Análise Funcional - Curso de Verão - 2020

Prof. Fabio Silva Botelho

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1 Metric Spaces

1.1 Introduction

In this chapter we present the main basic definitions and results relating the concept of metric spaces.

We recall that any Banach space is a metric one, so that the framework here introduced is suitable for a very large class of spaces.

1.2 The main definitions

We start this section presenting the metric definition and some concerning examples.

Definição 1.1 (Metric space)**.** *Let V be a non-empty set. We say that V is a metric space as it is possible to define a function $d : V \times V \to \mathbb{R}^+ = [0, +\infty)$ such that*

1. $d(u, v) > 0$ *if* $u \neq v$ *and* $d(u, u) = 0, \ \forall u, v \in V.$
2. $d(u, v) = d(v, u), \ \forall u, v \in V.$
3. $d(u, w) \leq d(u, v) + d(v, w), \ \forall u, v, w \in V.$

Such a function d is said to be a metric for V, so that the metric space in question is denoted by (V, d).

Exemplo 1.2. *$V = \mathbb{R}$ is a metric space with the metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ where*

$$d(u, v) = |u - v|, \forall u, v \in \mathbb{R}.$$

Exemplo 1.3. *$V = \mathbb{R}^2$ is a metric space with the metric $d : V \times V \to \mathbb{R}^+$ where*

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}, \ \forall \mathbf{u} = (u_1, u_2), \ \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2.$$

Exemplo 1.4. *$V = \mathbb{R}^n$ is a metric space with the metric $d : V \times V \to \mathbb{R}^+$ where*

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}, \ \forall \mathbf{u} = (u_1, \ldots, u_n), \ \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n.$$

Exemplo 1.5. *$V = C([a,b])$, where $C([a,b])$ is the metric space of continuous functions $u : [a,b] \to \mathbb{R}$ with the metric $d : V \times V \to \mathbb{R}^+$ where*

$$d(u,v) = \max_{x \in [a,b]} \{|u(x) - v(x)|\} \equiv \|u - v\|_\infty, \ \forall u, v \in V.$$

Exemplo 1.6. *$V = C([a,b])$, is a metric space with the metric $d : V \times V \to \mathbb{R}^+$ where*

$$d(u,v) = \int_a^b |u(x) - v(x)| \, dx, \ \forall u, v \in V.$$

1.3 The space l^∞

In this subsection we start to define some important classes of metric spaces.
The first definition presented is about the l^∞ space of sequences.

Definição 1.7. *We define the space l^∞ as*

$$l^\infty = \{\mathbf{u} = \{u_n\}_{n \in \mathbb{N}} \ : \ u_n \in \mathbb{C} \text{ and there exists } M > 0 \text{ such that } |u_n| < M, \ \forall n \in \mathbb{N}\}.$$

A metric for l^∞ may be defined by

$$d(\mathbf{u}, \mathbf{v}) = \sup_{j \in \mathbb{N}} \{|u_j - v_j|\},$$

where $\mathbf{u} = \{u_n\}$ e $\mathbf{v} = \{v_n\} \in l^\infty$.

1.4 Discrete metric

At this point we introduce the definition of discrete metric.

Definição 1.8. *Let V be a non-empty set. We define the discrete metric for V by*

$$d(u,v) = \begin{cases} 0, & \text{if } u = v, \\ 1, & \text{if } u \neq v. \end{cases} \tag{1}$$

In such a case we say that (V, d) is a discrete metric space.

Exercício 1.9. *Let $V = \mathbb{R}$ and let $d : V \times V \to \mathbb{R}$ be defined by*

$$d(u,v) = \sqrt{|u - v|}.$$

Show that d is a metric for V.

1.5 The metric space s

In the next lines we define one more metric space of sequences, namely, the space s.

Definição 1.10 (The metric space s)**.** *We define the metric space s as $s = (V, d)$, where*

$$V = \{\mathbf{u} = \{u_n\}, \ : \ u_n \in \mathbb{C}, \ \forall n \in \mathbb{N}\},$$

with the metric

$$d(\mathbf{u}, \mathbf{v}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{(1 + |u_n - v_n|)},$$

$\forall \mathbf{u} = \{u_n\}$ and $\mathbf{v} = \{v_n\} \in V$.

Exercício 1.11. *Show that this last function d is indeed a metric.*

1.6 The space $B(A)$

Another important metric space is the space of bounded functions defined on a set A, denoted by $B(A)$.

Definição 1.12. *Let A be a non-empty set and define*

$$B(A) = \{u : A \to \mathbb{R}, \ \text{such that there exists } M > 0 \text{ such that } |u(x)| < M, \ \forall x \in A\}.$$

$B(A)$ is said to be the space of bounded functions defined on A.

Exercício 1.13. *Show that $B(A)$ is a metric space with the metric*

$$d(u, v) = \sup_{x \in A}\{|u(x) - v(x)|\}.$$

1.7 The space l^p

Finally, one of most important metric space of sequences is the l^p one, whose definition is presented in the next lines.

Definição 1.14. *Let $p \geq 1$, $p \in \mathbb{R}$.*
We define the space l^p by

$$l^p = \left\{ \mathbf{u} = \{u_n\} \ : \ u_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |u_n|^p < \infty \right\}$$

with the metric

$$d(\mathbf{u}, \mathbf{v}) = \left(\sum_{n=1}^{\infty} |u_n - v_n|^p \right)^{1/p},$$

where $\mathbf{u} = \{u_n\}$ and $\mathbf{v} = \{v_n\} \in l^p$.

At this point we shall show that d is indeed a metric.

Let $p > 1$, $p \in \mathbb{R}$. Let $q > 1$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

that is

$$q = \frac{p}{p-1}.$$

Let $x, y \geq 0$, $x, y \in \mathbb{R}$.

We are going to show that

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

Observe that if $x = 0$ or $y = 0$ the inequality is immediate. Thus, suppose $x > 0$ e $y > 0$.

Fix $y > 0$ and define

$$h(x) = \frac{1}{p}x^p + \frac{1}{q}y^q - xy, \ \forall x > 0.$$

Observe that

$$h'(x) = x^{p-1} - y$$

and

$$h''(x) = (p-1)x^{p-2} > 0, \forall x > 0.$$

Therefore h is convex and its minimum on $(0, +\infty)$ is attained through the equation

$$h'(x) = x^{p-1} - y = 0,$$

that is, at $x_0 = y^{1/(p-1)}$.

Hence,

$$\begin{aligned}
\min_{x \in (0,+\infty)} h(x) &= h(x_0) \\
&= \frac{1}{p}(x_0)^p + \frac{1}{q}y^q - x_0 y \\
&= \frac{1}{p}y^{p/(p-1)} - y^{1/(p-1)}y + \frac{1}{q}y^q \\
&= (1/p - 1)\, y^q + \frac{1}{q}y^q \\
&= -\frac{1}{q}y^q + \frac{1}{q}y^q \\
&= 0. \qquad (2)
\end{aligned}$$

Thus,

$$h(x) = \frac{1}{p}x^p + \frac{1}{q}y^q - xy \geq h(x_0) = 0, \ \forall x > 0.$$

Therefore, since $y > 0$ is arbitrary, we obtain

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q, \ \forall x, y > 0.$$

so that

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q, \ \forall x, y \geq 0. \tag{3}$$

Let $\mathbf{u} = \{u_n\} \in l^p$ and $\mathbf{v} = \{v_n\} \in l^q$.

Denote

$$\|\mathbf{u}\|_p = \left(\sum_{n=1}^{\infty} |u_n|^p\right)^{1/p}$$

and

$$\|\mathbf{v}\|_q = \left(\sum_{n=1}^{\infty} |v_n|^q\right)^{1/q}.$$

Define also

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|_p} = \left\{\frac{u_n}{\left(\sum_{n=1}^{\infty} |u_n|^p\right)^{1/p}}\right\},$$

and

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|_q} = \left\{\frac{v_n}{\left(\sum_{n=1}^{\infty} |v_n|^q\right)^{1/q}}\right\}.$$

From this and (3) we obtain,

$$\begin{aligned} \sum_{n=1}^{\infty} |\hat{u}_n \hat{v}_n| &\leq \frac{1}{p}\sum_{n=1}^{\infty} |\hat{u}_n|^p + \frac{1}{q}\sum_{n=1}^{\infty} |\hat{v}_n|^q \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned} \tag{4}$$

Thus,

$$\sum_{n=1}^{\infty} |u_n v_n| \leq \|\mathbf{u}\|_p \|\mathbf{v}\|_q, \ \forall \mathbf{u} \in l^p, \ \mathbf{v} \in l^q.$$

This last inequality is well known as the Hölder one.

Exercício 1.15. *Prove the Minkowski inequality, namely*

$$\|\mathbf{u} + \mathbf{v}\|_p \leq \|\mathbf{u}\|_p + \|\mathbf{v}\|_p, \ \forall \mathbf{u}, \mathbf{v} \in l^p.$$

Hint

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|_p^p &= \sum_{n=1}^{\infty} |u_n+v_n|^p \\ &\leq \sum_{n=1}^{\infty} |u_n+v_n|^{p-1}(|u_n|+|v_n|). \end{aligned} \tag{5}$$

Apply the Hölder inequality to each part of the right hand side of the last inequality.

Use such an inequality to prove the triangle inequality concerning the metrics definition.

Prove also the remaining properties relating the metric definition and conclude that $d : l^p \times l^p \to \mathbb{R}^+$*, where*

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|_p,\ \forall \mathbf{u},\mathbf{v} \in l^p$$

is indeed a metric for the space l^p*.*

1.8 Some fundamental definitions

Definição 1.16 (neighborhood)**.** *Let* (U,d) *be a metric space. Let* $u \in U$ *and* $r > 0$*. We define the neighborhood of center* u *and radius* r*, denoted by* $V_r(u)$*, by*

$$V_r(u) = \{v \in U \mid d(u,v) < r\}.$$

Definição 1.17 (limit point)**.** *Let* (U,d) *be a metric space and* $E \subset U$*. A point* $u \in U$ *is said to be a limit point of* E *if for each* $r > 0$ *there exists* $v \in V_r(u) \cap E$ *such that* $v \neq u$*.*

We shall denote by E' the set of all limit points of E.

Exemplo 1.18. $U = \mathbb{R}^2$*,* $E = B_r(0)$*. Thus* $E' = \overline{B}_r(0)$*.*

Observação 1.19. *In the next definitions* U *shall denote a metric space with a metric* d*.*

Definição 1.20 (Isolated point)**.** *Let* $u \in E \subset U$*. We say that* u *is an isolated point of* E *if it is not a limit point of* E*.*

Exemplo 1.21.

$U = \mathbb{R}^2$*,* $E = B_1((0,0)) \cup \{(3,3)\}$*. Thus* $(3,3)$ *is an isolated point of* E*.*

Definição 1.22 (Closed set)**.** *Let* $E \subset U$ *and let* E' *be the set of limit points of* E*. We say that* E *is closed if* $E \supset E'$*.*

Exemplo 1.23.

Let $U = \mathbb{R}^2$ *and* $r > 0$*, thus* $E = \overline{B}_r((0,0))$ *is closed.*

Definição 1.24. *A point* $u \in E \subset U$ *is said to be an interior point of* E *if there exists* $r > 0$ *such that* $V_r(u) \subset E$*, where*

$$V_r(u) = \{v \in U \mid d(u,v) < r\}.$$

Exemplo 1.25.

For $U = \mathbb{R}^2$, *let* $E = B_1((0,0)) \cup \{(3,3)\}$, *for example* $u = (0.25, 0.25)$ *is an interior point of* E, *in fact, for* $r = 0.5$, *if* $v \in V_r(u)$ *then* $d(u,v) < 0.5$ *so that* $d(v,(0,0)) \leq d((0,0),u) + d(u,v) \leq \sqrt{1/8} + 0.5 < 1$ *that is,* $v \in B_1((0,0))$ *and thus* $V_r(u) \subset B_1((0,0))$. *We may conclude that* u *is an interior point of* $B_1((0,0))$. *In fact all points of* $B_1((0,0))$ *are interior.*

Definição 1.26 (Open set)**.** $E \subset U$ *is said to be open if all its points are interior.*

Exemplo 1.27.

For $U = \mathbb{R}^2$, *the ball* $B_1(0,0)$ *is open.*

Definição 1.28. *Let* $E \subset U$, *we define its complement, denoted by* E^c, *by:*

$$E^c = \{v \in U \mid v \notin E\}.$$

Definição 1.29. *A set* $E \subset U$ *is said to bounded if there exists* $M > 0$ *such that*

$$\sup\{d(u,v) \mid u, v \in E\} \leq M.$$

Definição 1.30. *A set* $E \subset U$ *is said to be dense in* U *if each point of* U *is either a point of* E *or it is a limit point of* E, *that is,* $U = E \cup E'$.

Exemplo 1.31.

The set $\mathbb{Q}$ *is dense in* $\mathbb{R}$. *Let* $u \in \mathbb{R}$ *and let* $r > 0$. *Thus, from a well known result in elementary analysis there exists* $v \in \mathbb{Q}$ *such that* $u < v < u + r$, *that is,* $v \in \mathbb{Q} \cap V_r(u)$ *and* $v \neq u$, *where* $V_r(u) = (u - r, u + r)$. *Therefore* u *is a limit point of* $\mathbb{Q}$. *Since* $u \in \mathbb{R}$ *is arbitrary, we may conclude that* $\mathbb{R} \subset \mathbb{Q}'$, *that is,* $\mathbb{Q}$ *is dense in* $\mathbb{R}$.

Teorema 1.32. *Let* (U, d) *be a metric space. Let* $u \in U$ *and* $r > 0$. *Then* $V_r(u)$ *is open.*

Proof. First we recall that

$$V_r(u) = \{v \in U \mid d(u,v) < r\}.$$

Let $v \in V_r(u)$. We have to show that v is an interior point of $V_r(u)$. Define $r_1 = r - d(u,v) > 0$. We shall show that que $V_{r_1}(v) \subset V_r(u)$.

Let $w \in V_{r_1}(v)$, thus $d(v,w) < r_1$. Hence

$$d(u,w) \leq d(u,v) + d(v,w) < d(u,v) + r_1 = r.$$

Therefore $w \in V_r(u), \forall w \in V_{r_1}(v)$, that is $V_{r_1}(v) \subset V_r(u)$, so that we may conclude that v is an interior point of $V_r(u), \forall v \in V_r(u)$, thus, $V_r(u)$ is open. The proof is complete.

Teorema 1.33. *Let* u *be a limit point of* $E \subset U$, *where* (U, d) *is a metric space. Then each neighborhood of* u *has an infinite number of points of* E, *distinct from* u.

