to the reader. □

Let us now briefly tackle the question of invariance of the Fredholm index under small perturbations. The proof, which we omit, is similar to the classical one (see, for example, 3.3.18 in [**12**]).

3.12. Proposition. *Let T in $\mathcal{L}_A(M, N)$ be Fredholm. Then there is a positive real number ε such that any T' satisfying $\|T' - T\| < \varepsilon$ is also Fredholm with* $\operatorname{ind}(T') = \operatorname{ind}(T)$. *In fact ε can be taken to be $\varepsilon = \|S\|^{-1}$ for any S in $\mathcal{L}_A(N, M)$ such that $I_M - ST$ is in $\mathcal{K}_A(M)$.*

It is equally easy to prove:

3.13. Proposition. *If $T_1 \in \mathcal{L}_A(M, N)$ and $T_2 \in \mathcal{L}_A(N, P)$ are Fredholm operators, then T_2T_1 is Fredholm and* $\operatorname{ind}(T_2T_1) = \operatorname{ind}(T_2) + \operatorname{ind}(T_1)$.

It is part of our goal to find an alternate definition of $K_0(A)$ in terms of Fredholm operators. For this reason it is important to have a sufficiently large collection of such operators. Specifically we will need to exhibit Fredholm operators with an arbitrary element of $K_0(A)$ as the index.

This is quite easy if A has a unit: given an arbitrary element $[M]_0 - [N]_0$ in $K_0(A)$, where M and N are finitely generated projective A-modules, consider the zero operator $\mathbf{0}\colon M \to N$. If both M and N are given the Hilbert module structure they possess, being isomorphic to direct summands of A^n, it follows that $\mathbf{0}$ is a Fredholm operator, its index obviously being $[M]_0 - [N]_0$. The situation is slightly more complicated in the non-unital case, and it is treated in our next result.

3.14. Proposition. *For any α in $K_0(A)$ there is an A-Fredholm operator T such that* $\operatorname{ind}(T)=\alpha$.

Proof. Regarding $K_0(A)$ as the kernel of the augmentation map ε_*, in $K_0(\tilde{A})$, write $\alpha = [p]_0 - [q]_0$, where p and q are self-adjoint idempotent $n \times n$ matrices over $\tilde{A}$ such that $\varepsilon_*([p]_0 - [q]_0) = 0$. It follows that $\varepsilon(p)$ and $\varepsilon(q)$ are similar complex matrices. Hence, after performing a conjugation of, say q, by a complex unitary matrix, we may assume that $\varepsilon(p)$ and $\varepsilon(q)$ are in fact equal. Therefore $p-q$ is in $M_n(A)$.

We now claim that the operator $T: pA^n \to qA^n$ given by $T(v) = qv$ is Fredholm and that its index is $[p]_0 - [q]_0$.

Let $S: qA^n \to pA^n$ be given by $S(v) = pv$. Denote by p_i the i^{th} column of p, viewed as a $n \times 1$ matrix, and let $(u_\lambda)_\lambda$ be an approximate identity for A. Define $\xi_i = p(p-q)p_i$ and $\eta_i^\lambda = p_i u_\lambda$ for $1 \le i \le n$. So, ξ_i and η_i^λ are elements of pA^n. Let $\xi = (\xi_i)_i$ and $\eta^\lambda = (\eta_i^\lambda)_i$. For ζ in pA^n we have

$$\Omega_\xi \Omega^*_{\eta^\lambda}(\zeta) = \sum_{i=1}^n \xi_i \langle \eta_i^\lambda, \zeta\rangle = \sum_{i=1}^n p(p-q)p_i u_\lambda p_i^* \zeta$$

which converges, uniformly in $\|\zeta\| \le 1$, as $\lambda \to \infty$, to

$$\sum_{i=1}^n p(p-q)p_i p_i^* \zeta = p(p-q)p\zeta = \zeta - pq\zeta = (I - ST)\zeta.$$

In a similar fashion we can show that $I - TS$ is A-compact as well. In order to compute the index of T, consider the operators $\tilde{T}$ in $\mathcal{L}_{\tilde{A}}(pA^n \oplus \tilde{A}^n, qA^n \oplus \tilde{A}^n)$ and $\tilde{S}$ in $\mathcal{L}_{\tilde{A}}(qA^n \oplus \tilde{A}^n, pA^n \oplus \tilde{A}^n)$ given by

$$\tilde{T} = \begin{pmatrix} qp & q(I-p) \\ (I-q)p & (I-q)(I-p) \end{pmatrix} \quad \text{and} \quad \tilde{S} = \begin{pmatrix} pq & p(I-q) \\ (I-p)q & (I-p)(I-q) \end{pmatrix}.$$

Direct computation shows that

$$\tilde{S}\tilde{T} = \begin{pmatrix} I & 0 \\ 0 & I-p \end{pmatrix} \quad \text{and} \quad \tilde{T}\tilde{S} = \begin{pmatrix} I & 0 \\ 0 & I-q \end{pmatrix},$$

from which it follows that $\tilde{S}$ is a pseudo-inverse for $\tilde{T}$ and hence that $\tilde{T}$ is a regular $\tilde{A}$-Fredholm operator. By definition we have

$$\operatorname{ind}(T) = \operatorname{ind}(\tilde{T}) = \operatorname{rank}(I - \tilde{S}\tilde{T}) - \operatorname{rank}(I - \tilde{T}\tilde{S}) = [p]_0 - [q]_0. \qquad \square$$

Our last result showed that any K_0 element is the index of some Fredholm operator. We would now like to discuss the question of when two Fredholm operators have the same index. As a first step, we classify operators with index zero.

