PARTIAL CROSSED PRODUCT DESCRIPTION OF THE
C*-ALGEBRAS ASSOCIATED WITH INTEGRAL DOMAINS

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ABSTRACT. Recently, Cuntz and Li introduced the C*-algebra $\mathcal{A}[R]$ associated to an
integral domain $R$ with finite quotients. In this paper, we show that $\mathcal{A}[R]$ is a partial
group algebra of the group $K \ltimes K^\times$ with suitable relations, where $K$ is the field of
fractions of $R$. We identify the spectrum of this relations and we show that it is
homeomorphic to the profinite completion of $R$. By using partial crossed product
theory, we reconstruct some results proved by Cuntz and Li. Among them, we prove
that $\mathcal{A}[R]$ is simple by showing that the action is topologically free and minimal.

1. Introduction

Fifteen years ago, motivated by the work of Julia [14], Bost and Connes constructed a
C*-dynamical system having the Riemann $\zeta$-function as partition function [2]. The C*-algebra
of the Bost-Connes system, denoted by $C_\mathbb{Q}$, is a Hecke C*-algebra obtained from
the inclusion of the integers into the rational numbers. In [19], Laca and Raeburn showed
that $C_\mathbb{Q}$ can be realized as a semigroup crossed product and, in [20], they characterized
the primitive ideal space of $C_\mathbb{Q}$.

In [1], [4] and [15], by observing that the construction of $C_\mathbb{Q}$ is based on the inclusion of
the integers into the rational numbers, Arledge, Cohen, Laca and Raeburn generalized
the construction of Bost and Connes. They replaced the field $\mathbb{Q}$ by an algebraic number
field $\mathbb{K}$ and $\mathbb{Z}$ by the ring of integers of $\mathbb{K}$. Many of the results obtained for $C_\mathbb{Q}$ were
generalized to arbitrary algebraic number fields (at least when the ideal class group of
the field is $h = 1$) [16], [17].

Recently, a new construction appeared. In [5], Cuntz defined two new C*-algebras:
$\mathcal{Q}_\mathbb{N}$ and $\mathcal{Q}_\mathbb{Z}$. Both algebras are simple and purely infinite and $\mathcal{Q}_\mathbb{N}$ can be seen as a
C*-subalgebra of $\mathcal{Q}_\mathbb{Z}$. These algebras encode the additive and multiplicative structure
of the semiring $\mathbb{N}$ and of the ring $\mathbb{Z}$. Cuntz showed that the algebra $\mathcal{Q}_\mathbb{N}$ is, essentially,
the algebra generated by $C_\mathbb{Q}$ and one unitary operator. In [25], Yamashita realized $\mathcal{Q}_\mathbb{N}$
as the C*-algebra of a topological higher-rank graph.

The next step was given by Cuntz and Li. In [6], they generalized the construction of $\mathcal{Q}_\mathbb{Z}$ by replacing $\mathbb{Z}$ by a unital commutative ring $R$ (which is an integral domain
with finite quotients by principal ideals). This algebra was called $\mathcal{A}[R]$. Cuntz and Li showed that $\mathcal{A}[R]$ is simple and purely infinite (when $R$ is not a field) and they related
a C*-subalgebra of its with the generalized Bost-Connes systems (when $R$ is the ring
of integers in an algebraic number field having $h = 1$ and, at most, one real place). In
[23], Li extended the construction of $\mathcal{A}[R]$ to an arbitrary unital ring.

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The aim of this text is to show that the algebra $\mathfrak{A}[R]$ can be seen as a partial crossed product (when $R$ is an integral domain with finite quotients). We show that $\mathfrak{A}[R]$ is isomorphic to a partial group algebra of the group $K \rtimes K^\times$ with suitable relations, where $K$ is the field of fractions of $R$. By using the relationship between partial group algebras and partial crossed products, we see that $\mathfrak{A}[R]$ is a partial crossed product by the group $K \rtimes K^\times$. We characterize the spectrum of the commutative algebra arising in the crossed product and show that this spectrum is homeomorphic to $\hat{R}$ (the profinite completion of $R$). Furthermore, we show that the partial action is topologically free and minimal. By using that the group $K \rtimes K^\times$ is amenable, we conclude that $\mathfrak{A}[R]$ is simple.

Recently, some similar results appeared. In [21] and [3], Brownlowe, an Huef, Laca and Raeburn showed that $Q_N$ is a partial crossed product by using a boundary quotient of the Toeplitz (or Wiener-Hopf) algebra of the quasi-lattice ordered group $(Q \rtimes Q^\times, N \rtimes N^\times)$ (see [24] and [18] for Toeplitz algebras of quasi-lattice ordered groups). We observe that our techniques are different from theirs. We don’t use Nica’s construction [24] (indeed, our group $K \rtimes K^\times$ is not a quasi-lattice, in general). From our results, in the case $R = \mathbb{Z}$, we see that $Q_\mathbb{Z}$ is a partial crossed product by $Q \rtimes Q^\times$. From this, it is immediate that $Q_N$ is a partial crossed product by $Q \rtimes Q^\times$ (as in [3]).

Before we go to the main result we give, in the section 2, a breifly review about the algebra $\mathfrak{A}[R]$ and the theories of partial crossed products and partial group algebras. In the section 3, we state our main theorem: the algebra $\mathfrak{A}[R]$ is isomorphic to a partial group algebra. In the section 4, we study $\mathfrak{A}[R]$ by using the techniques of partial crossed products. We recover the faithful conditional expectation constructed by Cuntz and Li in [6, Proposition 1] in a very natural way. Furthermore, we use the concepts of topological freeness and minimality of a partial action to show that $\mathfrak{A}[R]$ is simple.

2. Preliminaries

2.1. The $C^*$-algebra $\mathfrak{A}[R]$ of an Integral Domain. Throughout this text, $R$ will be an integral domain (unital commutative ring without zero divisors) with the property that the quotient $R/(m)$ is finite, for all $m \neq 0$ in $R$. We denote by $R^\times$ the set $R\setminus\{0\}$ and by $R^*$ the set of units in $R$.