Proof. Suppose to obtain contradiction, that there exists $r > 0$ such that $V_r(u)$ has a finite number of points of E distinct from u. Let $\{v_1, ..., v_n\}$ be such points of $V_r(u) \cap E$ distinct from u. Choose $0 < r_1 < \min\{d(u,v_1), d(u,v_2), ..., d(u,v_n)\}$. Hence $V_{r_1}(u) \subset V_r(u)$ and $v_i \notin V_{r_1}(u), \forall i \in \{1, 2, ..., n\}$. Therefore either $V_{r_1}(u) \cap E = \{u\}$ or $V_{r_1} \cap E = \emptyset$, which contradicts the fact that u is a limit point of E.

The proof is complete.

Corolário 1.34. *Let* $E \subset U$ *be a finite set. Then* E *has no limit points.*

1.9 Properties of open and closed sets in a metric space

In this section we present some basic properties of open and closed sets.

Proposição 1.35. *Let $\{E_\alpha,\ \alpha \in L\}$ be a collection of sets. Then*

$$(\cup_{\alpha\in L} E_\alpha)^c = \cap_{\alpha \in L} E_\alpha^c.$$

Proof. Observe that

$$\begin{aligned} u \in (\cup_{\alpha\in L} E_\alpha)^c &\Leftrightarrow u \notin \cup_{\alpha\in L} E_\alpha \\ &\Leftrightarrow u \notin E_\alpha, \forall \alpha \in L \\ &\Leftrightarrow u \in E_\alpha^c, \forall \alpha \in L \\ &\Leftrightarrow u \in \cap_{\alpha\in L} E_\alpha^c. \end{aligned} \tag{6}$$

Exercício 1.36.

Prove that

$$(\cap_{\alpha\in L} E_\alpha)^c = \cup_{\alpha \in L} E_\alpha^c.$$

Teorema 1.37. *Let (U, d) be a metric space and $E \subset U$. Thus, E is open if and only if E^c is closed.*

Proof. Suppose E^c is closed. Choose $u \in E$, thus $u \notin E^c$ and therefore u is not a limit point E^c. Hence there exists $r > 0$ such that $V_r(u) \cap E^c = \emptyset$. Hence, $V_r(u) \subset E$, that is, u is an interior point of E, $\forall u \in E$, so that E is open.

Reciprocally, suppose E is open. Let $u \in (E^c)'$. Thus for each $r > 0$ there exists $v \in V_r(u) \cap E^c$ such that $v \neq u$, so that

$$V_r(u) \nsubseteq E,\ \forall r > 0.$$

Therefore u is not an interior point of E. Since E is open we have that $u \notin E$, that is, $u \in E^c$. Hence $(E^c)' \subset E^c$, that is, E^c is closed.

The proof is complete.

Corolário 1.38. *Let (U, d) be a metric space, $F \subset U$ is closed if and only if F^c is open.*

Teorema 1.39. *Let (U, d) be a metric space.*

1. *If $G_\alpha \subset U$ and G_α is open $\forall \alpha \in L$, then*

$$\cup_{\alpha\in L} G_\alpha$$

   is open.

2. *If $F_\alpha \subset U$ and F_α is closed $\forall \alpha \in L$, then*

$$\cap_{\alpha\in L} F_\alpha$$

   is closed.

3. *If $G_1, ..., G_n \subset U$ and G_i is open $\forall i \in \{1, ..., n\}$, then*

$$\cap_{i=1}^n G_i$$

is open.

4. *If $F_1, ..., F_n \subset U$ and F_i is closed $\forall i \in \{1, ..., n\}$, then*

$$\cup_{i=1}^n F_i$$

is closed.

Proof. 1. Let $G_\alpha \subset U$, where G_α is open $\forall \alpha \in L$. Let $u \in \cup_{\alpha \in L} G_\alpha$. Thus $u \in G_{\alpha_0}$ for some $\alpha_0 \in L$. Since G_{α_0} is open, there exists $r > 0$ such that $V_r(u) \subset G_{\alpha_0} \subset \cup_{\alpha \in L} G_\alpha$. Hence, u is an interior point, $\forall u \in \cup_{\alpha \in L} G_\alpha$. Thus $\cup_{\alpha \in L} G_\alpha$ is open.

2. Let $F_\alpha \subset U$, where F_α is closed $\forall \alpha \in L$. Thus, F_α^c is open $\forall \alpha \in L$. From the last item, we have $\cup_{\alpha \in L} F_\alpha^c$ is open so that

$$\cap_{\alpha \in L} F_\alpha = (\cup_{\alpha \in L} F_\alpha^c)^c$$

is closed.

3. Let $G_1, ..., G_n \subset U$ be open sets. Let

$$u \in \cap_{i=1}^n G_i.$$

Thus,

$$u \in G_i, \forall i \in \{1, ..., n\}.$$

Since G_i is open, there exists $r_i > 0$ such that $V_{r_i}(u) \subset G_i$.

Define $r = \min\{r_1, ..., r_n\}$. Hence, $V_r(u) \subset V_{r_i}(u) \subset G_i, \forall i \in \{1, ..., n\}$ and therefore

$$V_r(u) \subset \cap_{i=1}^n G_i.$$

This means that u is an interior point of $\cap_{i=1}^n G_i$, and being $u \in \cap_{i=1}^n G_i$ arbitrary we obtain that $\cap_{i=1}^n G_i$ is open.

4. Let $F_1, ..., F_n \subset U$ be closed sets. Thus, $F_1^c, ..., F_n^c$ are open. Thus, from the last item, we obtain:

$$\cap_{i=1}^n F_i^c$$

is open, so that

$$\cup_{i=1}^n F_i = (\cap_{i=1}^n F_i^c)^c$$

is closed.

The proof is complete.

Exercício 1.40.

Let (U,d) be a metric space and let $u_0 \in U$. Show that $A = \{u_0\}$ is closed. Let $B = \{u_1, ..., u_n\} \subset U$. Show that B is closed.

Definição 1.41 (Closure)**.** *Let (U,d) be a metric space and let $E \subset U$. Denote the set of limit points of E by E'. We define the closure of E, denoted by $\overline{E}$, by:*

$$\overline{E} = E \cup E'.$$

Exemplos 1.42.

1. *Let $U = \mathbb{R}^2$, $E = B_1(0,0)$, we have that $E' = \overline{B}_1(0,0)$, so that in this example $\overline{E} = E \cup E' = E'$.*

2. *Let $U = \mathbb{R}$, $A = \{1/n \ : \ n \in \mathbb{N}\}$, we have that $A' = \{0\}$, and thus $\overline{A} = A \cup A' = A \cup \{0\}$.*

Teorema 1.43. *Let (U,d) be a metric space and $E \subset U$. Thus,*

1. *$\overline{E}$ is closed.*

2. *$E = \overline{E} \Leftrightarrow E$ is closed.*

3. *If $F \supset E$ and F is closed, then $F \supset \overline{E}$.*

Proof. 1. Observe that $\overline{E} = E \cup E'$. Let $u \in \overline{E}^c$. Thus $u \notin E$ and $u \notin E'$ (u is not a limit point of E). Therefore, there exists $r > 0$ such that $V_r(u) \cap E = \emptyset$, that is, $V_r(u) \subset E^c$, thus, u is an interior point of E^c.

We shall prove that $V_r(u) \cap \overline{E} = \emptyset$. Let $v \in V_r(u)$ and define $r_1 = r - d(u,v) > 0$. We shall show that

$$V_{r_1}(v) \subset V_r(u).$$

Let $w \in V_{r_1}(v)$, thus $d(v,w) < r_1$ and therefore

$$d(u,w) \leq d(u,v) + d(v,w) < d(u,v) + r_1 = r,$$

that is, $w \in V_r(u)$. Hence,

$$V_{r_1}(v) \subset V_r(u),$$

and thus v is not a limit point of E, that is, $v \in \overline{E}^c, \forall v \in V_r(u)$. Thus, $V_r(u) \subset \overline{E}^c$ which means that u is an interior point of $\overline{E}^c$, so that $\overline{E}^c$ is open, and hence $\overline{E}$ is closed.

2. Observe that $E \subset \overline{E} = E \cup E'$. Suppose that E is closed. Thus $E \supset E'$, that is $E \supset E \cup E' = \overline{E}$. Hence $E = \overline{E}$. Suppose $E = \overline{E}$. From the last item $\overline{E}$ is closed, and thus E is closed.

3. Let F be a closed set such that $F \supset E$. Thus, $F' \supset E'$.

   Hence

$$F = \overline{F} = F \cup F' \supset E \cup E' = \overline{E}.$$

The proof is complete.

Exercícios 1.44.

1. *In the proof of the last theorem we have used a result which now is requested to be proven in an exercise form.*

   Let U be a metric space. Assume $A \subset B \subset U$. Show that $A' \subset B'$.

2. *Let U be a metric space and let $A, B \subset U$. Show that*

$$A' \cup B' = (A \cup B)'.$$

3. *Let U be a metric space and let $E \subset U$. Show that E' is closed.*

4. *Let $B_1, B_2, ...$ be subsets of a metric space U.*

   (a) *Show that if*

$$A_n = \cup_{i=1}^n B_i, \text{ then } \overline{A}_n = \cup_{i=1}^n \overline{B}_i.$$

   (b) *Show that if*

$$B = \cup_{i=1}^\infty B_i \text{ then } \overline{B} \supset \cup_{i=1}^\infty \overline{B}_i.$$

5. *Let U be a metric space and let $E \subset U$. Recall that the interior of E, denoted by E°, is defined as the set of all interior points of E.*

   (a) *Show that E° is open.*

   (b) *Show that E is open, if and only if, $E = E^\circ$.*

   (c) *Show that if $G \subset E$ and G is open, then $G \subset E^\circ$.*

   (d) *Prove que $(E^\circ)^c = \overline{E^c}$.*

   (e) *Do E and $\overline{E}$ have always the same interior? If not, present a counter example.*

   (f) *Do E and E^0 have always the same closure? If not, present a counter example.*

6. *Prove that $\mathbb{Q}$, the rational set, has empty interior.*

7. *Prove that $\mathbb{I}$, the set of irrationals, has empty interior.*

8. *Prove that given $x, y \in \mathbb{R}$ such that $x < y$, there exists $\alpha \in \mathbb{I}$, such that*

$$x < \alpha < y.$$

9. *Prove that $\mathbb{I}$ is dense in $\mathbb{R}$.*

   Hint: Prove that

$$x \in \mathbb{I}', \ \forall x \in \mathbb{R},$$

   where $\mathbb{I}'$ denotes the set of limit of points of $\mathbb{I}$.

10. *Let $B \subset \mathbb{R}$ be an open set. Show that for all $x \in \mathbb{R}$ the set*

$$x + B = \{x + y \mid y \in B\}$$

   is open.

11. *Let* $A, B \subset \mathbb{R}$ *be open sets. Show that the set*

$$A + B = \{x + y \ : \ x \in A \text{ and } y \in B\},$$

is open.

12. *Let* $B \subset \mathbb{R}$ *be an open set. Show that for all* $x \in \mathbb{R}$ *such that* $x \neq 0$ *the set*

$$x \cdot B = \{x \cdot y \mid y \in B\}$$

is open.

13. *Let* $A, B \subset \mathbb{R}$, *show that*

    (a)
    $$(A \cap B)^\circ = A^\circ \cap B^\circ,$$

    (b)
    $$(A \cup B)^\circ \supset A^\circ \cup B^\circ,$$

    and give an example in which the inclusion is proper.

14. *Let* $A \subset \mathbb{R}$ *be an open set and* $a \in A$. *Prove that* $A \setminus \{a\}$ *is open.*

15. *Let* $A, B \subset \mathbb{R}$. *Prove that:*

    (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

    (b) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, *and give an example for which the last inclusion is proper.*

16. *Show that a set* A *is dense in* $\mathbb{R}$ *if, and only if,* A^c *has empty interior.*

17. *Let* $F \subset \mathbb{R}$ *be a closed set and let* $x \in F$. *Show that* x *is an isolated point of* F *if, and only if,* $F \setminus \{x\}$ *is closed.*

18. *Show that if* $A \subset \mathbb{R}$ *is uncountable, then so is* A'.

19. *Show that if* $A \subset \mathbb{R}$ *then* $\overline{A} \setminus A'$ *is countable.*

20. *Let* U *be a metric space and let* $A \subset U$ *be an open set. Assume* $a_1, .., a_n \in A$.
    Prove that $A \setminus \{a_1, ..., a_n\}$ *is open.*

21. *Let* U *be a metric space, let* $A \subset U$ *be an open set and let* $F \subset U$ *be a closed one.*
    Show that $A \setminus F$ *is open and* $F \setminus A$ *is closed.*

22. *Let* $A \subset \mathbb{R}$ *be an uncountable set. Prove that* $A \cap A' \neq \emptyset$.

1.10 Compact sets

Definição 1.45 (Open covering). *Let (U, d) be a metric space. We say that a collection of sets $\{G_\alpha, \alpha \in L\} \subset U$ is an open covering of $A \subset U$ if*

$$A \subset \cup_{\alpha \in L} G_\alpha$$

and G_α is open, $\forall \alpha \in L$.

Definição 1.46 (Compact set). *Let (U, d) be a metric space and $K \subset U$. We say that K is compact if each open covering $\{G_\alpha,\ \alpha \in L\}$ of K admits a finite sub-covering. That is, if $K \subset \cup_{\alpha \in L} G_\alpha$, and G_α is open $\forall \alpha \in L$, then there exist $\alpha_1, \alpha_2, ..., \alpha_n \in L$ such that $K \subset \cup_{i=1}^n G_{\alpha_i}$.*

Teorema 1.47. *Let (U, d) be a metric space. Let $K \subset U$ where K is compact. Then K is closed.*

Proof. Let us show that K^c is open. Let $u \in K^c$. For convenience, let us generically denote in this proof $V_r(u) = V(u, r)$.

For each $v \in K$ we have $d(u, v) > 0$. Define $r_v = d(u, v)/2$. Thus,

$$V(u, r_v) \cap V(v, r_v) = \emptyset, \forall v \in K. \tag{7}$$

Observe that

$$\cup_{v \in K} V(v, r_v) \supset K.$$

since K is compact, there exist $v_1, ..., v_n \in K$ such that

$$K \subset \cup_{i=1}^n V(v_i, r_{v_i}). \tag{8}$$

Define $r_0 = \min\{r_{v_1}, ..., r_{v_n}\}$, thus

$$V(u, r_0) \subset V(u, r_{v_i}), \forall i \in \{1, ..., n\},$$

so that from this and (7) we get

$$V(u, r_0) \cap V(v_i, r_{v_i}) = \emptyset, \forall i \in \{1, 2, ..., n\}.$$

Hence,

$$V(u, r_0) \cap (\cup_{i=1}^n V(v_i, r_{v_i})) = \emptyset.$$

From this and (8) we obtain, $V(u, r_0) \cap K = \emptyset$, that is $V(u, r_0) \subset K^c$. Therefore u is an interior point of K^c and being $u \in K^c$ arbitrary, K^c is open so that K is closed.