3.15. Lemma. *Let $T \in \mathcal{L}_A(M, N)$ be a Fredholm operator with* $\operatorname{ind}(T) = 0$. *Then there exists an integer n such that $T \oplus I_{A^n} \colon M \oplus A^n \to N \oplus A^n$ is an A-compact perturbation of an invertible operator.*

Proof. Let $\tilde{T}$ in $\mathcal{L}_{\tilde{A}}(M \oplus \tilde{A}^n, N \oplus \tilde{A}^n)$ be constructed as in 3.8 with $\tilde{A}$ playing the role of B. If $\tilde{S}$ is also as in 3.8 we have $I - \tilde{S}\tilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & I_{\tilde{A}^n} \end{pmatrix}$. The hypothesis that $\operatorname{ind}(T) = 0$ then says that $\operatorname{rank}(I - \tilde{T}\tilde{S}) = \operatorname{rank}(I - \tilde{S}\tilde{T})$. But, as we see above, the rank of $I - \tilde{S}\tilde{T}$ is the same as the rank of the free $\tilde{A}$-module $\tilde{A}^n$. So, we have that $\operatorname{Im}(I - \tilde{T}\tilde{S})$ is stably isomorphic, as $\tilde{A}$-modules, to $\tilde{A}^n$. By increasing n, if necessary, we may thus assume that $\operatorname{Im}(I - \tilde{T}\tilde{S})$ is, in fact, isomorphic to $\tilde{A}^n$.

This said, we may find a generating set $\{x_i\}_{i=1}^n$ for $\operatorname{Im}(I - \tilde{T}\tilde{S})$ which is orthonormal in the sense that $\langle x_i, x_j \rangle = \delta_{ij}$. Let each x_i be given by $x_i = (\xi_i, v_i)$. Each v_j is in $\tilde{A}^n$ and hence we may write $v_j = (v_{ij})_i$ with $v_{ij} \in \tilde{A}$. The fact that the x_i form an orthonormal set translates to

$$\langle \xi_i, \xi_j \rangle + \sum_{k=1}^{n} v_{ki}^* v_{kj} = \delta_{ij}.$$

Recalling that $\varepsilon \colon \tilde{A} \to \mathbf{C}$ denotes the augmentation homomorphism, observe that the complex matrix $u = \big(u_{ij}\big)_{i,j}$, where $u_{ij} = \varepsilon(v_{ij})$, is unitary. If we now define $x'_j = \sum_{i=1}^n u_{ji}^* x_i$ and write $x'_j = (\xi'_j, v'_j)$ with $v'_j = (v'_{ij})_i$ one can show that $\varepsilon(v'_{ij}) = \delta_{ij}$. In other words, we may assume, without loss of generality, that $\varepsilon(v_{ij}) = \delta_{ij}$. Otherwise, replace each x_i by x'_i.

Let $\xi = (\xi_i)_i$ and $v = (v_i)_i$ so that $\Omega_\xi \oplus \Omega_v$ is the isomorphism from $\tilde{A}^n$ onto $\operatorname{Im}(I - \tilde{T}\tilde{S})$ mentioned above. The operator U in $\mathcal{L}_{\tilde{A}}(M \oplus \tilde{A}^n, N \oplus \tilde{A}^n)$ given by

$$U = \begin{pmatrix} T & \Omega_\xi \\ \Omega_\mu^* & \Omega_v \end{pmatrix}$$

is therefore invertible (please note that μ is as in the proof of 3.8).

Note that

$$\overline{(M \oplus \tilde{A}^n) \cdot A} = M \oplus A^n$$

and similarly

$$\overline{(N \oplus \tilde{A}^n) \cdot A} = N \oplus A^n.$$

So, U gives, by restriction, an invertible operator from $M \oplus A^n$ to $N \oplus A^n$ and, denoting the latter by U, from now on, we have

$$U - \begin{pmatrix} T & 0 \\ 0 & I_{A^n} \end{pmatrix} = \begin{pmatrix} 0 & \Omega_\xi \\ \Omega_\mu^* & \Omega_v - I_{A^n} \end{pmatrix}.$$

The matrix on the right hand side represents an A-compact operator: the crucial point being that $\Omega_v - I_{A^n}$ is the operator on A^n given by multiplication by the matrix $\big(v_{ij} - \delta_{ij}\big)_{i,j}$ which is in $M_n(A)$ since $\varepsilon(v_{ij} - \delta_{ij})$ was seen to be zero (see [**7**], Lemma 2.4). $\square$

The following characterization of when two Fredholm operators have the same index is an immediate corollary of our last Lemma.