**Definition 2.1.** [6, Definition 1] The regular $C^*$-algebra of $R$, denoted by $\mathfrak{A}[R]$, is the universal $C^*$-algebra generated by isometries $\{s_m \mid m \in R^\times\}$ and unitaries $\{u^n \mid n \in R\}$ subject to the relations

(CL1) $s_ms_{m'} = s_{mm'}$;
(FL2) $u^nu^{n'} = u^{n+n'}$;
(FL3) $s_mu^n = u^{mn}s_m$;
(FL4) $\sum_{l+(m) \in R/(m)} u^ls_ms_{m}^*u^{-l} = 1$;

for all $m, m' \in R^\times$ and $n, n' \in R$.

We denote by $e_m$ the range projection of $s_m$, namely $e_m = s_ms_m^*$. It is easily seen that, under (FL2) and (FL3), $u^le_mu^{-l} = u^{l'}e_mu^{-l'}$ if $l+(m) = l' + (m)$. From this, we see that the sum in (FL4) is independent of the choice of $l$. 

Let \( \{ \xi_r \mid r \in R \} \) be the canonical basis of the Hilbert space \( \ell^2(R) \) and consider the operators \( S_m \) and \( U^n \) on \( \ell^2(R) \) given by \( S_m(\xi_r) = \xi_{mr} \) and \( U^n(\xi_r) = \xi^{n+r} \).

**Definition 2.2.** [6, Section 2] The reduced regular \( C^* \)-algebra of \( R \), denoted by \( \mathfrak{A}_r[R] \), is the \( C^* \)-subalgebra of \( \mathcal{B}(\ell^2(R)) \) generated by the operators \( \{ S_m \mid m \in R^\times \} \) and \( \{ U^n \mid n \in R \} \).

One can checks that \( S_m \) is an isometry, \( U^n \) is a unitary and satisfy (CL1)-(CL4). Hence, there exists a surjective \(*\)-homomorphism \( \mathfrak{A}[R] \twoheadrightarrow \mathfrak{A}_r[R] \).

In [6], Cuntz and Li showed that, when \( R \) is not a field, \( \mathfrak{A}[R] \) is simple; therefore the above \(*\)-homomorphism is a \(*\)-isomorphism. In the section 4, we will show that \( \mathfrak{A}[R] \) is simple (when \( R \) is not a field) by using the partial crossed product description of \( \mathfrak{A}[R] \).

For future references, we need the following lemma, proved by Cuntz and Li:

**Lemma 2.3.** [6, Lemma 1] For all \( n, n' \in R \) and \( m, m' \in R^\times \), the projections (in \( \mathfrak{A}[R] \)) \( u^ne_mu^{-n} \) and \( u^{n'}e_mu^{-n'} \) commute.

More details about these algebras can be found in [5], [6], [7], [8], [22], [23] and [25].

### 2.2. Partial Crossed Products

Here, we review some basic facts about partial actions and partial crossed products.

**Definition 2.4.** [9, Definition 1.1] A partial action \( \alpha \) of a (discrete) group \( G \) on a \( C^* \)-algebra \( \mathcal{A} \) is a collection \( \{ D_g \}_{g \in G} \) of ideals of \( \mathcal{A} \) and \(*\)-isomorphisms \( \alpha_g : D_{g^{-1}} \longrightarrow D_g \) such that

1. \( (PA1) \, D_e = \mathcal{A} \), where \( e \) represents the identity element of \( G \);
2. \( (PA2) \, \alpha_{h^{-1}}(D_h \cap D_{g^{-1}}) \subseteq D_{(gh)^{-1}} \);
3. \( (PA3) \, \alpha_g \circ \alpha_h(x) = \alpha_{gh}(x), \quad \forall x \in \alpha_{h^{-1}}(D_h \cap D_{g^{-1}}) \).

In the above definition, if we replace the \( C^* \)-algebra \( \mathcal{A} \) by a locally compact space \( X \), the ideals \( D_g \) by open sets \( X_g \) and the \(*\)-isomorphisms \( \alpha_g \) by homeomorphisms \( \theta_g : X_{g^{-1}} \longrightarrow X_g \), we obtain a partial action \( \theta \) of the group \( G \) on the space \( X \). A partial action \( \theta \) on a space \( X \) induces naturally a partial action \( \alpha \) on the \( C^* \)-algebra \( C_0(X) \). The ideals \( D_g \) are \( C_0(X_g) \) and \( \alpha_g(f) = f \circ \theta_{g^{-1}} \).

We say that a partial action \( \theta \) on a space \( X \) is **topologically free** if, for all \( g \in G \setminus \{ e \} \), the set \( F_g = \{ x \in X_{g^{-1}} \mid \theta_g(x) = x \} \) has empty interior. A subset \( V \) of \( X \) is **invariant** under the partial action \( \theta \) if \( \theta_g(V \cap X_{g^{-1}}) \subseteq V \), for every \( g \in G \). The partial action \( \theta \) is **minimal** if there are no invariant open subsets of \( X \) other than \( \emptyset \) and \( X \). It is easy to see that \( \theta \) is minimal if, and only if, every \( x \in X \) has dense orbit, namely \( O_x = \{ \theta_g(x) \mid g \in G \text{ for which } x \in X_{g^{-1}} \} \) is dense in \( X \).

**Definition 2.5.** [9, Definition 6.1] A partial representation \( \pi \) of a (discrete) group \( G \) into a unital \( C^* \)-algebra \( \mathcal{B} \) is a map \( \pi : G \longrightarrow \mathcal{B} \) such that, for all \( g, h \in G \),

1. \( (PR1) \, \pi(e) = 1; \)
2. \( (PR2) \, \pi(g^{-1}) = \pi(g)^*; \)
3. \( (PR3) \, \pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}). \)