The proof is complete.

Teorema 1.48. *Let (U, d) be a metric space. If $F \subset K \subset U$, K is compact and F is closed, then F is compact.*

Proof. Let $\{G_\alpha, \alpha \in L\}$ be an open covering of F, that is

$$F \subset \cup_{\alpha \in L} G_\alpha.$$

Observe that $U = F \cup F^c \supset K$, and thus,

$$F^c \cup (\cup_{\alpha \in L} G_\alpha) \supset K.$$

Therefore, since F^c is open $\{F^c,\ G_\alpha,\ \alpha \in L\}$ is an open covering of K, and since K is compact, there exist $\alpha_1, ..., \alpha_n \in L$ such that

$$F^c \cup G_{\alpha_1} \cup ... \cup G_{\alpha_n} \supset K \supset F.$$

Therefore

$$G_{\alpha_1} \cup ... \cup G_{\alpha_n} \supset F,$$

so that F is compact.

Exercício 1.49.

Show that if F is closed and K is compact, then $F \cap K$ is compact.

Teorema 1.50. *If $\{K_\alpha,\ \alpha \in L\}$ is a collection of compact sets in a metric space (U, d) such that the intersection of each finite sub-collection is non-empty, then*

$$\cap_{\alpha \in L} K_\alpha \neq \emptyset.$$

Proof. Suppose, to obtain contradiction, that

$$\cap_{\alpha \in L} K_\alpha = \emptyset. \tag{9}$$

Fix $\alpha_0 \in L$ and denote $L_1 = L \setminus \{\alpha_0\}$. From (9) we obtain

$$K_{\alpha_0} \cap (\cap_{\alpha \in L_1} K_\alpha) = \emptyset.$$

Hence

$$K_{\alpha_0} \subset (\cap_{\alpha \in L_1} K_\alpha)^c,$$

that is,

$$K_{\alpha_0} \subset \cup_{\alpha \in L_1} K_\alpha^c.$$

Since, K_{α_0} is compact and K_α^c is open, $\forall \alpha \in L$, there exist $\alpha_1, \alpha_2, ..., \alpha_n \in L_1$ such that

$$K_{\alpha_0} \subset \cup_{j=1}^n K_{\alpha_j}^c = \left(\cap_{j=1}^n K_{\alpha_j}\right)^c,$$

therefore,

$$K_{\alpha_0} \cap \left(\cap_{j=1}^n K_{\alpha_j}\right) = K_{\alpha_0} \cap K_{\alpha_1} \cap ... \cap K_{\alpha_n} = \emptyset,$$

which contradicts the hypotheses. The proof is complete.

Corolário 1.51. *Let (U,d) be a metric space. If $\{K_n,\ n \in \mathbb{N}\} \subset U$ is a sequence of compact non-empty sets such that $K_n \supset K_{n+1}, \forall n \in \mathbb{N}$ then $\cap_{n=1}^{\infty} K_n \neq \emptyset$.*

Teorema 1.52. *Let (U,d) be a metric space. If $E \subset K \subset U$, K is compact and E is infinite, then E has at least one limit point in K.*

Proof. Suppose, to obtain contradiction, that no point of K is a limit point of E. Then, for each $u \in K$ there exists $r_u > 0$ such that $V(u, r_u)$ has at most one point of E, namely, u if $u \in E$. Observe that $\{V(u, r_u),\ u \in K\}$ is an open covering of K and therefore of E. Since each $V(u, r_u)$ has at most one point of E which is infinite, no finite sub-covering (relating the open cover in question), covers E, and hence no finite sub-covering covers $K \supset E$, which contradicts the fact that K is compact. This completes the proof.

Teorema 1.53. *Let $\{I_n\}$ be a sequence of bounded closed non-empty real intervals, such that $I_n \supset I_{n+1}, \forall n \in \mathbb{N}$. Thus, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.*

Proof. Let $I_n = [a_n, b_n]$ and let $E = \{a_n,\ n \in \mathbb{N}\}$. Thus, $E \neq \emptyset$ and E is upper bounded by b_1. Let $x = \sup E$.

Observe that, given $m, n \in \mathbb{N}$ we have that

$$a_n \leq a_{n+m} \leq b_{n+m} \leq b_m,$$

so that

$$\sup_{n \in \mathbb{N}} a_n \leq b_m, \forall m \in \mathbb{N},$$

that is, $x \leq b_m, \forall m \in \mathbb{N}$. Hence,

$$a_m \leq x \leq b_m, \forall m \in \mathbb{N},$$

that is,

$$x \in [a_m, b_m],\ \forall m \in \mathbb{N},$$

so that $x \in \cap_{m=1}^{\infty} I_m$.

The proof is complete.

Teorema 1.54. *Let $I = [a,b] \subset \mathbb{R}$ be a bounded closed non-empty real interval. Under such hypotheses, I is compact.*

Proof. Observe that if $x, y \in [a,b]$ then $|x - y| \leq (b-a)$.. Suppose there exists an open covering of I, denoted by $\{G_\alpha,\ \alpha \in L\}$ for which there is no finite sub-covering.

Let $c = (a+b)/2$. Thus, either $[a,c]$ or $[c,b]$ has no finite sub-covering related to $\{G_\alpha,\ \alpha \in L\}$. Denote such an interval by I_1. Dividing I_1 into two connected closed sub-intervals of same size, we get an interval I_2 for which there is no finite sub-covering related to $\{G_\alpha,\ \alpha \in L\}$.

Proceeding in this fashion, we may obtain a sequence of closed intervals $\{I_n\}$ such that

1. $I_n \supset I_{n+1}, \forall n \in \mathbb{N}$.
2. No finite sub-collection of $\{G_\alpha, \alpha \in L\}$ covers I_n, $\forall n \in \mathbb{N}$.
3. If $x, y \in I_n$ then $|x - y| \leq 2^{-n}(b-a)$.

From the last theorem, there exists $x^* \in \mathbb{R}$ such that $x^* \in \cap_{n=1}^{\infty} I_n \subset I \subset \cup_{\alpha \in L} G_\alpha$. Hence, there exists $\alpha_0 \in L$ such that $x^* \in G_{\alpha_0}$. Since G_{α_0} is open, there exists $r > 0$ such that

$$V_r(x^*) = (x^* - r, x^* + r) \subset G_{\alpha_0}.$$

Choose $n_0 \in \mathbb{N}$ such that

$$2^{-n_0}(b-a) < r/2.$$

Hence, since $x^* \in I_{n_0}$, if $y \in I_{n_0}$ then from item 3 above, $|y - x^*| \leq 2^{-n_0}(b-a) < r/2$, that is $y \in V_r(x^*) \subset G_{\alpha_0}$.

Therefore

$$y \in I_{n_0} \Rightarrow y \in G_{\alpha_0},$$

so that $I_{n_0} \subset G_{\alpha_0}$, which contradicts the item 2 above indicated.

The proof is complete.

Teorema 1 (Heine-Borel)**.** *Let $E \subset \mathbb{R}$, thus the following three properties are equivalent.*

1. *E is closed and bounded.*
2. *E is compact.*
3. *Each infinite subset of E has a limit point of E.*

Proof.

- 1 implies 2: Let $E \subset \mathbb{R}$ be a closed and bounded. Thus, since E is bounded there exists $[a, b]$ a bounded closed interval such that $E \subset [a, b]$. From the last theorem $[a, b]$ is compact and since E is closed, from Theorem 1.48 we may infer that E is compact.

- 2 implies 3: This follows from Theorem1.52.

- 3 implies 1: We prove the contrapositive, that is, the negation of 1 implies the negation of 3.

  The negation of 1 is: E is not bounded or E is not closed. If $E \subset \mathbb{R}$ is not bounded, choosing $x_1 \in E$, for each $n \in \mathbb{N}$ there exists $x_{n+1} \in E$ such that $|x_{n+1}| > n + |x_n| \geq n$. Hence $\{x_n\}$ has no limit points so that we have got the negation of 3.

  On the other hand, suppose E is not closed. Thus there exists $x_0 \in \mathbb{R}$ such that $x_0 \in E'$ and $x_0 \notin E$.

  Since $x_0 \in E'$, for each $n \in \mathbb{N}$ there exists $x_n \in E$ such that $|x_n - x_0| < 1/n$ ($x_n \in V_{1/n}(x_0)$).

  Let $y \in E$, we are going to show that y is not limit point $\{x_n\} \subset E$. Observe that,

$$\begin{aligned} |x_n - y| &\geq |x_0 - y| - |x_n - x_0| \\ &> |x_0 - y| - 1/n \\ &> |x_0 - y|/2 > 0 \end{aligned} \tag{10}$$

  for all n sufficiently big.

  Hence y is not a limit point of $\{x_n\}$, $\forall y \in E$. Therefore $\{x_n\} \subset E$ is a infinite set with no limit point in E.

In any case, we have got the negation of 3. This completes the proof.

Exercícios 1.55.

1. *Let U be a metric space and let $\{K_\lambda,\ \lambda \in L\}$ be a collection of compact sets, such that $K_\lambda \subset U$, $\forall \lambda \in L$. Prove that $\cap_{\lambda \in L} K_\lambda$ is compact.*

2. *Let U be a metric space and let $K_1, K_2, ..., K_n \subset U$ be compact sets. Prove that*

$$\cup_{j=1}^{n} K_j$$

is compact.

Teorema 1.56 (Weierstrass)**.** *Any real set which is bounded and infinite has a limit point in $\mathbb{R}$.*

Proof. Let $E \subset \mathbb{R}$ be a bounded infinite set. Thus, there exists $r > 0$ such that $E \subset [-r, r] = I_r$. Since E is infinite and I_r is compact, from Theorem 1.52, E has a limit point in $I_r \subset \mathbb{R}$. The proof is complete. □

1.11 Separable metric spaces

Definição 1.57 (Separable metric space)**.** *Let (V, d) be a metric space. We say that a set $M \subset V$ is dense in V if*

$$\overline{M} = M \cup M' = V.$$

If V has dense subset which is countable, we say that V is separable.

Exemplo 1.58. *$V = \mathbb{R}$ is separable. Indeed $\mathbb{Q}$, the set of rational number, is dense in $\mathbb{R}$ and countable.*

Exemplo 1.59. *The space l^∞ is not separable.*

In fact, let $A \subset l^\infty$ be the set of all real sequences whose entries are only 0 and 1.

From elementary analysis it is well known that A is non-countable.

Let $0 < \varepsilon < 1/4$.

Suppose, to obtain contradiction, that

$$B = \{u_n\}_{n \in \mathbb{N}} \subset l^\infty$$

is dense in l^∞.

Thus, for each $v \in A$, we may select a $n_v \in \mathbb{N}$ such that

$$d(v, u_{n_v}) < \varepsilon.$$

Let $v, w \in A$ be such that $v \neq w$. Therefore,

$$d(v, w) = 1,$$

so that

$$d(v, w) \leq d(v, u_{n_v}) + d(u_{n_v}, w),$$

and thus

$$\begin{aligned} d(u_{n_v}, w) &\geq 1 - d(v, u_{n_v}) \\ &\geq 1 - \varepsilon \\ &> 1 - 1/4 \\ &= 3/4 \\ &> \varepsilon. \end{aligned} \tag{11}$$

So to summarize, if $v \neq w$, then

$$u_{n_v} \neq u_{n_w}.$$

Let $T : A \to B$, where

$$T(v) = u_{n_v}.$$

Thus T is a bijection on $I_m(T) \subset B$.
Therefore,

$$A \sim Im(T) \sim B \sim \mathbb{N}.$$

This contradicts A to be non-countable.
So, we may infer that l^∞ is non-separable.

Exercício 1.60. *Let $1 \leq p < +\infty$. Prove that l^p is separable.*

1.12 Complete metric spaces

Definição 1.61. *Let $\{u_n\} \subset V$ where (V, d) is a metric space.*
We say that $u_0 \in V$ is the limit of $\{u_n\}$ as n goes to infinity (∞), if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$, then

$$d(u_n, u_0) < \varepsilon.$$

In such a a case we denote,

$$\lim_{n \to \infty} u_n = u_0,$$

or

$$u_n \to u_0, \text{ as } n \to \infty$$

and say that the sequence $\{u_n\}$ is convergent.

Exercício 1.62. *Let (V, d) be a metric space and let $\{u_n\} \subset V$ be a convergent sequence.*
Show that $\{u_n\}$ is bounded.

Definição 1.63 (Cauchy sequence)**.** *Let (V, d) be a metric space and let $\{u_n\} \subset V$ be a sequence. We say that such a sequence is a Cauchy one as for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $m, n > n_0$, then*

$$d(u_n, u_m) < \varepsilon.$$

Exercício 1.64. *Let (V, d) be a metric space and let $\{u_n\} \subset V$ be a Cauchy sequence.*
Show that $\{u_n\}$ is bounded.

Definição 1.65 (Complete metric space)**.** *Let (V, d) be a metric space. We say that V is complete as each Cauchy sequence in V converges to an element of V.*

Exercício 1.66. *Let (V, d) be a metric space and let $M \subset V$.*

1. *Show that $u \in \overline{M}$ if, and only if, there exists a sequence $\{u_n\} \subset M$ such that*

$$u_n \to u, \text{ as } n \to \infty.$$

2. *Show that M is closed if, and only if, the following property is valid:*
   If $\{u_n\} \subset M$ and $u_n \to u$, then $u \in M$.

Exercício 1.67. *Let (V, d) be a metric space and let $M \subset V$. Show that M is complete if, and only if, M is closed in V.*

Exercício 1.68. *Prove that $\mathbb{R}^n$ is complete (with the Euclidean metric).*

Exercício 1.69. *Prove that c is complete, where c is the space of complex convergent sequences.*

Exercício 1.70. *Let (V, d) be a metric space where $V = C([a, b])$ whit the metric*

$$d(u, v) = \int_a^b |u(x) - v(x)| \, dx, \ \forall u, v \in V.$$

Show that V is not complete.

2 Completion of a metric space

Definição 2.1 (Isometries, isometric spaces)**.** *Let (V, d) and (V_1, d_1) be metric space. A function $T : V \to V_1$ is said to be a isometry of V in V_1 as*

$$d_1(T(u), T(v)) = d(u, v), \ \forall u, v \in V.$$

If there exists an isometry betwween V and V_1, we say that V and V_1 are isometric.

Teorema 2.2 (Completion)**.** *Let (V, d) be a metric space which is not complete.*

Under such hypotheses, there exists a metric space $(\hat{V}, \hat{d})$ such that V is isometric to a sub-space W of $\hat{V}$ which is dense in $\hat{V}$. Moreover, $\hat{V}$ is complete.

Proof. 1. First part: Construction of $\hat{V}$.

Let $\{u_n\}$ and $\{u'_n\}$ be Cauchy sequences in V.