3.16. Proposition. *If T_i in $\mathcal{L}_A(M_i, N_i)$ for $i = 1, 2$, are Fredholm operators such that* $\operatorname{ind}(T_1) = \operatorname{ind}(T_2)$*, then for some integer n, the operator*

$$T_1 \oplus T_2^* \oplus I_{A^n} \ : \ M_1 \oplus N_2 \oplus A^n \to N_1 \oplus M_2 \oplus A^n$$

is an A-compact perturbation of an invertible operator.

Using the machinery developed so far, we may provide an alternate definition of the K-theory group $K_0(A)$. Choose, once and for all, a cardinal number ω which is bigger than the cardinality of A^n for every integer n. We remark that the role of ω is merely to avoid set theoretical problems arising from the careless reference to the set of *all* A-Fredholm operator. Any choice of ω, as long as it is sufficiently large, will result in the same conclusions.

Denote by $F_0(A)$ the set of all A-Fredholm operators whose domain and codomain are Hilbert modules of cardinality no larger than ω (actually we should require these Hilbert modules to have a subset of ω as their carrier set). Declare two elements T_1 and T_2 of $F_0(A)$ equivalent, if there is an integer n such that $T_1 \oplus T_2^* \oplus I_{A^n}$ is an A-compact perturbation of an invertible operator.

The quotient $F(A)$ of $F_0(A)$ by the above equivalence relation is obviously a group with the operation of direct sum of Fredholm operators. The inverse of the class of T is given by that of T^* by 3.11 and 3.15.

3.17. Corollary. *The Fredholm index map, viewed as a map*

$$\operatorname{ind}\colon F(A) \to K_0(A),$$

is a group isomorphism.

Proof. Follows immediately from 3.14 and 3.16. □

We should remark that 3.17 is a generalization of the fact that $KK(\mathbf{C}, A)$ is isomorphic to $K_0(A)$. The new aspect is that no separability is involved. This is one of the crucial steps in achieving our main result as we shall see shortly.

4. Preliminaries on Hilbert Bimodules

We would now like to set the present section aside in order to present a few relevant aspects of the theory of Hilbert bimodules which will be important for our discussion of Morita equivalence. We adopt the definition of Hilbert bimodules given in [**4**], 1.8. Namely, if A and B are C^*-algebras, a Hilbert A-B-bimodule is a complex vector space X which is a left Hilbert A-module as well as a right Hilbert B-module, and such that the A-valued inner product

$$(\cdot|\cdot): X \times X \to A$$

and the B-valued inner product

$$\langle \cdot\,, \cdot \rangle: X \times X \to B$$

are related by the identity

$$(\xi|\eta)\mu = \xi\langle \eta, \mu\rangle, \quad \xi, \eta, \mu \in X.$$

Some authors prefer to use the notation $\langle \cdot\,, \cdot\rangle_A$ and $\langle \cdot\,, \cdot\rangle_B$ for these inner-products but we believe the notation indicated above makes some formulas much more readable. In particular, it is implicit that any inner-product denoted by $\langle \cdot\,, \cdot\rangle$ will be linear in the second variable while those denoted by $(\cdot|\cdot)$ are linear in the first variable. We should nevertheless remark that the differentiated notation is not meant to imply any asymmetry in the structure of bimodules. With the obvious interchange of left and right, any result that holds on the "left" will also hold on the "right" and vice-versa.

As mentioned in [**4**], Hilbert A-B-bimodules are nothing but Rieffel's imprimitivity bimodules (see [**13**], 6.10) for which it is *not* assumed that the range of the inner-products generate the coefficient algebras.

The closed span of the set $\{\langle x, y\rangle: x, y \in X\}$, which we denote by $\langle X, X\rangle$, is a two sided ideal in B and similarly, the closed span of $\{(x|y): x, y \in X\}$ is the ideal $(X|X)$ of A.

4.1. Definition. *The Hilbert A-B-bimodule X is said to be left-full (resp. right-full) if $(X|X)$ coincides with A (resp. if $\langle X, X\rangle$ coincides with B).*

Using the terminology just introduced, Rieffel's imprimitivity bimodules are precisely the Hilbert-bimodules that are simultaneously left-full and right-full.

Throughout this section we shall consider fixed two C^*-algebras A and B as well as a Hilbert A-B-bimodule X. As before, M and N will denote Hilbert A-modules. If M is a Hilbert A-module (we remind the reader of our convention according to which *module* without further adjectives, means *right module*), then the tensor product module $M \otimes_A X$ has a natural B-valued (possibly degenerated) inner-product specified by

$$\langle \xi_1 \otimes x_1, \xi_2 \otimes x_2\rangle = \langle x_1, \langle \xi_1, \xi_2\rangle x_2\rangle, \quad \xi_1, \xi_2 \in M,\ x_1, x_2 \in X.$$

After moding out the elements of norm zero and completing, we are left with a Hilbert B-module which we also denote, for simplicity, by $M \otimes_A X$. See [**6**], 1.2.3 for details, but please observe that the notation used there is not the same as the one just described. It should also be observed that one does not need the A-valued inner-product on X in order to perform this construction. It is enough that X be a left A-module in such a way that the representation of A, as left multiplication operators on X, be a $*$-homomorphism.