From a partial action \( \alpha \), we can construct two partial crossed products: \( \mathcal{A} \rtimes_{\alpha} G \) (full) and \( \mathcal{A} \rtimes_{\alpha,r} G \) (reduced). We can define both as follows: let \( \mathcal{L} \) be the normed \(*\)-algebra of the finite formal sums \( \sum_{g \in G} a_g \delta_g \), where \( a_g \in D_g \). The operations and the
norm in $\mathcal{L}$ are given by $(a\delta_g)(a\delta_h) = \alpha_g(\alpha_{g^{-1}}a_h)\delta_{gh}$, $(a\delta_g)^* = \alpha_{g^{-1}}(a^*_g)\delta_{g^{-1}}$ and $||\sum_{g\in G}a\delta_g|| = \sum_{g\in G}||a||$. If we denote by $B_g$ the vector subspace $\mathcal{D}_g\delta_g$ of $\mathcal{L}$, then the family $(B_g)_{g\in G}$ generates a Fell bundle. The full and the reduced crossed products are, respectively, the full and the reduced cross sectional algebra of $(B_g)_{g\in G}$. It is well known that $\mathcal{A} \rtimes_{\alpha} G$ is universal with respect to a covariant pair $(\varphi, \pi)$, where $\varphi : \mathcal{A} \to \mathcal{B}$ is a $\ast$-homomorphism ($\mathcal{B}$ is a unital $C^*$-algebra), $\pi : G \to \mathcal{B}$ is a partial representation of $G$ and the covariant equations are $\varphi(\alpha_g(x)) = \pi(g)\varphi(x)\pi(g^{-1})$ for $x \in \mathcal{D}_{g^{-1}}$ and $\varphi(x)\pi(g)\pi(g^{-1}) = \pi(g)\varphi(x)\pi(g^{-1})\varphi(x)$ for $x \in \mathcal{A}$.

There exists a faithful conditional expectation $E : \mathcal{A} \rtimes_{\alpha, r} G \to \mathcal{A}$ given by $E(a\delta_g) = a$ if $g = e$, and $E(a\delta_g) = 0$ if $g \neq e$. When the Fell bundle $(B_g)_{g\in G}$ is amenable ($G$ amenable implies its), the full and reduced constructions are isomorphic and, in this case, there exists a faithful conditional expectation of $\mathcal{A} \rtimes_{\alpha} G$ onto $\mathcal{A}$.

There is a close relation between topological freeness and minimality of the partial action and ideals of the reduced crossed product. If $\theta$ is a topologically free partial action on a space $X$ then $\theta$ is minimal if, and only if, $C_0(X) \rtimes_{\alpha, r} G$ is simple, where $\alpha$ is the action induced by $\theta$. Under the amenability hypothesis, this result is valid for the full crossed product too.

For more details about partial crossed products, see [9], [10], [11], [12] and [13].

2.3. **Partial Group Algebras.** Let $G$ be a discrete group, let $\mathcal{G}$ be the set $G$ without the group operations and denote the elements in $\mathcal{G}$ by $[g]$ (namely, $\mathcal{G} = \{[g] \mid g \in G\}$). The **partial group algebra of $G$**, denoted by $C^*_p(G)$, is defined to be the universal $C^*$-algebra generated by the set $\mathcal{G}$ with the relations

$$\mathcal{R}_p = \{[e] = 1\} \cup \{[g^{-1}] = [g]^*\}_{g \in G} \cup \{[g][h] = [gh]\}_{g, h \in G}.$$ 

The algebra $C^*_p(G)$ is universal with respect to a partial representation. Observe that the relations in $\mathcal{R}_p$ correspond to the partial representation axioms (PR1), (PR2) and (PR3). Sometimes, we will refer to a relation in $\mathcal{R}_p$ by indicating the corresponding axiom.

Consider the natural bijection between $\mathcal{P}(G)$ and $\{0, 1\}^G$, where $\mathcal{P}(G)$ is the power set of $G$. With the product topology, $\{0, 1\}^G$ is a compact Hausdorff space. Give to $\mathcal{P}(G)$ the topology of $\{0, 1\}^G$. Denote by $X_G$ the subset of $\mathcal{P}(G)$ of the subsets $\xi$ of $G$ such that $e \in \xi$. Clearly, with the induced topology of $\mathcal{P}(G)$, $X_G$ is a compact space. For each $g \in G$, let $X_g = \{\xi \in X_G \mid g \in \xi\}$. It is easy to see that $\theta_g : X_{g^{-1}} \to X_g$ given by $\theta_g(\xi) = g\xi$ is a homeomorphism. The collection of open sets $(X_g)_{g \in G}$ of $X_G$ with the homeomorphisms $\theta_g$ define a partial action $\theta$ of $G$ on $X_G$. The partial crossed product $C(X_G) \rtimes_{\alpha} G$ is isomorphic to $C^*_p(G)$ (where $\alpha$ is the partial action induced by $\theta$).

For each $g \in G$, we abbreviate $[g][g^{-1}]$ by $e_g$. Let $\mathcal{R}$ be a set of relations on $\mathcal{G}$ such that every relation is of the form

$$\sum_i \lambda_i \prod_j e_{g_{ij}} = 0.$$ 

The **partial group algebra of $G$ with relations $\mathcal{R}$**, denoted by $C^*_p(G, \mathcal{R})$, is defined to be the universal $C^*$-algebra generated by the set $\mathcal{G}$ with the relations $\mathcal{R}_p \cup \mathcal{R}$. Given a partial representation $\pi$ of $G$, we can extend $\pi$ naturally to sums of products of elements in $\mathcal{G}$. If this extension satisfies the relations $\mathcal{R}$, we say that $\pi$ is a **partial**
representation that satisfies $\mathcal{R}$. The algebra $C^*_p(G, \mathcal{R})$ is universal with respect to a partial representation that satisfies the relations $\mathcal{R}$.

Denote by $1_g$ the function in $C(X_G)$ given by $1_g(\xi) = 1$ if $g \in \xi$ and $1_g(\xi) = 0$ otherwise. By an abuse of notation, we also denote by $\mathcal{R}$ the subset of $C(X_G)$ given by the functions $\sum_i \lambda_i \prod_j 1_{g_{ij}}$, where $\sum_i \lambda_i \prod_j e_{g_{ij}} = 0$ is a relation in (the original) $\mathcal{R}$. The spectrum of the relations $\mathcal{R}$ is defined to be the compact Hausdorff space

$$\Omega_{\mathcal{R}} = \{ \xi \in X_G | f(g^{-1} \xi) = 0, \forall f \in \mathcal{R}, \forall g \in \xi \}.$$ 

Let $\Omega_g = \{ \xi \in \Omega_{\mathcal{R}} | g \in \xi \}$. By restricting the above $\theta_g$ to $\Omega_g^{-1}$, we obtain a partial action (again denoted by $\theta$) of $G$ on $\Omega_{\mathcal{R}}$ (the open sets are the $\Omega_g$’s and the homeomorphisms are the restrictions of the $\theta_g$’s). The main result concerning $C^*_p(G, \mathcal{R})$ says that this algebra is isomorphic to the partial crossed product $C(\Omega_{\mathcal{R}}) \rtimes_\alpha G$ (again, $\alpha$ is the partial action induced by $\theta$).