We define a relation of equivalence in the set of Cauchy sequences in V by declaring

$$\{u_n\} \sim \{u'_n\}$$

as

$$\lim_{n \to \infty} d(u_n, u'_n) = 0.$$

Let

$$\hat{V} = \{\widehat{\{u_n\}}\ :\ \{u_n\} \text{ is a Cauchy sequence in } V\},$$

and where

$$\widehat{\{u_n\}} = \{\{u_n'\} \subset V,\ \text{ such that } \{u_n'\}\text{is a Cauchy sequence and } \{u_n'\} \sim \{u_n\}\}.$$

For $u = \{u_n\}$ and $v = \{v_n\} \subset V$ define

$$\hat{d}(\hat{u}, \hat{v}) = \lim_{n\to\infty} d(u_n, v_n).$$

We shall show that this metric is well defined.

Let $\{u_n\} \in \hat{u}$ and $\{v_n\} \in \hat{v}$.

Observe that for, $m, n \in \mathbb{N}$ we have that

$$d(u_n, v_n) \leq d(u_n, u_m) + d(u_m, v_m) + d(v_m, v_n),$$

that is,

$$d(u_n, v_n) - d(u_m, v_m) \leq d(u_n, u_m) + d(v_m, v_n) \to 0,\ \text{as } m, n \to \infty.$$

Inverting the roles of m and n, we may similarly obtain:

$$d(u_m, v_m) - d(u_n, v_n) \leq d(u_n, u_m) + d(v_m, v_n) \to 0,\ \text{as } m, n \to \infty$$

so that

$$|d(u_m, v_m) - d(u_n, v_n)| \to 0,\ \text{as } m, n \to \infty.$$

Therefore $\{d(u_n, v_n)\}$ is a real Cauchy sequence and thus it is convergent.

Let $\{u_n'\} \in \hat{u}$ and $\{v_n'\} \in \hat{v}$.

Hence,

$$d(u_n, v_n) \leq d(u_n, u_n') + d(u_n', v_n') + d(v_n', v_n),$$

so that

$$d(u_n, v_n) - d(u_n', v_n') \leq d(u_n, u_n') + d(v_n, v_n') \to 0,\ \text{as } n \to \infty.$$

Inverting the roles of the sequences, we get

$$d(u_n', v_n') - d(u_n, v_n) \leq d(u_n, u_n') + d(v_n, v_n') \to 0,\ \text{as } n \to \infty,$$

so that

$$|d(u_n, v_n) - d(u_n', v_n')| \to 0,\ \text{as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} d(u_n, v_n) = \lim_{n\to\infty} d(u_n', v_n'),\ \forall \{u_n'\} \in \hat{u},\ \{v_n'\} \in \hat{v}.$$

Therefore, the candidate to metric in question is well defined.

Furthermore,

$$\begin{aligned} \hat{d}(\hat{u},\hat{v})=0 \quad &\Leftrightarrow \quad \lim_{n\to\infty} d(u_n,v_n)=0 \\ &\Leftrightarrow \quad \{v_n\}\in\hat{u} \\ &\Leftrightarrow \quad \hat{u}=\hat{v}. \end{aligned} \tag{12}$$

Finally, let $\hat{u},\hat{v}$ and $\hat{w}\in\hat{V}$.

Thus,

$$\begin{aligned} \hat{d}(\hat{u},\hat{w}) &= \lim_{n\to\infty} d(u_n,w_n) \\ &\leq \lim_{n\to\infty}[d(u_n,v_n)+d(v_n,w_n)] \\ &= \lim_{n\to\infty} d(u_n,v_n)+\lim_{n\to\infty} d(v_n,w_n) \\ &= d(\hat{u},\hat{v})+d(\hat{v},\hat{w}). \end{aligned} \tag{13}$$

From this we may conclude that $\hat{d}$ is in fact a metric for $\hat{V}$.

2. We shall show now that V is isometric a dense subspace of $\hat{V}$.

   Let $b\in V$. Define $\hat{b}$ by its representative

$$\{u_n\}=\{b,b,b,\ldots\}.$$

   Define

$$W=\{\hat{b}=\{b,\widehat{b,b,\ldots}\}\ :\ b\in V\}.$$

   Define also $T:V\to W$ by

$$T(b)=\hat{b}=\{b,\widehat{b,b,\ldots}\}.$$

   Thus,

$$\hat{d}(\hat{b},\hat{c})=\lim_{n\to\infty} d(b,c)=d(b,c).$$

   Therefore, T is a isometry.

   We are going to show that W is dense in $\hat{V}$.

   Let $\hat{u}\in\hat{V}$ and $\{u_n\}\in\hat{u}$. Let $\varepsilon>0$.

   Since $\{u_n\}$ is a Cauchy sequence, there exists $n_0\in\mathbb{N}$ such that if $m,n>n_0$, then

$$d(u_n,u_m)<\varepsilon/2.$$

   Choose $N>n_0$.

   Thus, $d(u_n,u_N)<\varepsilon/2, \forall n>n_0$.

Observe that

$$\hat{u}_N = \{u_N, \widehat{u_N, u_N}, \ldots\} \in W,$$

and

$$\hat{d}(\hat{u}, \hat{u}_N) = \lim_{n\to\infty} d(u_n, u_N) \leq \varepsilon/2 < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we may conclude that

$$\hat{u} \in W' \cup W, \forall \hat{u} \in \hat{V},$$

that is, W is dense in $\hat{V}$.

3. Now we shall show that $\hat{V}$ complete.

Let $\{\hat{u}_n\}$ be a sequence in $\hat{V}$.

Since W is dense in $\hat{V}$, for each $n \in \mathbb{N}$ there exists $\hat{z}_n \in W$ such that

$$\hat{d}(\hat{u}, \hat{z}_n) < \frac{1}{n}.$$

Observe that

$$\begin{aligned} \hat{d}(\hat{z}_n, \hat{z}_m) &\leq \hat{d}(\hat{z}_m, \hat{u}_m) + d(\hat{u}_m, \hat{u}_n) + \hat{d}(\hat{u}_n, \hat{z}_n) \\ &\leq \frac{1}{n} + d(\hat{u}_m, \hat{u}_n) + \frac{1}{m}. \end{aligned} \tag{14}$$

Let $\varepsilon > 0$ (a new one). Hence, there exists $n_0 \in \mathbb{N}$ such that if $m, n > n_0$, then

$$\hat{d}(\hat{u}_n, \hat{u}_m) < \frac{\varepsilon}{3}.$$

Thus if

$$m, n > \max\left\{\frac{3}{\varepsilon}, n_0\right\},$$

then

$$\hat{d}(\hat{z}_n, \hat{z}_m) < \varepsilon.$$

Therefore, $\{\hat{z}_m\}$ is a Cauchy sequence and since $T : V \to W$ is a isometry it follows that

$$\{z_m\} = \{T^{-1}(\hat{z}_m)\},$$

is also a Cauchy one.

Let $\hat{u}$ be the class of $\{z_m\}$. We will show that

$$\lim_{n\to\infty} \hat{d}(\hat{u}_n, \hat{u}) = 0.$$

Indeed,

$$\begin{aligned}\hat{d}(\hat{u}_n, \hat{u}) &\leq \hat{d}(\hat{u}_n, \hat{z}_n) + \hat{d}(\hat{z}_n, \hat{u}) \\ &\leq \frac{1}{n} + \lim_{m\to\infty} d(z_n, z_m). \qquad (15)\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \hat{d}(\hat{u}_n, \hat{u}) = 0.$$

Thus, $\hat{V}$ is complete.

The proof is complete.

3 Other topics on compactness in metric spaces

Definição 3.1 (Diameter of a set)**.** *Let (U, d) be a metric space and $A \subset U$. We define the diameter of A, denoted by $diam(A)$ by*

$$diam(A) = \sup\{d(u, v) \mid u, v \in A\}.$$

Definição 3.2. *Let (U, d) be a metric space. We say that $\{F_k\} \subset U$ is a nested sequence of sets if*

$$F_1 \supset F_2 \supset F_3 \supset \ldots.$$

Teorema 3.3. *If (U, d) is a complete metric space then every nested sequence of non-empty closed sets $\{F_k\}$ such that*

$$\lim_{k\to+\infty} diam(F_k) = 0$$

has non-empty intersection, that is

$$\cap_{k=1}^{\infty} F_k \neq \emptyset.$$

Proof. Suppose $\{F_k\}$ is a nested sequence and $\lim_{k\to\infty} \text{diam}(F_k) = 0$. For each $n \in \mathbb{N}$ select $u_n \in F_n$. Suppose given $\varepsilon > 0$. Since

$$\lim_{n\to\infty} \text{diam}(F_n) = 0,$$

there exists $N \in \mathbb{N}$ such that if $n \geq N$ then

$$\text{diam}(F_n) < \varepsilon.$$

Thus if $m, n > N$ we have $u_m, u_n \in F_N$ so that

$$d(u_n, u_m) < \varepsilon.$$

Hence $\{u_n\}$ is a Cauchy sequence. Being U complete, there exists $u \in U$ such that

$$u_n \to u \text{ as } n \to \infty.$$

Choose $m \in \mathbb{N}$. We have that $u_n \in F_m, \forall n > m$, so that

$$u \in \bar{F}_m = F_m.$$

Since $m \in \mathbb{N}$ is arbitrary we obtain

$$u \in \cap_{m=1}^{\infty} F_m.$$

The proof is complete.

Teorema 3.4. *Let (U,d) be a metric space. If $A \subset U$ is compact then it is closed and bounded.*

Proof. We have already proved that A is closed. Suppose, to obtain contradiction that A is not bounded. Thus for each $K \in \mathbb{N}$ there exists $u, v \in A$ such that

$$d(u,v) > K.$$

Observe that

$$A \subset \cup_{u \in A} B_1(u).$$

Since A is compact there exists $u_1, u_2, ..., u_n \in A$ such that

$$A = \subset \cup_{k=1}^{n} B_1(u_k).$$

Define

$$R = \max\{d(u_i, u_j) \mid i, j \in \{1, ..., n\}\}.$$

Choose $u, v \in A$ such that

$$d(u,v) > R + 2. \tag{16}$$

Observe that there exist $i, j \in \{1, ..., n\}$ such that

$$u \in B_1(u_i), \; v \in B_1(u_j).$$

Thus

$$\begin{aligned} d(u,v) &\leq d(u,u_i) + d(u_i,u_j) + d(u_j,v) \\ &\leq 2 + R, \end{aligned} \tag{17}$$

which contradicts (16). This completes the proof.

Definição 3.5 (Relative compactness)**.** *In a metric space (U,d) a set $A \subset U$ is said to be relatively compact if $\overline{A}$ is compact.*

Definição 3.6 (ε - nets)**.** *Let (U,d) be a metric space. A set $N \subset U$ is sat to be a ε-net with respect to a set $A \subset U$ if for each $u \in A$ there exists $v \in N$ such that*

$$d(u,v) < \varepsilon.$$

Definição 3.7. *Let (U, d) be a metric space. A set $A \subset U$ is said to be totally bounded if for each $\varepsilon > 0$ there exists a finite ε-net with respect to A.*

Proposição 3.8. *Let (U, d) be a metric space. If $A \subset U$ is totally bounded then it is bounded.*

Proof. Choose $u, v \in A$. Let $\{u_1, ..., u_n\}$ be the $1-net$ with respect to A. Define

$$R = \max\{d(u_i, u_j) \mid i, j \in \{1, ..., n\}\}.$$

Observe that there exist $i, j \in \{1, ..., n\}$ such that

$$d(u, u_i) < 1, \; d(v, u_j) < 1.$$

Thus

$$\begin{aligned} d(u, v) &\leq d(u, u_i) + d(u_i, u_j) + d(u_j, v) \\ &\leq R + 2. \end{aligned} \tag{18}$$

Since $u, v \in A$ are arbitrary, A is bounded.

Teorema 3.9. *Let (U, d) be a metric space. If from each sequence $\{u_n\} \subset A$ we can select a convergent subsequence $\{u_{n_k}\}$ then A is totally bounded.*

Proof. Suppose, to obtain contradiction, that A is not totally bounded. Thus there exists $\varepsilon_0 > 0$ such that there exists no ε_0-net with respect to A. Choose $u_1 \in A$, hence $\{u_1\}$ is not a ε_0-net, that is, there exists $u_2 \in A$ such that

$$d(u_1, u_2) > \varepsilon_0.$$

Again $\{u_1, u_2\}$ is not a ε_0-net for A, so that there exists $u_3 \in A$ such that

$$d(u_1, u_3) > \varepsilon_0 \text{ and } d(u_2, u_3) > \varepsilon_0.$$

Proceeding in this fashion we can obtain a sequence $\{u_n\}$ such that

$$d(u_n, u_m) > \varepsilon_0, \; \text{if } m \neq n. \tag{19}$$

Clearly we cannot extract a convergent subsequence of $\{u_n\}$, otherwise such a subsequence would be Cauchy contradicting (19). The proof is complete.

Definição 3.10 (Sequentially compact sets)**.** *Let (U, d) be a metric space. A set $A \subset U$ is said to be sequentially compact if for each sequence $\{u_n\} \subset A$ there exist a subsequence $\{u_{n_k}\}$ and $u \in A$ such that*

$$u_{n_k} \to u, \; \textit{as } k \to \infty.$$

Teorema 3.11. *A subset A of a metric space (U, d) is compact if and only if it is sequentially compact.*

Proof. Suppose A is compact. By Proposition 6.8 A is countably compact. Let $\{u_n\} \subset A$ be a sequence. We have two situations to consider.

1. $\{u_n\}$ has infinitely many equal terms, that is in this case we have

$$u_{n_1} = u_{n_2} = ... = u_{n_k} = ... = u \in A.$$

Thus the result follows trivially.

2. $\{u_n\}$ has infinitely many distinct terms. In such a case, being A countably compact, $\{u_n\}$ has a limit point in A, so that there exist a subsequence $\{u_{n_k}\}$ and $u \in A$ such that

$$u_{n_k} \to u, \text{ as } k \to \infty.$$

In both cases we may find a subsequence converging to some $u \in A$.

Thus A is sequentially compact.

Conversely suppose A is sequentially compact, and suppose $\{G_\alpha,\ \alpha \in L\}$ is an open cover of A. For each $u \in A$ define

$$\delta(u) = \sup\{r \mid B_r(u) \subset G_\alpha, \text{ for some } \alpha \in L\}.$$

First we prove that $\delta(u) > 0, \forall u \in A$. Choose $u \in A$. Since $A \subset \cup_{\alpha \in L} G_\alpha$, there exists $\alpha_0 \in L$ such that $u \in G_{\alpha_0}$. Being G_{α_0} open, there exists $r_0 > 0$ such that $B_{r_0}(u) \subset G_{\alpha_0}$.