If T is in $\mathcal{L}_A(M,N)$, we denote by $T \otimes I_X$ the linear transformation

$$T \otimes I_X \ : \ M \otimes_A X \to N \otimes_A X$$

given by $T \otimes I_X(\xi, x) = T(\xi) \otimes x$ for ξ in M and x in X. A slight modification of [**6**], 1.2.3 shows that $T \otimes I_X$ is in $\mathcal{L}_B(M \otimes_A X, N \otimes_A X)$ and that $\|T \otimes I_X\| \le \|T\|$.

Let us now present one of our most important technical results. Although quite a simple fact, with an equally simple proof, it is a crucial ingredient in this work. Compare [**6**], 1.2.8.

4.2. Theorem. *If the Hilbert A-B-bimodule X is left-full and if T is in $\mathcal{K}_A(M,N)$, then $T \otimes I_X$ is B-compact.*

Proof. It obviously suffices to prove the statement in case $T = \Omega_\nu \Omega_\mu^*$ with μ in M and ν in N. Given $\xi \otimes x$ in $M \otimes_A X$ we have

$$T \otimes I_X(\xi \otimes x) = \nu\langle \mu, \xi\rangle \otimes x = \nu \otimes \langle \mu, \xi\rangle x.$$

Observe that, since X is left-full and also by [**6**], Lemma 1.1.4, there is no harm in assuming that $\nu = \nu_1(y|z)$ for some y, z in X and ν_1 in N. So

$$T \otimes I_X(\xi \otimes x) = \nu_1 \otimes (y|z)\langle \mu, \xi\rangle x = \nu_1 \otimes y\langle z, \langle \mu, \xi\rangle x\rangle$$

$$= (\nu_1 \otimes y)\langle \mu \otimes z, \xi \otimes x\rangle = \Omega_{\nu_1 \otimes y}\Omega^*_{\mu \otimes z}(\xi \otimes x).$$

This concludes the proof. □

One of the main uses we shall have for this result is recorded in:

4.3. Corollary. *If X is left-full and if $T \in \mathcal{L}_A(M,N)$ is an A-Fredholm operator, then $T \otimes I_X$ is B-Fredholm.*

Proof. If $S \in \mathcal{L}_A(N,N)$ is such that $I_M - ST$ is in $\mathcal{K}_A(M)$ and $I_N - TS$ is in $\mathcal{K}_A(N)$, then

$$I_{M \otimes_A X} - (S \otimes I_X)(T \otimes I_X) = (I_M - ST) \otimes I_X$$

which is a B-compact operator by 4.2. The same reasoning applies to $I_{M \otimes_A X} - (T \otimes I_X)(S \otimes I_X)$. □

At this point, the reader may have already anticipated our strategy of using a bimodule to create a homomorphism on K_0-groups: given an element α in $K_0(A)$, we may find, using 3.14, an A-Fredholm operator T whose index is α. The index, in $K_0(B)$, of the B-Fredholm operator $T \otimes I_X$ is the image of α under the homomorphism we have in mind. In order to make this picture work, we need to tackle the question of well definedness, which we now do.

4.4. Proposition. *If $T_1, T_2 \in \mathcal{L}_A(M,N)$ are A-Fredholm operators such that* $\operatorname{ind}(T_1) = \operatorname{ind}(T_2)$, *then* $\operatorname{ind}(T_1 \otimes I_X) = \operatorname{ind}(T_2 \otimes I_X)$.

Proof. According to 3.16 there is an integer n such that

$$T_1 \oplus T_2^* \oplus I_{A^n} \ : \ M_1 \oplus N_2 \oplus A^n \to N_1 \oplus M_2 \oplus A^n$$

is an A-compact perturbation of an invertible operator. By 4.2 and by the fact that the tensor product distributes over direct sums, we have that $(T_1 \otimes I_X) \oplus (T_2^* \otimes I_X) \oplus (I_{A^n} \otimes I_X)$ is a B-compact perturbation of an invertible operator. By 3.11 its index is therefore zero. On the other hand, also by 3.11 we have

$$\operatorname{ind}(T_1 \otimes I_X) + \operatorname{ind}(T_2^* \otimes I_X) + \operatorname{ind}(I_{A^n} \otimes I_X) = 0$$

which says that $\operatorname{ind}(T_1 \otimes I_X) = \operatorname{ind}(T_2 \otimes I_X)$. □

A important ingredient for the functoriality properties of left-full Hilbert bimodules is the notion of tensor product of bimodules. In order to avoid endless calculations that arise in an abstract treatment of tensor products, we shall provide an alternative picture for bimodules, as concrete operators between Hilbert spaces, in which the coefficient algebras are represented. The notion of representation of bimodules is described next. Compare [**4**], Definition 2.1.