The above results are proved in [12] and [13].

3. Partial Group Algebra Description of $\mathfrak{A}[R]$

Let $R$ be an integral domain satisfying the conditions stated in the previous section. Denote by $K$ the field of fractions of $R$ and consider the semidirect product $K \rtimes K^\times$. The elements of $K \rtimes K^\times$ will be denoted by a pair $(u, w)$, where $u \in K$ and $w \in K^\times$. Recall that $(u, w)(u', w') = (u + u'w, uw')$ and $(u, w)^{-1} = (-u/w, 1/w)$. We denote by $[u, w]$ an element of set $K \rtimes K^\times$ without the group operations (as the set $G$ associated to $G$ in the previous section).\(^1\) Again, denote $[g][g^{-1}]$ by $e_g$. Consider the sets of relations

$$\mathcal{R}_1 = \{ e_{(n,1)} = 1 \mid n \in R \}, \quad \mathcal{R}_2 = \{ e_{(0,\frac{1}{m})} = 1 \mid m \in R^\times \},$$

$$\mathcal{R}_3 = \left\{ \sum_{n+(m) \in R/(m)} e_{(n,m)} = 1 \mid m \in R^\times \right\}$$

and $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$. We observe that, under the relations $\mathcal{R}_1$ and $\mathcal{R}_p$ (relations stated in the previous section), the sum in $\mathcal{R}_3$ does not depend on the choice of $n$. Indeed, for $k \in R$,

\[
e_{(n+km,m)} = [n + km, m][(n + km, m)^{-1}] \overset{\mathcal{R}_1}{=} [(n, m)(k, 1)]e_{(-k,1)}[(k, 1)^{-1}(n, m)^{-1}] = [n, m][(k, 1)^{-1}][k, 1][(k, 1)^{-1}(n, m)^{-1}] \overset{(PR3)}{=} [n, m][k, 1][(k, 1)^{-1}][k, 1][(k, 1)^{-1}][(n, m)^{-1}] = [n, m]e_{(k,1)}e_{(k,1)}[(n, m)^{-1}] = e_{(n,m)}.
\]

**Remark 3.1.** The relations in $\mathcal{R}_1$ are unnecessary. They can be obtained from $\mathcal{R}_3$ with $m = 1$.

Consider the partial group algebra $C^*_p(K \rtimes K^\times, \mathcal{R})$. We will show that this algebra is isomorphic to $\mathfrak{A}[R]$.

**Proposition 3.2.** There exists a $\ast$-homomorphism $\Psi : \mathfrak{A}[R] \longrightarrow C^*_p(K \rtimes K^\times, \mathcal{R})$ such that $\Psi(u^n) = [n, 1]$ and $\Psi(s_m) = [0, m]$.

\(^1\)Sometimes, we work with the element $(u, w)^{-1}$ or the element $(u_1, w_1)(u_2, w_2)$. For these elements, our corresponding notations will be $[(u, w)^{-1}]$ and $[(u_1, w_1)(u_2, w_2)]$. 

Proof. We need to show that \([n, 1]\) is a unitary (for \(n \in R\)), that \([0, m]\) is an isometry (for \(m \in R^\times\)) and that the relations (CL1)-(CL4) are satisfied. From \(\mathcal{R}_1\) and (PR2), we have \([n, 1][n, 1]^* = e_{(n, 1)} = 1\) and \([n, 1]^*[n, 1] = e_{(-n, 1)} = 1\), ie, \([n, 1]\) is a unitary. Similarly, from \(\mathcal{R}_2\) and (PR2) we see that \([0, m]\) is an isometry. By using this fact,

\[
\Psi(s_{m}s_{m'}) = [0, m][0, m'] = [0, m][0, m']^*[0, m'] \quad (\text{PR3})
\]

\[
[0, mm'][0, m']^*[0, m'] = [0, mm'] = \Psi(s_{mm'}),
\]
hence (CL1) is satisfied. We can prove (CL2) in the same way. To show (CL3), note that

\[
\Psi(s_{m}u^n) = [0, m][n, 1] = [0, m][n, 1][n, 1]^*[n, 1] \quad (\text{PR3}) = [mn, m][n, 1][n, 1]^*[n, 1] = [mn, m],
\]
because \([n, 1]\) is a unitary. On the other hand,

\[
\Psi(u^{mn}s_{m}) = [mn, 1][0, m] = [mn, 1][mn, 1]^*[mn, 1][0, m] \quad (\text{PR3}) = [mn, 1][mn, 1]^*[mn, m] = [mn, m].
\]

Finally, (CL4) follows from \(\mathcal{R}_3\) and\(^2\)

\[
\Psi(u^n e_m u^{-n}) = [n, 1][0, m][0, m]^*[-n, 1] = [n, m][0, 1/m][-n, 1][-n, 1]^*[-n, 1] \quad (\text{PR3}) = [n, m][(n, m)^{-1}][-n, 1][-n, 1]^*[-n, 1] = [n, m][(n, m)^{-1}] = e_{(n,m)}.
\]

Now, we will construct an inverse for \(\Psi\). In the next claim, note that every element in \(K \times K^\times\) can be written under the form \((\frac{n}{m'}, \frac{m}{m'})\), where \(n \in R\) and \(m, m' \in R^\times\).

Claim 3.3. The map \(\pi : K \times K^\times \longrightarrow \mathfrak{A}[R]\) given by \(\pi \left(\left(\frac{n}{m'}, \frac{m}{m'}\right)\right) = s^*_m u^n s_m\) is independent of the representation of \((\frac{n}{m'}, \frac{m}{m'})\).