Thus

$$\delta(u) \geq r_0 > 0.$$

Now define δ_0 by

$$\delta_0 = \inf\{\delta(u) \mid u \in A\}.$$

Therefore, there exists a sequence $\{u_n\} \subset A$ such that

$$\delta(u_n) \to \delta_0 \text{ as } n \to \infty.$$

Since A is sequentially compact, we may obtain a subsequence $\{u_{n_k}\}$ and $u_0 \in A$ such that

$$\delta(u_{n_k}) \to \delta_0 \text{ and } u_{n_k} \to u_0,$$

as $k \to \infty$. Therefore, we may find $K_0 \in \mathbb{N}$ such that if $k > K_0$ then

$$d(u_{n_k}, u_0) < \frac{\delta(u_0)}{4}. \tag{20}$$

We claim that

$$\delta(u_{n_k}) \geq \frac{\delta(u_0)}{4}, \text{ if } k > K_0.$$

To prove the claim, suppose

$$z \in B_{\frac{\delta(u_0)}{4}}(u_{n_k}), \forall k > K_0,$$

(observe that in particular from (20)

$$u_0 \in B_{\frac{\delta(u_0)}{4}}(u_{n_k}), \forall k > K_0).$$

Since
$$\frac{\delta(u_0)}{2} < \delta(u_0),$$
there exists some $\alpha_1 \in L$ such that
$$B_{\frac{\delta(u_0)}{2}}(u_0) \subset G_{\alpha_1}.$$
However, since
$$d(u_{n_k}, u_0) < \frac{\delta(u_0)}{4}, \text{ if } k > K_0,$$
we obtain
$$B_{\frac{\delta(u_0)}{2}}(u_0) \supset B_{\frac{\delta(u_0)}{4}}(u_{n_k}), \text{ if } k > K_0,$$
so that
$$\delta(u_{n_k}) \geq \frac{\delta(u_0)}{4}, \forall k > K_0.$$
Therefore
$$\lim_{k\to\infty} \delta(u_{n_k}) = \delta_0 \geq \frac{\delta(u_0)}{4}.$$
Choose $\varepsilon > 0$ such that
$$\delta_0 > \varepsilon > 0.$$
From the last theorem since A it is sequentially compact, it is totally bounded. For the $\varepsilon > 0$ chosen above, consider an ε-net contained in A (the fact that the ε-net may be chosen contained in A is also a consequence of last theorem) and denote it by N that is,
$$N = \{v_1, ..., v_n\} \in A.$$
Since $\delta_0 > \varepsilon$, there exists
$$\alpha_1, ..., \alpha_n \in L$$
such that
$$B_\varepsilon(v_i) \subset G_{\alpha_i}, \forall i \in \{1, ..., n\},$$
considering that
$$\delta(v_i) \geq \delta_0 > \varepsilon > 0, \forall i \in \{1, ..., n\}.$$
For $u \in A$, since N is an ε-net we have
$$u \in \cup_{i=1}^n B_\varepsilon(v_i) \subset \cup_{i=1}^n G_{\alpha_i}.$$
Since $u \in U$ is arbitrary we obtain
$$A \subset \cup_{i=1}^n G_{\alpha_i}.$$
Thus
$$\{G_{\alpha_1}, ..., G_{\alpha_n}\}$$
is a finite subcover for A of
$$\{G_\alpha, \ \alpha \in L\}.$$
Hence A is compact.

The proof is complete;

Teorema 3.12. *Let (U, d) be a metric space. Thus $A \subset U$ is relatively compact if and only if for each sequence in A, we may select a convergent subsequence.*

Proof. Suppose A is relatively compact. Thus $\overline{A}$ is compact so that from the last Theorem, $\overline{A}$ is sequentially compact.

Thus from each sequence in $\overline{A}$ we may select a subsequence which converges to some element of $\overline{A}$. In particular, for each sequence in $A \subset \overline{A}$ we may select a subsequence that converges to some element of $\overline{A}$.

Conversely, suppose that for each sequence in A we may select a convergent subsequence. It suffices to prove that $\overline{A}$ is sequentially compact. Let $\{v_n\}$ be a sequence in $\overline{A}$. Since A is dense in $\overline{A}$, there exists a sequence $\{u_n\} \subset A$ such that

$$d(u_n, v_n) < \frac{1}{n}.$$

From the hypothesis we may obtain a subsequence $\{u_{n_k}\}$ and $u_0 \in \overline{A}$ such that

$$u_{n_k} \to u_0, \text{ as } k \to \infty.$$

Thus,

$$v_{n_k} \to u_0 \in \overline{A}, \text{ as } k \to \infty.$$

Therefore $\overline{A}$ is sequentially compact so that it is compact.

Teorema 3.13. *Let (U, d) be a metric space.*

1. *If $A \subset U$ is relatively compact then it is totally bounded.*

2. *If (U, d) is a complete metric space and $A \subset U$ is totaly bounded then A is relatively compact.*

Proof. 1. Suppose $A \subset U$ is relatively compact. From the last theorem, from each sequence in A we can extract a convergent subsequence. From Theorem 3.9 A is totally bounded.

2. Let (U, d) be a metric space and let A be a totally bounded subset of U.

   Let $\{u_n\}$ be a sequence in A. Since A is totally bounded for each $k \in \mathbb{N}$ we find a ε_k-net where $\varepsilon_k = 1/k$, denoted by N_k where

   $$N_k = \{v_1^{(k)}, v_2^{(k)}, ..., v_{n_k}^{(k)}\}.$$

   In particular for $k = 1$ $\{u_n\}$ is contained in the 1-net N_1. Thus at least one ball of radius 1 of N_1 contains infinitely many points of $\{u_n\}$. Let us select a subsequence $\{u_{n_k}^{(1)}\}_{k \in \mathbb{N}}$ of this infinite set (which is contained in a ball of radius 1). Similarly, we may select a subsequence here just partially relabeled $\{u_{n_l}^{(2)}\}_{l \in \mathbb{N}}$ of $\{u_{n_k}^{(1)}\}$ which is contained in one of the balls of the $\frac{1}{2}$-net. Proceeding in this fashion for each $k \in \mathbb{N}$ we may find a subsequence denoted by $\{u_{n_m}^{(k)}\}_{m \in \mathbb{N}}$ of the original sequence contained in a ball of radius $1/k$.

Now consider the diagonal sequence denoted by $\{u_{n_k}^{(k)}\}_{k\in\mathbb{N}} = \{z_k\}$. Thus

$$d(z_n, z_m) < \frac{2}{k}, \text{ if } m, n > k,$$

that is $\{z_k\}$ is a Cauchy sequence, and since (U, d) is complete, there exists $u \in U$ such that

$$z_k \to u \text{ as } k \to \infty.$$

From Theorem 3.12, A is relatively compact.
The proof is complete.

4 The Arzela-Ascoli Theorem

In this section we present a classical result in analysis, namely the Arzela-Ascoli theorem.

Definição 4.1 (Equi-continuity). *Let $\mathcal{F}$ be a collection of complex functions defined on a metric space (U, d). We say that $\mathcal{F}$ is equicontinuous if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in U$ and $d(u, v) < \delta$ then*

$$|f(u) - f(v)| < \varepsilon, \forall f \in \mathcal{F}.$$

Furthermore, we say that $\mathcal{F}$ is point-wise bounded if for each $u \in U$ there exists $M(u) \in \mathbb{R}$ such that

$$|f(u)| < M(u), \forall f \in \mathcal{F}.$$

Teorema 4.2 (Arzela-Ascoli). *Suppose $\mathcal{F}$ is a point-wise bounded equicontinuous collection of complex functions defined on a metric space (U, d). Also suppose that U has a countable dense subset E. Thus, each sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence that converges uniformly on every compact subset of U.*

Proof. Let $\{u_n\}$ be a countable dense set in (U, d). By hypothesis, $\{f_n(u_1)\}$ is a bounded sequence, therefore it has a convergent subsequence, which is denoted by $\{f_{n_k}(u_1)\}$. Let us denote

$$f_{n_k}(u_1) = \tilde{f}_{1,k}(u_1), \forall k \in \mathbb{N}.$$

Thus there exists $g_1 \in \mathbb{C}$ such that

$$\tilde{f}_{1,k}(u_1) \to g_1, \text{ as } k \to \infty.$$

Observe that $\{f_{n_k}(u_2)\}$ is also bounded and also it has a convergent subsequence, which similarly as above we will denote by $\{\tilde{f}_{2,k}(u_2)\}$. Again there exists $g_2 \in \mathbb{C}$ such that

$$\tilde{f}_{2,k}(u_1) \to g_1, \text{ as } k \to \infty.$$

$$\tilde{f}_{2,k}(u_2) \to g_2, \text{ as } k \to \infty.$$

Proceeding in this fashion for each $m \in \mathbb{N}$ we may obtain $\{\tilde{f}_{m,k}\}$ such that

$$\tilde{f}_{m,k}(u_j) \to g_j, \text{ as } k \to \infty, \forall j \in \{1, ..., m\},$$

where the set $\{g_1, g_2, ..., g_m\}$ is obtained as above. Consider the diagonal sequence

$$\{\tilde{f}_{k,k}\},$$

and observe that the sequence

$$\{\tilde{f}_{k,k}(u_m)\}_{k>m}$$

is such that

$$\tilde{f}_{k,k}(u_m) \to g_m \in \mathbb{C}, \text{ as } k \to \infty, \forall m \in \mathbb{N}.$$

Therefore we may conclude that from $\{f_n\}$ we may extract a subsequence also denoted by

$$\{f_{n_k}\} = \{\tilde{f}_{k,k}\}$$

which is convergent in

$$E = \{u_n\}_{n \in \mathbb{N}}.$$

Now suppose $K \subset U$, being K compact. Suppose given $\varepsilon > 0$. From the equi-continuity hypothesis there exists $\delta > 0$ such that if $u, v \in U$ and $d(u, v) < \delta$ we have

$$|f_{n_k}(u) - f_{n_k}(v)| < \frac{\varepsilon}{3}, \forall k \in \mathbb{N}.$$

Observe that

$$K \subset \cup_{u \in K} B_{\frac{\delta}{2}}(u),$$

and being K compact we may find $\{\tilde{u}_1, ..., \tilde{u}_M\}$ such that

$$K \subset \cup_{j=1}^{M} B_{\frac{\delta}{2}}(\tilde{u}_j).$$

Since E is dense in U, there exists

$$v_j \in B_{\frac{\delta}{2}}(\tilde{u}_j) \cap E, \forall j \in \{1, ..., M\}.$$

Fixing $j \in \{1, ..., M\}$, from $v_j \in E$ we obtain that

$$\lim_{k \to \infty} f_{n_k}(v_j)$$

exists as $k \to \infty$. Hence there exists $K_{0_j} \in \mathbb{N}$ such that if $k, l > K_{0_j}$ then

$$|f_{n_k}(v_j) - f_{n_l}(v_j)| < \frac{\varepsilon}{3}.$$

Pick $u \in K$, thus

$$u \in B_{\frac{\delta}{2}}(\tilde{u}_{\hat{j}})$$

for some $\hat{j} \in \{1, ..., M\}$, so that

$$d(u, v_{\hat{j}}) < \delta.$$

Therefore if

$$k, l > \max\{K_{0_1}, ..., K_{0_M}\},$$

then

$$\begin{aligned} |f_{n_k}(u) - f_{n_l}(u)| &\leq |f_{n_k}(u) - f_{n_k}(v_{\hat{j}})| + |f_{n_k}(v_{\hat{j}}) - f_{n_l}(v_{\hat{j}})| \\ &\quad + |f_{n_l}(v_{\hat{j}}) - f_{n_l}(u)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{21}$$

Since $u \in K$ is arbitrary, we conclude that $\{f_{n_k}\}$ is uniformly Cauchy on K.

The proof is complete.

5 Topological Vector Spaces

5.1 Introduction

The main objective of this chapter is to present an outline of the basic tools of analysis necessary to develop the subsequent chapters. We assume the reader has a background in linear algebra and elementary real analysis at an undergraduate level. The main references for this chapter are the excellent books on functional analysis, Rudin [6], Bachman and Narici [1] and Reed and Simon [5]. All proofs are developed in details.

5.2 Vector spaces

We denote by $\mathbb{F}$ a scalar field. In practice this is either $\mathbb{R}$ or $\mathbb{C}$, the set of real or complex numbers.

Definição 5.1 (Vector spaces)**.** *A vector space over $\mathbb{F}$ is a set which we will denote by U whose elements are called vectors, for which are defined two operations namely, addition denoted by $(+) : U \times U \to U$, and scalar multiplication denoted by $(\cdot) : \mathbb{F} \times U \to U$, so that the following relations are valid*

1. $u + v = v + u, \forall u, v \in U$,
2. $u + (v + w) = (u + v) + w, \forall u, v, w \in U$,
3. *there exists a vector denoted by θ such that $u + \theta = u$, $\forall u \in U$,*
4. *for each $u \in U$, there exists a unique vector denoted by $-u$ such that $u + (-u) = \theta$,*
5. $\alpha \cdot (\beta \cdot u) = (\alpha \cdot \beta) \cdot u, \forall \alpha, \beta \in \mathbb{F},\ u \in U$,
6. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v, \forall \alpha \in \mathbb{F},\ u, v \in U$,
7. $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u, \forall \alpha, \beta \in \mathbb{F},\ u \in U$,

8. $1 \cdot u = u, \forall u \in U$.

Observação 5.2. *From now on we may drop the dot $(\cdot)$ in scalar multiplications and denote $\alpha \cdot u$ simply as αu.*

Definição 5.3 (Vector subspace)**.** *Let U be a vector space. A set $V \subset U$ is said to be a vector subspace of U if V is also a vector space with the same operations as those of U. If $V \neq U$ we say that V is a proper subspace of U.*

Definição 5.4 (Finite dimensional space)**.** *A vector space is said to be of finite dimension if there exists fixed $u_1, u_2, ..., u_n \in U$ such that for each $u \in U$ there are corresponding $\alpha_1, ..., \alpha_n \in \mathbb{F}$ for which*

$$u = \sum_{i=1}^{n} \alpha_i u_i. \tag{22}$$

Definição 5.5 (Topological spaces)**.** *A set U is said to be a topological space if it is possible to define a collection σ of subsets of U called a topology in U, for which are valid the following properties:*

1. *$U \in \sigma$,*
2. *$\emptyset \in \sigma$,*
3. *if $A \in \sigma$ and $B \in \sigma$ then $A \cap B \in \sigma$, and*
4. *arbitrary unions of elements in σ also belong to σ.*

Any $A \in \sigma$ is said to be an open set.