4.5. Definition. *A representation of the Hilbert A-B-bimodule X consists of the following data:*

(i) *a representation π_A of A on a Hilbert space H_A,*

(ii) *a representation π_B of B on a Hilbert space H_B and*

(iii) *a bounded linear transformation π_X from X into the Banach space $\mathcal{B}(H_B, H_A)$ of all bounded linear operators from H_B to H_A.*

Furthermore it is required that, for $a \in A$, $b \in B$, and $x, x_1, x_2 \in X$,

(a) $\pi_X(ax) = \pi_A(a)\pi_X(x)$

(b) $\pi_X(xb) = \pi_X(x)\pi_B(b)$

(c) $\pi_A\big((x_1|x_2)\big) = \pi_X(x_1)\pi_X(x_2)^*$

(d) $\pi_B\big(\langle x_1, x_2\rangle\big) = \pi_X(x_1)^*\pi_X(x_2)$.

At this point it is necessary to remark that for x in X, one has that $\|(x|x)\| = \|\langle x, x\rangle\|$ (see [**4**], Remark 1.9). So, when we speak of $\|x\|$, we mean the square root of that common value. In particular, this is the norm we have in mind when we require, in (iii), that π_X be a bounded map on X.

4.6. Proposition. *Let (π_A, π_B, π_X) be any representation of X. If either π_A or π_B are faithful, then π_X is isometric.*

Proof. Let $x \in X$. We have $\|\pi_X(x)\|^2 = \|\pi_X(x)\pi_X(x)^*\| = \|\pi_A\big((x|x)\big)\|$. So, assuming that π_A is faithful, we have $\|\pi_X(x)\|^2 = \|(x|x)\| = \|x\|^2$. A similar argument applies if π_B is assumed to be faithful, instead. □

Given a representation π_B of B, it is natural to ask whether or not π_B is part of the data forming a representation of X. To answer this question we need to bring in the conjugate module and the linking algebra. The conjugate module of X is the bimodule one obtains by reversing its structure so as to produce a Hilbert B-A-bimodule as explained in [**13**], 6.17, or [**4**], 1.4. We shall denote the conjugate module by X^* (although $\tilde{X}$ is used in [**13**]). The linking algebra of X, introduced in [**3**], 1.1 in the special case when X is both left and right-full, and in [**4**], 2.2 in general, is the C^*-algebra

$$L = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

equipped with the multiplication

$$\begin{pmatrix} a_1 & x_1 \\ y_1^* & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2^* & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + (x_1|y_2) & a_1x_2 + x_1b_2 \\ y_1^*a_2 + b_1y_2^* & \langle y_1, x_2\rangle + b_1b_2 \end{pmatrix}$$

and involution

$$\begin{pmatrix} a & x \\ y^* & b \end{pmatrix}^* = \begin{pmatrix} a^* & y \\ x^* & b^* \end{pmatrix}$$

for $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $x, x_1, x_2, y, y_1, y_2 \in X$. Here, x^* denotes the element x of X when it is viewed in X^*.

4.7. Proposition. *Let π_B be a non-degenerate representation of B on the Hilbert space H_B. Then there exists a Hilbert space H_A, a non-degenerate representation π_A of A on H_A and a bounded linear map $\pi_X\colon X \to \mathcal{B}(H_B, H_A)$ which, when put together, form a representation of X.*

Proof. Let L be the linking algebra of X. Thus π_B becomes a representation of a subalgebra of L, namely B. Let π be a representation of L on a Hilbert space H which contains a copy of H_B, such that H_B is invariant under the operators $\pi(b)$, for b in B, and, finally, such that $\pi(b)|_{H_B} = \pi_B(b)$. The existence of π, in the case that π_B is cyclic, follows from the Theorem on extension of states and the GNS construction. In the general case, it follows from the fact that any representation is a direct sum of cyclic representations.

Viewing X as the subset of L formed by the matrices with the only non-zero entry lying in the upper right hand corner, let $H_A = \overline{\pi(X)H_B}$. Here, and in the sequel, products of sets, as in "$\pi(X)H_B$", will always mean the linear span of the set of individual products.

Since X is a left A-module, it is clear that H_A is invariant under $\pi(A)$. Denote by π_A the sub-representation of A on H_A given in this way. Observe that $\overline{AX} = X$, by [**6**], 1.1.4, so we conclude that π_A is non-degenerate.

For each x in X we have, by definition, that $\pi(x)H_B \subseteq H_A$. By totally different reasons we also have that $\pi_X(x)^* H_A \subseteq H_B$. In fact, let $\xi \in H_A$. Without loss of generality we may assume that $\xi = \pi(y)\eta$ where y is in X and $\eta \in H_B$. We then have

$$\pi(x)^*\xi = \pi(x)^*\pi(y)\eta = \pi\big(\langle x, y\rangle\big)\eta \in H_B.$$

For each x in X let $\pi_X(x)$ denote the element of $\mathcal{B}(H_B, H_A)$ obtained by restriction of $\pi(x)$. The properties of Definition 4.5 may now be easily checked. If the reader does decide to do so, we suggest the formal definition $\pi_X(x) := i_A^*\pi(x)i_B$ where i_A and i_B are the inclusion operators from H_A and H_B into H. This has the advantage of taking care of the subtle issue of reducing the size of the co-domain of an operator. $\square$