Proof. Let \((\frac{n}{m'}, \frac{m}{m'}) = \left(\frac{q}{p'}, \frac{p}{p'}\right)\), ie, \(pm' = p^n\) and \(m'q = p^n\). Hence,

\[
s^*_p u^q s_p = s^*_p s^*_m s_{m'} u^q s_p \quad (\text{CL3}) = s^*_p s^*_m u^{m'q} s_{m'} s_p \quad (\text{CL1}) = s^*_p s^*_m u^{m'q} s_{m'} s_p \quad (\text{CL1}) = s^*_p s^*_m u^{m'q} s_{m'} s_p \quad (\text{CL1}) = s^*_p u^{m'q} s_{m'} s_p \quad (\text{CL3}) = s^*_p s^*_m s_{m'} s_{m'} u^n s_m = s^*_m u^n s_m.
\]

\[\square\]

Proposition 3.4. The map \(\pi\) defined above is a partial representation of \(K \times K^\times\) that satisfies \(\mathcal{R}\).

Proof. First, we will show that \(\pi\) is a partial representation. Since \(\pi((0, 1)) = s^*_1 u^0 s_1 = 1\), we have (PR1). Observe that

\[
\pi \left(\left(\frac{n}{m'}, \frac{m}{m'}\right)^{-1}\right) = \pi \left(\left(-\frac{n}{m}, \frac{m'}{m}\right)\right) = s^*_m u^{-n} s_{m'} = \pi \left(\left(\frac{n}{m'}, \frac{m}{m'}\right)^*\right),
\]

\[\square\]

\(^2\)Be careful with the \(e\)'s! The notation \(e_m\) represents \(s_m s^*_m\) in \(\mathfrak{A}[R]\) and \(e_{(n,m)}\) represents \([n, m][n, m]^*\) in \(C^*_b(K \times K^\times; \mathcal{R})\).
which shows (PR2). To see (PR3), let \( g = \left( \frac{a}{p}, \frac{b}{p} \right) \) and \( h = \left( \frac{n}{m'}, \frac{m}{m'} \right) \). We have \( gh = \left( \frac{m'q + pm}{pm'}, \frac{pm}{pm'} \right) \) and, therefore,

\[
\pi(gh) \pi(h^{-1}) = \pi(g) \pi(h)^* = (s^*_{p'm'} u^{m'q + pm} s_{pm})(s_m^* u^{-n} s_{m'})^{(\text{CL1}), (\text{CL2}), (\text{CL3})} \]

\[
s^*_{p'} u^g s^*_m s_p u^n s_m s^*_{m'} u^{-n} s_{m'} = s^*_{p'} u^g s^*_m s_p u^n s_m s^*_{m'} u^{-n} s_{m'} 
\]

Lemma 2.3.

This shows that \( \pi \) is a partial representation. It remains to show that the extension of \( \pi \) satisfies the relations in \( \mathcal{R} \). By remark 3.1, it suffices to show that the relations in \( \mathcal{R}_2 \) and \( \mathcal{R}_3 \) are satisfied. It follows from

\[
\pi(e_{(0, 1/m)}) = \pi([0, 1/m][0, m]) = s_m^* u^0 s_1^* u^0 s_m = 1
\]

and

\[
\pi \left( \sum_{n+\langle m \rangle \in \mathbb{R}/\langle m \rangle} e_{(n, m)} \right) = \sum_{n+\langle m \rangle \in \mathbb{R}/\langle m \rangle} s_1^* u^n s_m s^*_m u^{-n} s_1 = 1.
\]

\[\square\]

**Remark 3.5.** We can define \( \pi \) for a general representation of a element in \( K \times K^\times \) by \( \pi \left( \left( \frac{n}{m'}, \frac{m}{m'} \right) \right) = s^*_{m'} u^n s^*_m s_{m'} u^m \).

**Theorem 3.6.** The \(*\)-homomorphism \( \Psi \) defined above is a \(*\)-isomorphism. Its inverse \( \Phi : C^*_p(K \times K^\times, \mathcal{R}) \to \mathfrak{A}[R] \) is given by \( \Phi \left( \left[ \frac{n}{m'}, \frac{m}{m'} \right] \right) = s^*_{m'} u^n s_m \).

**Proof.** The existence of \( \Phi \) follows from \( \pi \) and the universal property of \( C^*_p(K \times K^\times, \mathcal{R}) \). It remains to show that \( \Phi \) and \( \Psi \) are inverses each other. Indeed, \( \Phi(\Psi(u^n)) = \Phi([n, 1]) = s_1^* u^n s_1 = u^n \), \( \Phi(\Psi(s_m)) = \Phi([0, m]) = s_1^* u^0 s_m = s_m \) and

\[
\Psi \left( \Phi \left( \left[ \frac{n}{m'}, \frac{m}{m'} \right] \right) \right) = \Psi(s^*_{m'} u^n s_m) = [0, 1/m'] [n, 1] [0, m] = [0, 1/m'] [0, 1/m']^* [0, 1/m'] [n, 1] [n, 1]^* [n, 1] [0, m] = \left[ \frac{n}{m'}, \frac{m}{m'} \right].
\]

\[\square\]

4. **Partial Crossed Product Description of \( \mathfrak{A}[R] \)**

Before characterizing \( \mathfrak{A}[R] \) as a partial crossed product, note that the group \( K \times K^\times \) is solvable and, hence, amenable. Therefore, there exists a faithful conditional expectation (imported from the partial crossed product realization) \( E : C^*_p(K \times K^\times, \mathcal{R}) \to C^*(\{e_g\}_{g \in K \times K^\times}) \) given by

\[E([g_1] [g_2] \cdots [g_k]) = \delta_{g_1 g_2 \cdots g_k} e [g_1] [g_2] \cdots [g_k].\]

In [6, Proposition 1], Cuntz and Li constructed a faithful conditional expectation \( \Theta \) on \( \mathfrak{A}[R] \) given by \( \Theta(s^*_{m'} u^n s_m s^*_m u^{-n} s_{m'}) = \delta_{m', m} \delta_{n, m'} s^*_{m'} u^n s_m s^*_m u^{-n} s_{m'} \). The next proposition shows that, under the \(*\)-isomorphism \( \Psi \), \( E \) and \( \Theta \) are the same conditional expectation.
Proposition 4.1. $E \circ \Psi = \Psi \circ \Theta$.