Observação 5.6. *When necessary, to clarify the notation, we shall denote the vector space U endowed with the topology σ by (U, σ).*

Definição 5.7 (Closed sets)**.** *Let U be a topological space. A set $A \subset U$ is said to be closed if $U \setminus A$ is open. We also denote $U \setminus A = A^c = \{u \in U \mid u \notin A\}$.*

Observação 5.8. *For any sets $A, B \subset U$ we denote*

$$A \setminus B = \{u \in A \mid u \notin B\}.$$

Proposição 5.9. *For closed sets we have the following properties:*

1. *U and $\emptyset$ are closed,*
2. *If A and B are closed sets then $A \cup B$ is closed,*
3. *Arbitrary intersections of closed sets are closed.*

Proof. 1. Since $\emptyset$ is open and $U = \emptyset^c$, by Definition 5.7 U is closed. Similarly, since U is open and $\emptyset = U \setminus U = U^c$, $\emptyset$ is closed.

2. A, B closed implies that A^c and B^c are open, and by Definition 5.5, $A^c \cap B^c$ is open, so that $A \cup B = (A^c \cap B^c)^c$ is closed.

3. Consider $A = \cap_{\lambda \in L} A_\lambda$, where L is a collection of indices and A_λ is closed, $\forall \lambda \in L$. We may write $A = (\cup_{\lambda \in L} A_\lambda^c)^c$ and since A_λ^c is open $\forall \lambda \in L$ we have, by Definition 5.5, that A is closed.

Definição 5.10 (Closure)**.** *Given $A \subset U$ we define the closure of A, denoted by $\bar{A}$, as the intersection of all closed sets that contain A.*

Observação 5.11. *From Proposition 5.9 Item 3 we have that $\bar{A}$ is the smallest closed set that contains A, in the sense that, if C is closed and $A \subset C$ then $\bar{A} \subset C$.*

Definição 5.12 (Interior)**.** *Given $A \subset U$ we define its interior, denoted by A°, as the union of all open sets contained in A.*

Observação 5.13. *It is not difficult to prove that if A is open then $A = A^\circ$.*

Definição 5.14 (Neighborhood)**.** *Given $u_0 \in U$ we say that $\mathcal{V}$ is a neighborhood of u_0 if such a set is open and contains u_0. We denote such neighborhoods by $\mathcal{V}_{u_0}$.*

Proposição 5.15. *If $A \subset U$ is a set such that for each $u \in A$ there exists a neighborhood $\mathcal{V}_u \ni u$ such that $\mathcal{V}_u \subset A$, then A is open.*

Proof. This follows from the fact that $A = \cup_{u \in A} \mathcal{V}_u$ and any arbitrary union of open sets is open.

Definição 5.16 (Function)**.** *Let U and V be two topological spaces. We say that $f : U \to V$ is a function if f is a collection of pairs $(u, v) \in U \times V$ such that for each $u \in U$ there exists only one $v \in V$ such that $(u, v) \in f$.*

In such a case we denote

$$v = f(u).$$

Definição 5.17 (Continuity at a point)**.** *A function $f : U \to V$ is continuous at $u \in U$ if for each neighborhood $\mathcal{V}_{f(u)} \subset V$ of $f(u)$ there exists a neighborhood $\mathcal{V}_u \subset U$ of u such that $f(\mathcal{V}_u) \subset \mathcal{V}_{f(u)}$.*

Definição 5.18 (Continuous function)**.** *A function $f : U \to V$ is continuous if it is continuous at each $u \in U$.*

Proposição 5.19. *A function $f : U \to V$ is continuous if and only if $f^{-1}(\mathcal{V})$ is open for each open $\mathcal{V} \subset V$, where*

$$f^{-1}(\mathcal{V}) = \{u \in U \mid f(u) \in \mathcal{V}\}. \tag{23}$$

Proof. Suppose $f^{-1}(\mathcal{V})$ is open whenever $\mathcal{V} \subset V$ is open. Pick $u \in U$ and any open $\mathcal{V}$ such that $f(u) \in \mathcal{V}$. Since $u \in f^{-1}(\mathcal{V})$ and $f(f^{-1}(\mathcal{V})) \subset \mathcal{V}$, we have that f is continuous at $u \in U$. Since $u \in U$ is arbitrary we have that f is continuous. Conversely, suppose f is continuous and pick $\mathcal{V} \subset V$ open. If $f^{-1}(\mathcal{V}) = \emptyset$ we are done, since $\emptyset$ is open. Thus, suppose $u \in f^{-1}(\mathcal{V})$, since f is continuous, there exists $\mathcal{V}_u$ a neighborhood of u such that $f(\mathcal{V}_u) \subset \mathcal{V}$. This means $\mathcal{V}_u \subset f^{-1}(\mathcal{V})$ and therefore, from Proposition 5.15, $f^{-1}(\mathcal{V})$ is open.

Definição 5.20. *We say that (U,σ) is a Hausdorff topological space if, given u_1, $u_2 \in U$, $u_1 \neq u_2$, there exists $\mathcal{V}_1$, $\mathcal{V}_2 \in \sigma$ such that*

$$u_1 \in \mathcal{V}_1 \; , \; u_2 \in \mathcal{V}_2 \text{ and } \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset. \tag{24}$$

Definição 5.21 (Base). *A collection $\sigma' \subset \sigma$ is said to be a base for σ if every element of σ may be represented as a union of elements of σ'.*

Definição 5.22 (Local base). *A collection $\hat{\sigma}$ of neighborhoods of a point $u \in U$ is said to be a local base at u if each neighborhood of u contains a member of $\hat{\sigma}$.*

Definição 5.23 (Topological vector spaces). *A vector space endowed with a topology, denoted by (U,σ), is said to be a topological vector space if and only if*

1. *Every single point of U is a closed set,*
2. *The vector space operations (addition and scalar multiplication) are continuous with respect to σ.*

More specifically, addition is continuous if, given $u, v \in U$ and $\mathcal{V} \in \sigma$ such that $u + v \in \mathcal{V}$ then there exists $\mathcal{V}_u \ni u$ and $\mathcal{V}_v \ni v$ such that $\mathcal{V}_u + \mathcal{V}_v \subset \mathcal{V}$. On the other hand, scalar multiplication is continuous if given $\alpha \in \mathbb{F}$, $u \in U$ and $\mathcal{V} \ni \alpha \cdot u$, there exists $\delta > 0$ and $\mathcal{V}_u \ni u$ such that, $\forall \beta \in \mathbb{F}$ satisfying $|\beta - \alpha| < \delta$ we have $\beta \mathcal{V}_u \subset \mathcal{V}$.

Given (U,σ), let us associate with each $u_0 \in U$ and $\alpha_0 \in \mathbb{F}$ ($\alpha_0 \neq 0$) the functions $T_{u_0} : U \to U$ and $M_{\alpha_0} : U \to U$ defined by

$$T_{u_0}(u) = u_0 + u \tag{25}$$

and

$$M_{\alpha_0}(u) = \alpha_0 \cdot u. \tag{26}$$

The continuity of such functions is a straightforward consequence of the continuity of vector space operations (addition and scalar multiplication). It is clear that the respective inverse maps, namely T_{-u_0} and M_{1/α_0} are also continuous. So if $\mathcal{V}$ is open then $u_0 + \mathcal{V}$, that is $(T_{-u_0})^{-1}(\mathcal{V}) = T_{u_0}(\mathcal{V}) = u_0 + \mathcal{V}$ is open. By analogy $\alpha_0 \mathcal{V}$ is open. Thus σ is completely determined by a local base, so that the term local base will be understood henceforth as a local base at θ. So to summarize, a local base of a topological vector space is a collection Ω of neighborhoods of θ, such that each neighborhood of θ contains a member of Ω.

Now we present some simple results, namely:

Proposição 5.24. *If $A \subset U$ is open, then $\forall u \in A$ there exists a neighborhood $\mathcal{V}$ of θ such that $u + \mathcal{V} \subset A$*

Proof. Just take $\mathcal{V} = A - u$.

Proposição 5.25. *Given a topological vector space (U,σ), any element of σ may be expressed as a union of translates of members of Ω, so that the local base Ω generates the topology σ.*

Proof. Let $A \subset U$ open and $u \in A$. $\mathcal{V} = A - u$ is a neighborhood of θ and by definition of local base, there exists a set $\mathcal{V}_{\Omega u} \subset \mathcal{V}$ such that $\mathcal{V}_{\Omega_u} \in \Omega$. Thus, we may write

$$A = \cup_{u \in A}(u + \mathcal{V}_{\Omega_u}). \tag{27}$$

5.3 Some properties of topological vector spaces

In this section we study some fundamental properties of topological vector spaces. We start with the following proposition:

Proposição 5.26. *Any topological vector space U is a Hausdorff space.*

Proof. Pick $u_0, u_1 \in U$ such that $u_0 \neq u_1$. Thus $\mathcal{V} = U \setminus \{u_1 - u_0\}$ is an open neighborhood of zero. As $\theta + \theta = \theta$, by the continuity of addition, there exist $\mathcal{V}_1$ and $\mathcal{V}_2$ neighborhoods of θ such that

$$\mathcal{V}_1 + \mathcal{V}_2 \subset \mathcal{V} \tag{28}$$

define $\mathcal{U} = \mathcal{V}_1 \cap \mathcal{V}_2 \cap (-\mathcal{V}_1) \cap (-\mathcal{V}_2)$, thus $\mathcal{U} = -\mathcal{U}$ (symmetric) and $\mathcal{U} + \mathcal{U} \subset \mathcal{V}$ and hence

$$u_0 + \mathcal{U} + \mathcal{U} \subset u_0 + \mathcal{V} \subset U \setminus \{u_1\} \tag{29}$$

so that

$$u_0 + v_1 + v_2 \neq u_1, \quad \forall v_1, v_2 \in \mathcal{U}, \tag{30}$$

or

$$u_0 + v_1 \neq u_1 - v_2, \quad \forall v_1, v_2 \in \mathcal{U}, \tag{31}$$

and since $\mathcal{U} = -\mathcal{U}$

$$(u_0 + \mathcal{U}) \cap (u_1 + \mathcal{U}) = \emptyset. \tag{32}$$

Definição 5.27 (Bounded sets)**.** *A set $A \subset U$ is said to be bounded if to each neighborhood of zero $\mathcal{V}$ there corresponds a number $s > 0$ such that $A \subset t\mathcal{V}$ for each $t > s$.*

Definição 5.28 (Convex sets)**.** *A set $A \subset U$ such that*

$$\text{if } u, v \in A \text{ then } \lambda u + (1 - \lambda)v \in A, \quad \forall \lambda \in [0, 1], \tag{33}$$

is said to be convex.

Definição 5.29 (Locally convex spaces)**.** *A topological vector space U is said to be locally convex if there is a local base Ω whose elements are convex.*

Definição 5.30 (Balanced sets)**.** *A set $A \subset U$ is said to be balanced if $\alpha A \subset A$, $\forall \alpha \in \mathbb{F}$ such that $|\alpha| \leq 1$.*

Teorema 5.31. *In a topological vector space U we have:*

1. *Every neighborhood of zero contains a balanced neighborhood of zero,*
2. *Every convex neighborhood of zero contains a balanced convex neighborhood of zero.*

Proof. 1. Suppose $\mathcal{U}$ is a neighborhood of zero. From the continuity of scalar multiplication, there exist $\mathcal{V}$ (neighborhood of zero) and $\delta > 0$, such that $\alpha\mathcal{V} \subset \mathcal{U}$ whenever $|\alpha| < \delta$. Define $\mathcal{W} = \cup_{|\alpha|<\delta}\alpha\mathcal{V}$, thus $\mathcal{W} \subset \mathcal{U}$ is a balanced neighborhood of zero.

2. Suppose $\mathcal{U}$ is a convex neighborhood of zero in U. Define

$$A = \{\cap\alpha\mathcal{U} \mid \alpha \in \mathbb{C},\ |\alpha| = 1\}. \tag{34}$$

As $0 \cdot \theta = \theta$ (where $\theta \in U$ denotes the zero vector) from the continuity of scalar multiplication there exists $\delta > 0$ and there is a neighborhood of zero $\mathcal{V}$ such that if $|\beta| < \delta$ then $\beta\mathcal{V} \subset \mathcal{U}$. Define $\mathcal{W}$ as the union of all such $\beta\mathcal{V}$. Thus $\mathcal{W}$ is balanced and $\alpha^{-1}\mathcal{W} = \mathcal{W}$ as $|\alpha| = 1$, so that $\mathcal{W} = \alpha\mathcal{W} \subset \alpha\mathcal{U}$, and hence $\mathcal{W} \subset A$, which implies that the interior A° is a neighborhood of zero. Also $A^\circ \subset \mathcal{U}$. Since A is an intersection of convex sets, it is convex and so is A°. Now will show that A° is balanced and complete the proof. For this, it suffices to prove that A is balanced. Choose r and β such that $0 \leq r \leq 1$ and $|\beta| = 1$. Then

$$r\beta A = \cap_{|\alpha|=1} r\beta\alpha\mathcal{U} = \cap_{|\alpha|=1} r\alpha\mathcal{U}. \tag{35}$$

Since $\alpha\mathcal{U}$ is a convex set that contains zero, we obtain $r\alpha\mathcal{U} \subset \alpha\mathcal{U}$, so that $r\beta A \subset A$, which completes the proof.

Proposição 5.32. *Let U be a topological vector space and $\mathcal{V}$ a neighborhood of zero in U. Given $u \in U$, there exists $r \in \mathbb{R}^+$ such that $\beta u \in \mathcal{V}$, $\forall \beta$ such that $|\beta| < r$.*

Proof. Observe that $u + \mathcal{V}$ is a neighborhood of $1 \cdot u$, then by the continuity of scalar multiplication, there exists $\mathcal{W}$ neighborhood of u and $r > 0$ such that

$$\beta\mathcal{W} \subset u + \mathcal{V}, \forall \beta \text{ such that } |\beta - 1| < r, \tag{36}$$

so that

$$\beta u \in u + \mathcal{V}, \tag{37}$$

or

$$(\beta - 1)u \in \mathcal{V}, \text{ where } |\beta - 1| < r, \tag{38}$$

and thus

$$\hat{\beta} u \in \mathcal{V}, \forall \hat{\beta} \text{ such that } |\hat{\beta}| < r, \tag{39}$$

which completes the proof.

Corolário 5.33. *Let $\mathcal{V}$ be a neighborhood of zero in U, if $\{r_n\}$ is a sequence such that $r_n > 0, \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = \infty$, then $U \subset \cup_{n=1}^{\infty} r_n\mathcal{V}$.*

Proof. Let $u \in U$, then $\alpha u \in \mathcal{V}$ for any α sufficiently small, from the last proposition $u \in \frac{1}{\alpha}\mathcal{V}$. As $r_n \to \infty$ we have that $r_n > \frac{1}{\alpha}$ for n sufficiently big, so that $u \in r_n\mathcal{V}$, which completes the proof.