4.8. Proposition. *There exists a representation (π_A, π_B, π_X) such that both π_A and π_B are faithful (and hence π_X is isometric by 4.6).*

Proof. Let π be a faithful representation of the linking algebra L on the Hilbert space H. Define $H_A = \overline{\pi(A)H}$ and $H_B = \overline{\pi(B)H}$ and let π_A and π_B be the corresponding sub-representations of A and B on H_A and H_B, respectively. Since $\overline{AX} = X$ and $\overline{BX^*} = X^*$ it is clear that $\pi(X)H \subseteq H_A$ and that $\pi(X)^*H \subseteq H_B$. If, for each x in X, we denote by $\pi_X(x)$ the element of $\mathcal{B}(H_B, H_A)$ given by restriction of $\pi(x)$, the proof can be completed as in 4.7. $\square$

4.9. Lemma. *The set $\big\{\sum_{i=1}^n (x_i|x_i) \colon n \in \mathbf{N},\ x_i \in X\big\}$ is dense in the positive cone of $(X|X)$.*

Proof. For $a = \sum_{i=1}^n (x_i|y_i)$ we have

$$a^*a = \sum_{i,j} (y_i|x_i)(x_j|y_j) = \sum_{i,j} \big((y_i|x_i)x_j|y_j\big) = \sum_{i,j} \big(y_i\langle x_i, x_j\rangle|y_j\big).$$

The matrix $\big(\langle x_i, x_j\rangle\big)_{i,j} \in M_n(B)$ is a positive matrix as observed in [**6**], 1.2.4. So, there is $\big(b_{ij}\big)_{i,j}$ in $M_n(B)$ such that

$$\langle x_i, x_j\rangle = \sum_{k=1}^n b_{ik}b_{jk}^*.$$

We then have

$$a^*a = \sum_{i,j,k} (y_i b_{ik}|y_j b_{jk}).$$

If we now define $z_k = \sum_{i=1}^n y_i b_{ik}$ we have

$$a^*a = \sum_{k=1}^n (z_k|z_k).$$

Since the set of elements a^*a, with a as above, is clearly dense in the positive cone of $(X|X)$, the proof is complete. $\square$

4.10. Proposition. *Let (π_A, π_B, π_X) be a representation of X. If π_B is faithful then π_A is faithful on $(X|X)$.*

Proof. Using 4.9, it is enough to show that $\|\pi_A\big(\sum_{i=1}^n (x_i|x_i)\big)\| = \|\sum_{i=1}^n (x_i|x_i)\|$. Thus, let $h = \sum_{i=1}^n (x_i|x_i)$ and observe that

$$\|\pi_A(h)\| = \|\sum_{i=1}^n \pi_X(x_i)\pi_X(x_i)^*\| = \left\| \big(\pi_X(x_1), \ldots, \pi_X(x_n)\big) \begin{pmatrix} \pi_X(x_1)^* \\ \vdots \\ \pi_X(x_n)^* \end{pmatrix} \right\|$$

where, in the last term above, we mean the product of a row matrix by a column matrix. Since the identity $\|T^*T\| = \|TT^*\|$ holds for general operators, the above equals

$$\left\| \begin{pmatrix} \pi_X(x_1)^* \\ \vdots \\ \pi_X(x_n)^* \end{pmatrix} \big(\pi_X(x_1), \ldots, \pi_X(x_n)\big) \right\| = \|\big(\pi_X(x_i)^*\pi_X(x_j)\big)_{i,j}\|$$

$$= \|\big(\pi_B(\langle x_i, x_j\rangle)\big)_{i,j}\| = \|\big(\langle x_i, x_j\rangle\big)_{i,j}\|.$$

The exact same computations done so far can obviously be repeated, in reverse order, for a representation (ρ_A, ρ_B, ρ_X) in which all components are isometric, as for example, the representation provided by 4.8. This shows that the last term above equals $\|\rho_A(h)\| = \|h\|$. So, $\|\pi_A(h)\| = \|h\|$. ☐

From this point on, and until the end of this section, we shall consider fixed another C^*-algebra, denoted C, and a Hilbert B-C-bimodule Y. Our goal is to make sense of $X \otimes_B Y$ as a Hilbert A-C-bimodule. So, for the time being, let us denote by $X \otimes_B Y$, the algebraic tensor product of X and Y over B which provides us with an A-C-bimodule.

4.11. Definition. *Let $(\cdot|\cdot)$ and $\langle \cdot, \cdot\rangle$ be the sesqui-linear forms on $X \otimes_B Y$ (the first one being linear in the first variable and vice-versa) specified by*

$$(x_1 \otimes y_1 | x_2 \otimes y_2) = (x_1(y_1|y_2)|x_2)$$

and

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle y_1, \langle x_1, x_2\rangle y_2\rangle.$$

The only steps that are not entirely trivial in checking that this can be made into a Hilbert B-C-bimodule are that

(a) both sesqui-linear forms above are positive and

(b) $\|(z|z)\| = \|\langle z, z\rangle\|$ for all z in $X \otimes_B Y$.