Proof. First of all, observe that $(\frac{n}{m'}, \frac{m}{m''}) (\frac{-n'}{m', \frac{m}{m''}}) = (0, 1)$ if, and only if, $m' = m''$ and $n = n'$. Hence,

$$E \circ \Psi(s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m''}) = E \left( \left[ \frac{n}{m''}, \frac{m}{m''} \right] \left[ -\frac{n'}{m', \frac{m}{m''}} \right] \right) = \delta_{m', m''} \delta_{n, n'} \left[ \frac{n}{m'}, \frac{m}{m''} \right] \left[ -\frac{n'}{m', \frac{m}{m''}} \right].$$

On the other hand

$$\Psi \circ \Theta(s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m''}) = \Psi(\delta_{m', m''} \delta_{n, n'} s_{m''}^* u^n s_m s_m^* u^{-n'} s_{m''}) = \delta_{m', m''} \delta_{n, n'} \left[ \frac{n}{m'}, \frac{m}{m''} \right] \left[ -\frac{n'}{m', \frac{m}{m''}} \right].$$

\[ \square \]

We already know that $\mathfrak{A}[R]$ is a partial crossed product. Indeed, every partial group algebra is a partial crossed product (see section 2.3). From now on, our goal is to study $\mathfrak{A}[R]$ by this way.

There exists a natural partial order on $R^\times$ given by the divisibility: we say that $m \leq m'$ if there exists $r \in R$ such that $m' = rm$. Whenever $m \leq m'$, we can consider the canonical projection $p_{m, m'} : R/(m') \rightarrow R/(m)$. Since $(R^\times, \leq)$ is a directed set, we can consider the inverse limit

$$\hat{R} = \lim_{\leftarrow} \{ R/(m), p_{m, m'} \},$$

which is the profinite completion of $R$. In this text, we shall use the following concrete description of $\hat{R}$:

$$\hat{R} = \left\{ (r_m + (m))_m \in \prod_{m \in R^\times} R/(m) \mid p_{m, m'}(r_m + (m')) = r_m + (m), \text{ if } m \leq m' \right\}.$$ Give to $R/(m)$ the discrete topology, to $\prod_{m \in R^\times} R/(m)$ the product topology and to $\hat{R}$ the induced topology of $\prod_{m \in R^\times} R/(m)$. With the operations defined componentwise, $\hat{R}$ is a compact topological ring. There exists a canonical inclusion of $R$ into $\hat{R}$ given by $r \mapsto (r + (m))_m$ (to see injectivity, take $r \neq 0$, $m$ non-invertible and note that $r \not\in (rm)$).

The above partial order can be extended to $K^\times$. For $w, w' \in K^\times$, we say that $w \leq w'$ if there exists $r \in R$ such that $w' = wr$. Denote by $(w)$ the fractional ideal generated by $w$, namely $(w) = wR \subseteq K$. As before, if $w \leq w'$, we can consider the canonical projection $p_{w, w'} : (R + (w'))/(w') \rightarrow (R + (w))/(w)$. As before, we consider the inverse limit

$$\hat{R}_K = \lim_{\leftarrow} \{ (R + (w))/(w), p_{w, w'} \} \cong \left\{ (u_w + (w))_w \in \prod_{w \in K^\times} (R + (w))/(w) \mid p_{w, w'}(u_{w'} + (w')) = u_w + (w), \text{ if } w \leq w' \right\}.$$

\[ \text{By the second isomorphism theorem, it could be } p_{w, w'} : R/(R \cap (w')) \rightarrow R/(R \cap (w)). \]
It is a compact crossed product ring too. In fact, \( \hat{R}_K \) is naturally isomorphic to \( \hat{R} \) as topological ring. In this text, we use \( \hat{R}_K \) instead of \( \hat{R} \) to simplify our proofs.

It is easy to see that, when \( R \) is a field, then \( \hat{R} \cong \hat{R}_K \cong \{0\} \).

Let \( \Omega \) be the spectrum of the relations \( \mathcal{R} \) (see section 2.3). We will show that \( \Omega \) is homeomorphic to \( \hat{R}_K \) (hence, homeomorphic to \( \hat{R} \)). Define

\[
\rho : \hat{R}_K \longrightarrow \mathcal{P}(K \rtimes K^x) \quad (u_w + (w))_w \longmapsto \{(u_w + rw, w) \mid w \in K^x, r \in R\}.
\]

Note that the definition is independent of the choice of \( u_w \) in \( u_w + (w) \).

**Claim 4.2.** \( \rho(\hat{R}_K) \subseteq \Omega \).

**Proof.** Let \((u_w + (w))_w \in \hat{R}_K\). By the definition of \( \hat{R}_K \), if \( w \leq w' \), then \( u_{w'} = u_w + kw \) for some \( k \in R \). Denote \( \rho((u_w + (w))) \) by \( \xi \). Clearly, \((0, 1) \in \xi \). We need to show that \( f(g^{-1}\xi) = 0 \), for all \( f \in \mathcal{R} \) and \( g \in \xi \). Fix \( g = (u_w + rw, w) \in \xi \). Let \( f = 1_{(n, 1)} - 1 \) in \( \mathcal{R}_1 \) and note that \( f(g^{-1}\xi) = 0 \) is equivalent to \( g(n, 1) \in \xi \). Since \( g(n, 1) = (u_w + rw, w)(n, 1) = (u_w + (r + n)w, w) \), we have \( g(n, 1) \in \xi \). Now, let \( f = 1_{(0, 1/m)} - 1 \) in \( \mathcal{R}_2 \). Similarly, we must show that \( g(0, 1/m) \in \xi \). Observe that \( g(0, 1/m) = (u_w + rw, w)(0, 1/m) = (u_w + rw, w/m) \). Since \( w/m \leq w \), then \( g(0, 1/m) = (u_w/m + k(w/m) + rw, w/m) = (u_w/m + (k + rm)(w/m)) \) and, for it belongs to \( \xi \), we must have \((n + r - k)w \in (w/m)\). Hence, \( n \equiv k - r \mod m \), in other words, there exists only one class \( n + (m) \) such that \( g(n, m) \in \xi \). Indeed, \( g(n, m) = (u_w + rw, w)(n, m) = (u_w + (n + r)w, wm) = (u_{wm} + (n + r - k)w, wm) \) and, for it belongs to \( \xi \), we must have \((n + r - k)w \in (wm)\). Hence, \( n \equiv k - r \mod m \), in other words, there exists only one class \( n + (m) \) such that \( g(n, m) \in \xi \). Since \( \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \), the proof is completed. \( \square \)

**Proposition 4.3.** \( \rho : \hat{R}_K \longrightarrow \Omega \) is a homeomorphism.

**Proof.**

**Injectivity.** Let \((u_w + (w))_w, (v_w + (w))_w \in \hat{R}_K \) such that \( \rho((u_w + (w))) = \rho(v_w + (w)) \). By the definition of \( \rho \), the elements in \( \rho((u_w + (w))) \) whose second component equals \( w \) are of the form \((u_w + rw, w) \). Since \((v_w, w) \in \rho((u_w + (w))) \) and, therefore, \((v_w, w) \in \rho((u_w + (w))) \), we must have \( v_w = u_w + rw \) for some \( r \in R \). This show that \((u_w + (w))_w = (v_w + (w))_w \).