Proposição 5.34. *Suppose $\{\delta_n\}$ is sequence such that $\delta_n \to 0$, $\delta_n < \delta_{n-1}$, $\forall n \in \mathbb{N}$ and $\mathcal{V}$ a bounded neighborhood of zero in U, then $\{\delta_n\mathcal{V}\}$ is a local base for U.*

Proof. Let $\mathcal{U}$ be a neighborhood of zero, as $\mathcal{V}$ is bounded, there exists $t_0 \in \mathbb{R}^+$ such that $\mathcal{V} \subset t\mathcal{U}$ for any $t > t_0$. As $\lim_{n\to\infty} \delta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\delta_n < \frac{1}{t_0}$, so that $\delta_n\mathcal{V} \subset \mathcal{U}, \forall n$ such that $n \geq n_0$.

Definição 5.35 (Convergence in topological vector spaces)**.** *Let U be a topological vector space. We say $\{u_n\}$ converges to $u_0 \in U$, if for each neighborhood $\mathcal{V}$ of u_0 then, there exists $N \in \mathbb{N}$ such that*

$$u_n \in \mathcal{V}, \forall n \geq N.$$

Definição 5.36 (Dense set)**.** *Let (V, σ) be a topological vector space (T.V.E.). Let $A, B \subset V$. We say that A is dense in B as*

$$B \subset \overline{A}.$$

Definição 5.37. *We say that topological vector space V is separable as it has a dense and countable set.*

5.4 Nets and convergence

Definição 5.38. *A directed system is a set of indices I, with a order relation $\prec$, which satisfies the following properties:*

1. *If α, $\beta \in I$, then there exists $\gamma \in I$ such that*

$$\alpha \prec \gamma \text{ and } \beta \prec \gamma.$$

2. *$\prec$ é a partial order relation.*

Definição 5.39 (Net)**.** *Let (V, σ) be a topological space. A net in (V, σ) is a function defined on a directed system I with range in V, where we denote such a net by*

$$\{u_\alpha\}_{\alpha \in I},$$

and where

$$u_\alpha \in V, \ \forall \alpha \in I.$$

Definição 5.40 (Convergent net)**.** *Let (V, σ) be a topological space and let $\{u_\alpha\}_{\alpha\in I}$ be a net in V.*

We say that such a net converges $u \in V$ as for each neighborhood $W \in \sigma$ of u there exists $\beta \in I$ such that if $\alpha \succ \beta$, then

$$u_\alpha \in W.$$

Definição 5.41. *Let (V, σ) be a topological space and let $\{u_\alpha\}_{\alpha \in I}$ be a net in V. We say that $u \in V$ is a cluster point of the net in question as for each neighborhood $W \in \sigma$ of u and each $\beta \in I$, there exists $\alpha \succ \beta$ such that*

$$u_\alpha \in W.$$

Definição 5.42 (Limit point)**.** *Let (V, σ) be a topological space e let $A \subset V$. We say that $u \in V$ is a limit point of A as for each neighborhood $W \in \sigma$ of u, there exists $v \in W \cap A$ such that $v \neq u$.*

Teorema 5.43. *Let (V, d) be a topological space and let $A \subset V$.*

Under such hypotheses,

$$\overline{A} = A \cup A'$$

where A' denotes the set of limit points of A.

Proof. Let $u \in A \cup A'$.

If $u \in A$, then $u \in \overline{A}$.

Thus, suppose $u \in A' \setminus A$.

Hence, for each neighborhood $W \in \sigma$ of u, there exists $u_w \in A \setminus \{u\}$ such that $u_w \in W$.

Denote by I the set of all neighborhoods of u, partially order by the relation

$$W_1 \prec W_2 \Leftrightarrow W_2 \subset W_1.$$

From the exposed above we may obtain a net $\{u_w\}_{W \in I}$ such that

$$u_w \to u.$$

Assume, to obtain contradiction, that $u \notin \overline{A}$.

Hence $u \in \overline{A}^c$ which is an open set. Since $u_w \to u$, there exists $W_1 \in I$ such that if $W_2 \succ W_1$, then

$$u_{w_2} \in \overline{A}^c.$$

In particular $u_{w_2} \in A^c$, if $W_2 \succ W_1$, which contradicts

$$u_{w_2} \in A \setminus \{u\}.$$

Summarizing,

$$u \in \overline{A}, \ \forall u \in A \cup A'.$$

Therefore,

$$A \cup A' \subset \overline{A}. \tag{40}$$

Reciprocally, suppose $u \in \overline{A}$.

If $u \in A$, then $u \in A \cup A'$.

Suppose, to obtain contradiction, that $u \notin A$ and $u \notin A'$.

Thus there exists a neighborhood $W \in \sigma$ of u such that

$$W \cap A = \emptyset.$$

Thus, $A \subset W^c$ and W^c is closed, so that

$$\overline{A} \subset W^c.$$

From this and $u \in W$ we get

$$u \notin \overline{A},$$

a contradiction. Therefore $u \in A$ or $u \in A'$, $\forall u \in \overline{A}$.

Thus

$$\overline{A} \subset A \cup A'. \tag{41}$$

From (40) e (41), we obtain

$$\overline{A} = A \cup A'.$$

This complete the proof. □

Teorema 5.44. *Let (V_1, σ_1) and (V_2, σ_2) be topological spaces.*

Let $f : V_1 \to V_2$ be a function.

Let $u \in V_1$. Thus f is continuous at u if, and only if, for each net $\{u_\alpha\}_{\alpha \in I} \subset V_1$ such that $u_\alpha \to u$, we have that

$$f(u_\alpha) \to f(u).$$

Proof. Suppose f is continuous at u. Let $\{u_\alpha\}_{\alpha \in I}$ be a net such that

$$u_\alpha \to u.$$

Let $W_{f(u)} \in \sigma_2$ be such that $f(u) \in W_{f(u)}$.

From the hypotheses, there exists $V_u \in \sigma_1$ such that $u \in V_u$ and

$$f(V_u) \subset W_{f(u)}.$$

From $u_\alpha \to u$, there exists $\beta \in I$ such that if $\alpha \succ \beta$, then

$$u_\alpha \in V_u.$$

Therefore,

$$f(u_\alpha) \in W_{f(u)}, \text{ if } \alpha \succ \beta.$$

Since $W_{f(u)}$ is arbitrary, it follows that

$$f(u_\alpha) \to f(u).$$

Reciprocally, suppose

$$f(u_\alpha) \to f(u)$$

whenever

$$u_\alpha \to u.$$

Suppose, to obtain contradiction, that f is not continuous at u.

Thus there exists $W_{f(u)} \in \sigma_2$ such that $f(u) \in W_{f(u)}$ and so that for each neighborhood $W \in \sigma_1$ of u there exists $u_W \in W$ such that

$$f(u_W) \notin W_{f(u)}.$$

Denote by I the set of all neighborhoods of u, partially ordered by the relation

$$W_1 \prec W_2 \Leftrightarrow W_2 \subset W_1.$$

Thus, the net $\{u_W\}_{W \in I}$ is such that

$$u_W \to u.$$

However

$$f(u_W) \notin V_{f(u)}, \; \forall W \in I.$$

Hence, $\{f(u_W)\}$ does not converges to $f(u)$, which contradicts the reciprocal hypothesis.

The proof is complete.

□

6 Compactness in topological vector spaces

We start this section with the definition of open covering.

Definição 6.1 (Open Covering)**.** *Given $B \subset U$ we say that $\{\mathcal{O}_\alpha, \quad \alpha \in \mathrm{A}\}$ is a covering of B if $B \subset \cup_{\alpha \in \mathrm{A}} \mathcal{O}_\alpha$. If $\mathcal{O}_\alpha$ is open $\forall \alpha \in \mathrm{A}$ then $\{\mathcal{O}_\alpha\}$ is said to be an open covering of B.*

Definição 6.2 (Compact Sets)**.** *A set $B \subset U$ is said to be compact if each open covering of B has a finite sub-covering. More explicitly, if $B \subset \cup_{\alpha \in \mathrm{A}} \mathcal{O}_\alpha$, where $\mathcal{O}_\alpha$ is open $\forall \alpha \in \mathrm{A}$, then, there exist $\alpha_1, ..., \alpha_n \in \mathrm{A}$ such that $B \subset \mathcal{O}_{\alpha_1} \cup ... \cup \mathcal{O}_{\alpha_n}$, for some n, a finite positive integer.*

Teorema 6.3. *Let (V, σ) be a topological space. Let $K \subset V$.*

Under such hypotheses, K is compact if, and only if, each net $\{u_\alpha\}_{\alpha \in I} \subset K$ has a limit point in K.

Proof. Suppose K is compact. Let $\{u_\alpha\}_{\alpha \in I} \subset K$ be a net with infinite distinct terms (otherwise the result is immediate).

Denote $E = \{u_\alpha\}_{\alpha \in I}$. Suppose, to obtain contradiction, that no point of K is a limit point of E.

Hence, for each $u \in K$, there exists a neighborhood W_u of u such that

$$W_u \cap E = \emptyset,$$

or

$$W_u \cap E = \{u\} \text{ if } u \in E.$$

In any case each W_u has no more than one point of E.

Observe that $\cup_{u \in K} W_u \supset K$. Since K is compact, there exist $u_1, \ldots, u_n \in K$ such that

$$E \subset K \subset \cup_{j=1}^n W_{u_j}.$$

From this we may conclude that E has no more than n distinct elements, which contradicts E to have infinity distinct terms.

Reciprocally, suppose that each net $\{u_\alpha\}_{\alpha \in I} \subset K$ has at least one limit point in K.

Suppose, to obtain contradiction K is not compact.

Thus there exists an open covering $\{G_\alpha,\ \alpha \in L\}$ of K which admits no finite sub-covering.

Denote by F the finite sub-collections of $\{G_\alpha,\ \alpha \in L\}$.

Hence, for a $W \in F$ we may select a $u_W \notin W$ where $u_W \in K$.

Let us partially order F through the relation

$$W_1 \prec W_2 \Leftrightarrow W_1 \subset W_2.$$

From the hypotheses, the net $\{u_W\}_{W \in F}$ has a limit point $u \in K$.

Observe that

$$u \in K \subset \cup_{\alpha \in L} G_\alpha.$$

Thus, there exists $\alpha_0 \in L$ such that

$$u \in G_{\alpha_0}.$$

Since u is a limit point of $\{u_W\}_{W \in F} \subset K$, there exists $W_1 \succ G_{\alpha_0}$ such that

$$u_{W_1} \in G_{\alpha_0} \subset W_1.$$

This contradicts $u_{W_1} \notin W_1$. Therefore, K is compact.

The proof is complete. □

Proposição 6.4. *A compact subset of a Hausdorff space is closed.*

Proof. Let U be a Hausdorff space and consider $A \subset U$, A compact. Given $x \in A$ and $y \in A^c$, there exist open sets $\mathcal{O}_x$ and $\mathcal{O}_y^x$ such that $x \in \mathcal{O}_x$, $y \in \mathcal{O}_y^x$ and $\mathcal{O}_x \cap \mathcal{O}_y^x = \emptyset$. It is clear that $A \subset \cup_{x \in A} \mathcal{O}_x$ and since A is compact, we may find $\{x_1, x_2, ..., x_n\}$ such that $A \subset \cup_{i=1}^n \mathcal{O}_{x_i}$. For the selected $y \in A^c$ we have $y \in \cap_{i=1}^n \mathcal{O}_y^{x_i}$ and $(\cap_{i=1}^n \mathcal{O}_y^{x_i}) \cap (\cup_{i=1}^n \mathcal{O}_{x_i}) = \emptyset$. Since $\cap_{i=1}^n \mathcal{O}_y^{x_i}$ is open, and y is an arbitrary point of A^c we have that A^c is open, so that A is closed, which completes the proof. The next result is very useful.

Teorema 6.5. *Let $\{K_\alpha,\ \alpha \in L\}$ be a collection of compact subsets of a Hausdorff topological vector space U, such that the intersection of every finite sub-collection (of $\{K_\alpha,\ \alpha \in L\}$) is non-empty.*

Under such hypotheses

$$\cap_{\alpha \in L} K_\alpha \neq \emptyset.$$

Proof. Fix $\alpha_0 \in L$. Suppose, to obtain contradiction that

$$\cap_{\alpha \in L} K_\alpha = \emptyset.$$

That is,

$$K_{\alpha_0} \cap [\cap_{\alpha \in L}^{\alpha \neq \alpha_0} K_\alpha] = \emptyset.$$

Thus,

$$\cap_{\alpha \in L}^{\alpha \neq \alpha_0} K_\alpha \subset K_{\alpha_0}^c,$$

so that

$$K_{\alpha_0} \subset [\cap_{\alpha \in L}^{\alpha \neq \alpha_0} K_\alpha]^c,$$

$$K_{\alpha_0} \subset [\cup_{\alpha \in L}^{\alpha \neq \alpha_0} K_\alpha^c].$$

However K_{α_0} is compact and K_α^c is open, $\forall \alpha \in L$.

Hence, there exist $\alpha_1, ..., \alpha_n \in L$ such that

$$K_{\alpha_0} \subset \cup_{i=1}^n K_{\alpha_i}^c.$$

From this we may infer that

$$K_{\alpha_0} \cap [\cap_{i=1}^n K_{\alpha_i}] = \emptyset,$$

which contradicts the hypotheses.

The proof is complete.

Proposição 6.6. *Let U be a topological Hausdorff space and let $A \subset B$ where A is closed and B is compact.*

Under such hypotheses, A is compact.

Proof. Consider $\{\mathcal{O}_\alpha, \alpha \in L\}$ an open cover of A. Thus $\{A^c, \ \mathcal{O}_\alpha, \ \alpha \in L\}$ is a cover of U, so that it is a cover of B. As B is compact, there exist $\alpha_1, \alpha_2, ..., \alpha_n$ such that $A^c \cup (\cup_{i=1}^n \mathcal{O}_{\alpha_i}) \supset B \supset A$, so that $\{\mathcal{O}_{\alpha_i}, \ \ i \in \{1, ..., n\}\}$ covers A. From this we may infer that A is compact. The proof is complete. □

Definição 6.7 (Countably compact sets)**.** *A set A is said to be countably compact if every infinite subset of A has a limit point in A.*

Proposição 6.8. *Every compact subset of a Hausdorff topological space U is countably compact.*

Proof. Let B an infinite subset of A compact and suppose B has no limit point in A, so that there is no any limit point. Choose a countable infinite set $\{x_1, x_2, x_3, ...\} \subset B$ and define $F = \{x_1, x_2, x_3, ...\}$. It is clear that F has no limit point. Thus for each $n \in \mathbb{N}$, there exist $\mathcal{O}_n$ open such that $\mathcal{O}_n \cap F = \{x_n\}$. Also, for each $x \in A \setminus F$, there exist $\mathcal{O}_x$ such that $x \in \mathcal{O}_x$ and $\mathcal{O}_x \cap F = \emptyset$. Thus $\{\mathcal{O}_x, \ x \in A \setminus F; \ \mathcal{O}_1, \mathcal{O}_2, ...\}$ is an open cover of A without a finite subcover, which contradicts the fact that A is compact.