In order to check these, fix a faithful non-degenerate representation π_B of B in some Hilbert space H_B. Using 4.7 we may find a representation π_A on a space H_A as well as a representation π_X of X, by bounded operators from H_B to H_A, satisfying the axioms described in Definition 4.5.

Using the symmetric version of 4.7 we can also find a Hilbert space H_C as well as π_C and π_Y satisfying the conditions of 4.5. Note that, by 4.6, both π_X and π_Y are isometric. By 4.10 it follows that π_A is faithful on $(X|X)$ and that π_C is faithful on $\langle Y, Y\rangle$. Consider the map

$$\rho\colon X \otimes_B Y \to \mathcal{B}(H_C, H_A)$$

given by $\rho(x \otimes y) = \pi_X(x)\pi_Y(y)$ (meaning composition of operators). Observe that, for x_1, x_2 in X and y_1, y_2 in Y, we have

$$\pi_A\big((x_1 \otimes y_1|x_2 \otimes y_2)\big) = \pi_X(x_1(y_1|y_2))\pi_X(x_2)^*$$
$$= \pi_X(x_1)\pi_Y(y_1)\pi_Y(y_2)^*\pi_X(x_2)^* = \rho(x_1 \otimes y_1)\rho(x_2 \otimes y_2)^*.$$

Similarly we have $\pi_C\big(\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle\big) = \rho(x_1 \otimes y_1)^*\rho(x_2 \otimes y_2)$. If $z = \sum_{i=1}^n x_i \otimes y_i$ is an arbitrary element of $X \otimes_B Y$ we then have that

$$\pi_A\big((z|z)\big) = \Big(\sum \rho(x_i \otimes y_i)\Big)\Big(\sum \rho(x_i \otimes y_i)\Big)^*$$

which is clearly a positive element in $\mathcal{B}(H_A)$. Therefore, since π_A is faithful on $(X|X)$, we conclude that $(z|z)$ is positive. Similarly it can be shown that $\langle z, z\rangle$ is positive as well. This proves (a) above. With respect to (c) we have

$$\|(z|z)\| = \|\pi_A\big((z|z)\big)\| = \Big\|\Big(\sum \rho(x_i \otimes y_i)\Big)\Big(\sum \rho(x_i \otimes y_i)\Big)^*\Big\|$$
$$= \Big\|\Big(\sum \rho(x_i \otimes y_i)\Big)^*\Big(\sum \rho(x_i \otimes y_i)\Big)\Big\| = \|\pi_C\big(\langle z, z\rangle\big)\| = \|\langle z, z\rangle\|.$$

This said, we may define unambiguously, a semi-norm in $X \otimes_B Y$ by $\|z\| = \|(z|z)\| = \|\langle z, z\rangle\|$. After moding out by the elements of norm zero and completing, we are left with a Hilbert A-C-bimodule, which we denote, for simplicity, by $X \otimes_B Y$, as well. Observe that the procedure of moding out the null elements and completing is equivalent to considering the closure of $\rho(X \otimes_B Y)$ in $\mathcal{B}(H_C, H_A)$, which, incidentally is the same as $\overline{\pi_X(X)\pi_Y(Y)}$.

In light of 4.5, it is easy to see that the A-C-bimodule structure is reproduced as the usual composition of operators. Furthermore, the A-valued inner product becomes $(T|S) = TS^*$ while $\langle T, S\rangle = T^*S$, as long as the appropriate identifications are made.

4.12. Proposition. *If both X and Y are left-full then so is $X \otimes_B Y$.*

Proof. By [**6**], 1.1.4, it follows that $\overline{XB} = X$. But, since we are assuming that $(Y|Y) = B$ we get $\overline{X(Y|Y)} = X$. It follows that

$$(X \otimes_B Y|X \otimes_B Y) = \overline{(X(Y|Y)|X)} = (X|X) = A. \qquad \square$$

Recall that X^* denotes the conjugate module of X. Clearly X^* is left-full (resp. right-full) if and only if X is right-full (resp. left-full).

4.13 Proposition. *The tensor product Hilbert A-A bimodule $X \otimes_B X^*$ is isomorphic to $(A|A)$ (once $(A|A)$ is given its obvious structure of Hilbert A-A-bimodule, as any ideal of A).*

Proof. Choosing a faithful representation as in 4.8 we may assume that $A \subseteq \mathcal{B}(H_A)$, $B \subseteq \mathcal{B}(H_B)$ and $X \subseteq \mathcal{B}(H_B, H_A)$. In addition the bimodule structure is composition of operators and the inner products are given by $(x|y) = xy^*$ and $\langle x, y\rangle = x^*y$. The tensor product $X \otimes_B X^*$ is moreover identified with $\overline{XX^*}$ (where the last occurrence of "$*$" should be interpreted as the usual operator involution). This said, $X \otimes_B X^* = (A|A)$. ☐

5. K-Theory and Hilbert Bimodules

The stage is now set for the presentation of the main section of this work. Given C^*-algebras A and B as well as a left-full Hilbert A-B-bimodule X, we want to define a group homomorphism $X_*: K_0(A) \to K_0(B)$ which will be proven to be an isomorphism if X is right-full as well.