**Surjectivity.** Let \( \xi \in \Omega \). The relations in \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) together implies that if \( g \in \xi \), then \( g(q/p, 1/p) \in \xi \) for all \( q \in R \) and \( p \in R^x \) (fix \( g \) and apply \( f(g^{-1}\xi) = 0 \) for various \( f \)). For each \( m \in R^x \), let \( f = \sum_{n+(m)} 1_{(n, m)} - 1 \) in \( \mathcal{R}_3 \) and apply \( f(g^{-1}\xi) = 0 \) with \( g = (0, 1) \) to see that there exists only one class \( n + (m) \) such that \( (n, m) \in \xi \). Denote this class by \( u_{m} + (m) \). Since \( g(0, 1/p) \in \xi \) if \( g \in \xi \), then \( p_{m, mp}(u_{mp} + (mp)) = (u_{m} + (m)) \). From this, we can define unambiguously \( u_w + (w) = u_m + (w) \) for \( w = m/m' \in K^x \). One can see that the classes \( u_w + (w) \) are compatible with the projections \( p_{w, w'} \) by using that \( g(q/p, 1/p) \in \xi \) if \( g \in \xi \). Hence, we have constructed \((u_w + (w))_w \in \hat{R}_K \). We claim that \( \rho((u_w + (w))) = \xi \). Since \((u_w, w) \in \xi \), \((u_w, w)(q, 1) = (u_w + qw, w) \) must belongs to \( \xi \). This shows that \( \rho((u_w + (w))) \subseteq \xi \). Suppose, by contradiction, \( \rho((u_w + (w))) \neq \xi \). Hence, there exists \( h \in \xi \) such that \( h \notin \rho((u_w + (w))) \). If we write \( h = (n'/m', m/m') \), then \( h \notin \rho((u_w + (w))) \) is equivalent to \( n' - m'u_m \notin (m) \). Let
\( g = (u_m, 1/m'), h' = (u_m, m/m') \) and note that both belong to \( \rho((u_w + (w))) \) (hence, belong to \( \xi \)). Since \( g^{-1}h = (-m' u_m, m'(n'/m', m/m')) = (n' - m'u_m, m) \) and \( n' - m'u_m \notin (m) \), then \( f(g^{-1} \xi) \neq 0 \) if \( f = \sum_{n+(m)} 1_{(n,m)} - 1 \), which contradicts the fact that \( \xi \in \Omega \). Hence, \( \rho((u_w + (w))) = \xi \).

To finish the proof, observe that \( \hat{R}_K \) and \( \Omega \) are compact Hausdorff, therefore it suffices to show that \( \rho \) (or \( \rho^{-1} \)) is continuous to conclude that \( \rho \) is a homeomorphism. We will prove that \( \rho^{-1} \) is continuous by showing that \( \pi_w \circ \rho^{-1} \) is continuous for all \( w \in K^\times \), where \( \pi_w : \hat{R}_K \longrightarrow (R + (w))/(w) \) is the canonical projection. Since \( (R + (w))/(w) \) is discrete, it suffices to show that \( \rho \circ \pi_w^{-1}(\{u_w + (w)\}) \) is an open set of \( \Omega \), for all \( u_w + (w) \in (R + (w))/(w) \). To see this, note that

\[
\rho \circ \pi_w^{-1}(\{u_w + (w)\}) = \{\xi \in \Omega \mid (u_w, w) \in \xi\},
\]

which is an open set of \( \Omega \) (recall that the topology on \( \Omega \) is induced by the product topology of \( \{0,1\}^{K \times K^\times} \)).

Following the section 2.3, there exists a partial action of \( K \times K^\times \) on \( \Omega \). By the above proposition, we can define this partial action on \( \hat{R}_K \). Let \( \hat{R}_g = \rho^{-1}(\Omega_g) \), where \( \Omega_g = \{\xi \in \Omega \mid g \in \xi\} \), and \( \theta_g \) be the homeomorphism between \( \hat{R}_{g^{-1}} \) and \( \hat{R}_g \). It is easy to see that

\[
\hat{R}_{(u,w)} = \{(u_w' + (w'))_{w'} \in \hat{R}_K \mid u_w + (w) = u + (w)\}
\]

and

\[
\theta_{(u,w)}((u_w' + (w'))_{w'}) = (u + uw_{w'} + (ww'))_{ww'} = (u + uw_{w^{-1}w'} + (w'))_{w'},
\]

ie, \( \theta_{(u,w)} \) acts on \( \hat{R}_{(u,w^{-1})} \) by the affine transformation corresponding to \( (u, w) \). The next proposition, whose proof is trivial, will be useful later.

**Proposition 4.4.** We have that

(i) \( \hat{R}_{(u,w)} = \emptyset \iff u \notin R + (w) \);

(ii) \( \hat{R}_{(u,w)} = \hat{R}_K \iff R \subseteq u + (w) \).