5.1. Definition. *Let X be a left-full Hilbert A-B-bimodule. If α is an element of the group $K_0(A)$, which we identify with $F(A)$ under the isomorphism of 3.17, let T be a Fredholm operator representing α in the sense that* $\operatorname{ind}(T) = \alpha$. *We denote by $X_*(\alpha)$ the element of $K_0(B)$ defined by*

$$X_*(\alpha) = \operatorname{ind}(T \otimes I_X).$$

Clearly, by 4.4, X_* is well defined and it is not hard to see that it is a group homomorphism.

5.2. Proposition. *Let X be a left-full Hilbert A-B-bimodule and Y be a left-full Hilbert B-C-bimodule, then $Y_* \circ X_* = (X \otimes_B Y)_*$.*

Proof. This follows from the easy fact that tensor products are associative, even if we drag along all the extra structure of Hilbert bimodules. ☐

Recall that the C^*-algebras A and B are said to be strongly Morita equivalent if there exists a Hilbert A-B-bimodule which is simultaneously left and right-full. Such a module is called an imprimitivity bimodule. Given an imprimitivity bimodule X, we may therefore consider the homomorphisms $X_*: K_0(A) \to K_0(B)$ and $(X^*)_*: K_0(B) \to K_0(A)$, which compose to the identity in either order by 5.2 and 4.13. The immediate outcome of these facts is our main result.

5.3. Theorem. *If A and B are strongly Morita equivalent and X is an imprimitivity bimodule, then X_* is an isomorphism from $K_0(A)$ onto $K_0(B)$.*

In order to treat K_1-groups we shall use the usual argument of taking suspensions. The formalism of Hilbert bimodules developed in section 4 makes is very easy to discuss Hilbert bimodules over tensor product C^*-algebras. So, before we embark on a study of K_1, let us briefly deal with "external tensor products".

Let A_i and B_i be C^*-algebras and X_i be Hilbert A_i-B_i-bimodules for $i = 1, 2$. Under faithful representations we may assume that $A_i \subseteq \mathcal{B}(H_{A_i})$, $B \subseteq \mathcal{B}(H_{B_i})$ and $X_i \subseteq \mathcal{B}(H_{B_i}, H_{A_i})$.

Denote by $X_1 \otimes X_2$ the closed linear span of the set $\{x_1 \otimes x_2 : x_1 \in X_1, x_2 \in X_2\}$ of operators in $\mathcal{B}(H_{B_1} \otimes H_{B_2}, H_{A_1} \otimes H_{A_2})$. It is not hard to see that $X_1 \otimes X_2$ is a Hilbert $(A_1 \otimes A_2)$-$(B_1 \otimes B_2)$-bimodule. Here $A_1 \otimes A_2$ and $B_1 \otimes B_2$ mean the spatial tensor products. If $x_i, y_i \in X_i$ for $i = 1, 2$, observe that

$$(x_1 \otimes y_1 | x_2 \otimes y_2) = (x_1 \otimes y_1)(x_2 \otimes y_2)^* = (x_1 x_2^*) \otimes (y_1 y_2^*) = (x_1 | x_2) \otimes (y_1 | y_2)$$

so we see that the concrete inner-products on $X_1 \otimes X_2$ may be defined abstractly, at least on the algebraic tensor product of X_1 by X_2, without mentioning the representations. In fact it is easy to see that the definition of $X_1 \otimes X_2$ given above does not depend (up to the obvious notion of isomorphism) on the particular faithful representation chosen.

The object so defined is called the "external" tensor product of X_1 and X_2 (compare [**6**], 1.2.4). It is readily apparent that if each X_i is left-full (resp. right-full) then so is $X_1 \otimes X_2$. As an obvious consequence we have:

5.4. Proposition. *If A_i and B_i are C^*-algebras and if A_i is strongly Morita equivalent to B_i under the imprimitivity bimodule X_i, for $i = 1, 2$, then $A_1 \otimes B_1$ is strongly Morita equivalent to $A_2 \otimes B_2$under the imprimitivity bimodule $X_1 \otimes X_2$.*

Let $C_0(\mathbf{R})$ denote the C^*-algebra of all continuous complex valued functions on the real line. If A and B are strongly Morita equivalent under the imprimitivity bimodule X, it follows that the suspension of A, namely $SA = C_0(\mathbf{R}) \otimes A$ and and SB are strongly Morita equivalent to each other under the imprimitivity bimodule $C_0(\mathbf{R}) \otimes X$. Using the standard isomorphism between $K_1(A)$ and $K_0(SA)$, we have:

5.5. Theorem. *If A and B are strongly Morita equivalent and X is an imprimitivity bimodule, then $(C_0(\mathbf{R}) \otimes X)_*$ is an isomorphism from $K_1(A)$ onto $K_1(B)$.*

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