Now, we describe the topology on \( \hat{R}_K \). Since \( \hat{R}_K \) is a singleton set when \( R \) is a field, we shall assume that \( R \) is not a field in this paragraph. For \( w \in K^\times \) and \( C_w \subseteq (R + (w))/(w) \), we define the open set

\[
V^C_w = \{(u_w' + (w'))_{w'} \in \hat{R}_K \mid u_w + (w) \in C_w\}.
\]

Clearly, if \( w \leq w' \), then \( V^C_w = V^C_{w'} \), where \( C_{w'} = \{u + (w') \in (R + (w'))/(w') \mid u + (w) \in C_w\} \). From the product topology, we know that the finite intersections of open sets \( V^C_w \) form a basis for the topology on \( \hat{R}_K \). By taking a common multiple of the \( w \)'s in the intersection, we see that every basic open set is of the form \( V^C_w \) (since \( V^C_{w_1} \cap V^C_{w_2} = V^C_{w_1 \cap w_2} \)). Furthermore, if \( C_w \neq \emptyset \), \( r \) is a non-invertible element in \( R \) and \( V^C_w = V^C_{wr} \), then \( C_{wr} \) has, at least, two elements. Indeed, let \( u + (w) \in C_w \) and \( r_1, r_2 \in R \) such that \( r_1 + (r) \neq r_2 + (r) \). It is easy to see that \( u + wr_1 + (wr) \) and \( u + wr_2 + (wr) \) are in \( C_{wr} \) and that \( u + w + (wr) \neq u + wr + (wr) \). This says that, if \( V^C_w \) is non-empty, we can suppose that \( C_w \) has more than one element.

**Proposition 4.5.** The partial action \( \theta \) on \( \hat{R}_K \) is topologically free if, and only if, \( R \) is not a field.
Proof. If $R$ is a field, then $\hat{R}_K = \{0\}$ and, hence, $\theta$ is not topologically free. Conversely, suppose that $R$ is not a field. We need to show that $F_g = \{x \in \hat{R}_{g^{-1}} \mid \theta_g(x) = x\}$ has empty interior, for all $g \in K \times K^\times \setminus \{(0,1)\}$. We shall consider two cases: $g = (u,1)$ and $g = (u,w), w \neq 1$.

Case 1. If $u \notin R$, then the proposition 4.4 says that $\hat{R}_{g^{-1}} = \emptyset$. So, we can suppose $u \in R$. If $F_g \neq \emptyset$, then equation $\theta_g(x) = x$ implies that $u \in (m)$ for every $m \in R^\times$. Since $R$ is not a field, then $u = 0$. This show that $F_g = \emptyset$ if $g = (u,1)$ and $u \neq 0$.

Case 2. Let $g = (u,w)$ such that $w \neq 1$ and $u \in R + (w)$ (if $u \notin R + (w)$, then $\hat{R}_{g^{-1}} = \emptyset$). Let $V$ be a non-empty open set contained in $\hat{R}_{g^{-1}}$. We will show that there exists $x \in V$ such that $\theta_g(x) \neq x$. By shrinking $V$ if necessary, we can suppose that $V = V_w^{C_{w'}}$. Futhermore, we can assume that $C_{w'}$ has more than one element. Let $u_1 + (w')$ and $u_2 + (w')$ be distinct elements of $C_{w'}$, hence $u_1 - u_2 \notin (w')$. Suppose, by contradiction, $\theta_g(x) = x$ for all $x \in V$. Since $(u_i + (w''))_{w''} \in V$, $i = 1, 2$, then

$$\theta_{(u,w)}((u_i + (w''))_{w''}) = (u_i + (w''))_{w''} \implies (u + uw_i + (w''))_{w''} = (u_i + (w''))_{w''}.$$  

By choosing $w'' = (w-1)w'$ (note that $w \neq 1$), we see that $u + (w-1)u_i \in ((w-1)w')$, for $i = 1, 2$. By subtracting the equations (for different $i$’s), we have $(w-1)(u_1 - u_2) \in ((w-1)w')$ and, therefore $u_1 - u_2 \in (w')$; which is a contradiction! This show that $F_g$ has empty interior. \hfill \Box

**Proposition 4.6.** The partial action $\theta$ is minimal.

Proof. If $R$ is a field, then the result is trivial. Now, suppose that $R$ is not a field. We will prove that every $x \in \hat{R}_K$ has dense orbit (see section 2.2) by showing that if $V$ is a non-empty open set, then there exists $g \in K \times K^\times$ such that $x \in \hat{R}_{g^{-1}}$ and $\theta_g(x) \in V$. Let $x = (u_w + (w))_{w} \in \hat{R}_K$ and $V = V_w^{C_{w'}}$ non-empty. Take $u' + (w') \in C_{w'}$ and observe that we can suppose, without loss of generality, $u' \in R$ and $u_w \in R$. Let $g = (u' - u_w, 1)$. By the proposition 4.1, $\hat{R}_{g^{-1}} = \hat{R}_K$ and, hence, $x \in \hat{R}_{g^{-1}}$. To finish, note that $\theta_g(x) = \theta_{(w'-u_w,1)}((u_{w'} + (w'))_{w'}) = (u' - u_w + u_w + (w'))_{w'} \in V$. \hfill \Box

Following, we summarize the results of this section.

**Theorem 4.7.** The algebra $\mathfrak{A}[R]$ is $*$-isomorphic to the partial crossed product $C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times$, where $\alpha$ is the partial action induced by $\theta$. The $*$-isomorphism is given by $u^a \mapsto 1_{(0,1)}$ and $s_m \mapsto 1_{(0,m)}\delta_{(0,m)}$, where $1_{(0,m)}$ is the characteristic function of $\hat{R}_g$.

**Theorem 4.8.** $\mathfrak{A}[R]$ is simple.

Proof. By the propositions 4.5 and 4.6, the reduced crossed product $C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times$ is simple. Since $K \rtimes K^\times$ is amenable, then $C(\hat{R}_K) \rtimes_\alpha K \rtimes K^\times \cong C(\hat{R}_K) \rtimes_\alpha K \times K^\times$ and, therefore, $C(\hat{R}_K) \rtimes_\alpha K \times K^\times$ is simple. The result follows from the previous theorem. \hfill \Box

**Corollary 4.9.** $\mathfrak{A}[R] \cong \mathfrak{A}_r[R]$.

When $R = \mathbb{Z}$, we can restrict our partial action to the subgroup $\mathbb{Q} \times \mathbb{Q}^*_+$ of $\mathbb{Q} \times \mathbb{Q}^*$ and the corresponding partial crossed product is the algebra $\mathcal{Q}_N$ introduced by Cuntz in [5] and realized as a partial crossed product in [3] by Brownlowe, an Huef, Laca and Raeburn.
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