6.1 A note on convexity in topological vector spaces

Definição 6.9. *Let (V, σ) be a topological vector space. Let $A \subset V$ be such that $A \neq \emptyset$.*

We define the convex hull of A, denoted by $Conv(A)$, as

$$Conv(A) = \left\{ \sum_{k=1}^n \lambda_k u_k \ : \ n \in \mathbb{N}, \ \lambda_k \geq 0, \ u_k \in A, \ \forall k \in \{1, \ldots, n\} \ and \ \sum_{k=1}^n \lambda_k = 1 \right\}.$$

Teorema 6.10. *let (V, σ) be a topological vector space. Let $A \subset V$ be such that $A \neq \emptyset$.*

Under such hypotheses, $Conv(A)$ is convex.

Proof. Let $u, v \in Conv(A)$ and let $\lambda \in [0,1]$. Thus, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$u = \sum_{k=1}^{n_1} \lambda_k u_k \text{ and } v = \sum_{k=1}^{n_2} \tilde{\lambda}_k v_k,$$

where $u_k \in A$ e $\lambda_k \geq 0$, $\forall k \in \{1, \ldots n_1\}$ and also $\sum_{k=1}^{n_1} \lambda_k = 1$.

Moreover, $v_k \in A$, $\tilde{\lambda}_k \geq 0$, $\forall k \in \{1, \ldots n_2\}$ and $\sum_{k=1}^{n_2} \tilde{\lambda}_k = 1$.

Thus, we have that

$$\lambda u + (1-\lambda)v = \sum_{k=1}^{n_1} \lambda\lambda_k u_k + \sum_{k=1}^{n_2} (1-\lambda)\tilde{\lambda}_k v_k,$$

where

$$\lambda\lambda_k \geq 0, u_k \in A, \ \forall k \in \{1, \ldots, n_1\}$$

e

$$(1-\lambda)\tilde{\lambda}_k \geq 0, v_k \in A, \ \forall k \in \{1, \ldots, n_2\}$$

so that

$$\sum_{k=1}^{n_1} \lambda\lambda_k + \sum_{k=1}^{n_2} (1-\lambda)\tilde{\lambda}_k = \lambda + (1-\lambda) = 1.$$

Therefore,

$$\lambda u + (1 - \lambda v) \in Conv(A), \ \forall u, v \in Conv(A), \ \lambda \in [0,1].$$

Hence, $Conv(A)$ is convex.

The proof is complete.

$\square$

Teorema 6.11. *Let (V, σ) be a topological vector space. Let $A \subset V$ be such that $A \neq \emptyset$.*

Under such hypotheses, A if, and only if, $Conv(A) = A$.

Proof. Suppose that A is convex. We shall prove that

$$A = B_n \equiv \left\{ \sum_{k=1}^{n} \lambda_k u_k \ : \ \lambda_k \geq 0, \ u_k \in A, \ \forall k \in \{1, \ldots, n\} \text{ and } \sum_{k=1}^{n} \lambda_k = 1 \right\}, \forall n \in \mathbb{N}.$$

We shall do it by induction on n.

Observe that for $n = 1$ and $n = 2$, from the convexity of A we obtain $A = B_1$ and $A = B_2$.

Let $n \in \mathbb{N}$. Suppose $A = B_n$. We are going to prove that $A = B_{n+1}$ which will complete the induction.

Clearly $B_n \subset B_{n+1}$, so that $A \subset B_{n+1}$.

Reciprocally, let $u \in B_{n+1}$. Thus, there exist $u_1, \ldots, u_{n+1} \in A$ and $\lambda_1, \ldots \lambda_{n+1}$ such that $\lambda_k \geq 0, \forall k \in \{1, \ldots, n+1\}$, $\sum_{k=1}^{n+1} \lambda_k = 1$, and

$$u = \sum_{k=1}^{n+1} \lambda_k u_k.$$

With no loss in generality, assume $0 < \lambda_{n+1} < 1$ (otherwise the conclusion is immediate). Thus,

$$\lambda_1 + \cdots + \lambda_n = (1 - \lambda_{n+1}) > 0.$$

Hence,

$$\frac{\lambda_1}{1 - \lambda_{n+1}} + \cdots + \frac{\lambda_n}{1 - \lambda_{n+1}} = 1.$$

Therefore, defining

$$\tilde{\lambda}_k = \frac{\lambda_k}{1 - \lambda_{n+1}} \geq 0, \ \forall k \in \{1, \ldots, n\}$$

we have that

$$\sum_{k=1}^{n} \tilde{\lambda}_k = 1,$$

so that

$$w = \sum_{k=1}^{n} \tilde{\lambda}_k u_k \in B_n = A.$$

Since A convex, we obtain

$$w_1 = (1 - \lambda_{n+1})w + \lambda_{n+1}u_{n+1} \in A,$$

that is,

$$w_1 = \sum_{k=1}^{n+1} \lambda_k u_k = u \in A, \forall u \in B_{n+1}.$$

Thus,

$$B_{n+1} \subset A,$$

and hence

$$B_{n+1} = A.$$

This completes the induction, that is,

$$A = B_n, \ \forall n \in \mathbb{N}.$$

Hence,

$$A = \cup_{n=1}^{\infty} B_n = Conv(A).$$

Reciprocally, assume $A = Conv(A)$. Since $Conv(A)$ is convex, A is convex.
The proof is complete.

□

Observação 6.12. *Let $A \subset B \subset V$. Clearly $Conv(A) \subset Conv(B)$. In particular, if B is convex, then*

$$Conv(A) \subset B = Conv(B).$$

Proposição 6.13. *Let (V, σ) be a topological vector space. Suppose that a non-empty $A \subset V$ is open. Under such hypotheses, $Conv(A)$ is open.*

Proof. Let $u \in Conv(A)$. Thus, there exist $n \in \mathbb{N}$, $u_k \in A$, $\lambda_k \geq 0$, $\forall k \in \{1, \ldots, n\}$ such that $\sum_{k=1}^{n} \lambda_k = 1$, and $u = \sum_{k=1}^{n} \lambda_k u_k$.

With no loss in generality, assume $\lambda_1 \neq 0$ (redefine the indices, if necessary).

Since $u_1 \in A$ and A is open, there exists a neighborhood V_{u_1} of u_1 such that $V_{u_1} \subset A$.

Thus, $W = \lambda_1 V_{u_1} + \lambda_2 u_2 + \cdots + \lambda_n u_n \subset Conv(A)$.

Observe that W is open and $u \in W \subset Conv(A)$.

Therefore u is an interior point of $Conv(A), \forall u \in Conv(A)$. Thus, $Conv(A)$ is open.

This completes the proof. □

Proposição 6.14. *Let (V, σ) be a topological vector space. Suppose $A \subset V$ is convex and $A^\circ \neq \emptyset$. Under such hypotheses, A° é convexo.*

Proof. Let $u, v \in A^\circ$ and $\lambda \in [0, 1]$. Thus, there exist neighborhoods V_u of u and V_v of V such that $V_u \subset A$ and $V_v \subset A$. Hence,

$$B \equiv V_u \cup V_v \subset A.$$

Therefore, since A is convex, we obtain

$$Conv(B) \subset Conv(A) = A.$$

From the last proposition $Conv(B)$ is open and moreover $Conv(B) \subset A^\circ$. Thus,

$$\lambda u + (1 - \lambda)v \in Conv(B) \subset A^\circ, \ \forall u, v \in A^\circ, \ \lambda \in [0, 1].$$

From this we may infer that A° is convex.

The proof is complete.

□

Observação 6.15. *Let (V, σ) be a topological vector space a let $A \subset V$ be a non-empty open set..*

Thus, tA is open, $\forall t \in \mathbb{F}$ such that $t \neq 0$.

Let $B \subset V$ be a balanced set such that $\mathbf{0} \in B^\circ$.

Let $\alpha \in \mathbb{F}$ be such that $0 < |\alpha| \leq 1$. Thus,

$$\alpha B^\circ \subset \alpha B \subset B.$$

Since αB° is open, we have that $\alpha B^\circ \subset B^\circ, \forall \alpha \in \mathbb{F}$ such that $|\alpha| \leq 1$.

From this we may infer that B° is balanced.

7 Normed and metric spaces

The idea here is to prepare a route for the study of Banach spaces defined below. We start with the definition of norm.

Definição 7.1 (Norm)**.** *A vector space U is said to be a normed space, if it is possible to define a function $\|\cdot\|_U : U \to \mathbb{R}^+ = [0, +\infty)$, called a norm, which satisfies the following properties:*

1. $\|u\|_U > 0$, *if* $u \neq \theta$ *and* $\|u\|_U = 0 \Leftrightarrow u = \theta$
2. $\|u + v\|_U \leq \|u\|_U + \|v\|_U, \forall \ \ u, v \in U$,
3. $\|\alpha u\|_U = |\alpha| \|u\|_U, \forall u \in U, \alpha \in \mathbb{F}$.

Now we recall the definition of metric.

Definição 7.2 (Metric Space)**.** *A vector space U is said to be a metric space if it is possible to define a function $d : U \times U \to \mathbb{R}^+$, called a metric on U, such that*

1. $0 \leq d(u, v), \quad \forall u, v \in U$,
2. $d(u, v) = 0 \Leftrightarrow u = v$,
3. $d(u, v) = d(v, u), \quad \forall u, v \in U$,
4. $d(u, w) \leq d(u, v) + d(v, w), \forall u, v, w \in U$.

A metric can be defined through a norm, that is

$$d(u, v) = \|u - v\|_U. \tag{42}$$

In this case we say that the metric is induced by the norm.

The set $B_r(u) = \{v \in U \mid d(u, v) < r\}$ is called the open ball with center at u and radius r. A metric $d : U \times U \to \mathbb{R}^+$ is said to be invariant if

$$d(u + w, v + w) = d(u, v), \forall u, v, w \in U. \tag{43}$$

The following are some basic definitions concerning metric and normed spaces:

Definição 7.3 (Convergent sequences)**.** *Given a metric space U, we say that $\{u_n\} \subset U$ converges to $u_0 \in U$ as $n \to \infty$, if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that if $n \geq n_0$ then $d(u_n, u_0) < \varepsilon$. In this case we write $u_n \to u_0$ as $n \to +\infty$.*

Definição 7.4 (Cauchy sequence)**.** *$\{u_n\} \subset U$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(u_n, u_m) < \varepsilon, \forall m, n \geq n_0$*

Definição 7.5 (Completeness)**.** *A metric space U is said to be complete if each Cauchy sequence related to $d : U \times U \to \mathbb{R}^+$ converges to an element of U.*

Definição 7.6 (Limit point)**.** *Let (U, d) be a metric space and let $E \subset U$. We say that $v \in U$ is a limit point of E if for each $r > 0$ there exists $w \in B_r(v) \cap E$ such that $w \neq v$.*

Definição 7.7 (Interior point, topology for (U, d))**.** *Let (U, d) be a metric space and let $E \subset U$. We say that $u \in E$ is interior point if there exists $r > 0$ such that $B_r(u) \subset E$. If a point of E is not a limit one is said to be isolated. We may define a topology for a metric space (U, d), by declaring as open all set $E \subset U$ such that all its points are interior. Such a topology is said to be induced by the metric d.*

Proposição 7.8. *Let (U, d) be a metric space. The set σ of all open sets, defined through the last definition, is indeed a topology for (U, d).*

Proof.

1. Obviously $\emptyset$ and U are open sets.

2. Assume A and B are open sets and define $C = A \cap B$. Let $u \in C = A \cap B$, thus from $u \in A$, there exists $r_1 > 0$ such that $B_{r_1}(u) \subset A$. Similarly from $u \in B$ there exists $r_2 > 0$ such that $B_{r_2}(u) \subset B$.

   Define $r = \min\{r_1, r_2\}$. Thus $B_r(u) \subset A \cap B = C$, so that u is an interior point of C. Since $u \in C$ is arbitrary, we may conclude that C is open.

3. Suppose $\{A_\alpha,\ \alpha \in L\}$ is a collection of open sets. Define $E = \cup_{\alpha \in L} A_\alpha$ and we shall show that E is open.

   Choose $u \in E = \cup_{\alpha \in L} A_\alpha$. Thus there exists $\alpha_0 \in L$ such that $u \in A_{\alpha_0}$. Since A_{α_0} is open there exists $r > 0$ such that $B_r(u) \subset A_{\alpha_0} \subset \cup_{\alpha \in L} A_\alpha = E$. Hence u is an interior point of E, since $u \in E$ is arbitrary, $E = \cup_{\alpha \in L} A_\alpha$ is open.

The proof is complete. □

Definição 7.9. *Let (U, d) be a metric space and let $E \subset U$. We define E' as the set of all the limit points of E.*

Teorema 7.10. *Let (U, d) be a metric space and let $E \subset U$. Then E is closed if and only if $E' \subset E$.*

Proof. Suppose $E' \subset E$. Let $u \in E^c$, thus $u \notin E$ and $u \notin E'$. Therefore there exists $r > 0$ such that $B_r(u) \cap E = \emptyset$, so that $B_r(u) \subset E^c$. Therefore u is an interior point of E^c. Since $u \in E^c$ is arbitrary we may infer that E^c is open, so that $E = (E^c)^c$ is closed.

Conversely, suppose that E is closed, that is E^c is open.

If $E' = \emptyset$ we are done.

Thus assume $E' \neq \emptyset$ and choose $u \in E'$. Thus for each $r > 0$ there exists $v \in B_r(u) \cap E$ such that $v \neq u$. Thus $B_r(u) \not\subset E^c, \forall r > 0$ so that u is not a interior point of E^c. Since E^c is open, we have that $u \notin E^c$ so that $u \in E$. We have thus obtained, $u \in E, \forall u \in E'$, so that $E' \subset E$.

The proof is complete.

Observação 7.11. *From this last result, we may conclude that in a metric space $E \subset U$ is closed if and only if $E' \subset E$.*

At this point we recall the definition of Banach space.

Definição 7.12 (Banach Spaces)**.** *A normed vector space U is said to be a Banach Space if each Cauchy sequence related to the metric induced by the norm converges to an element of U.*

Observação 7.13. *Let (U, σ) be a topological space. We say that the topology σ is compatible with a metric $d : U \times U \to \mathbb{R}^+$ if σ coincides with the topology generated by such a metric. In this case we say that $d : U \times U \to \mathbb{R}^+$ induces the topology σ.*

Definição 7.14 (Metrizable spaces)**.** *A topological vector space (U, σ) is said to be metrizable if σ is compatible with some metric d.*

Definição 7.15 (Normable spaces)**.** *A topological vector space (U, σ) is said to be normable if the induced metric (by this norm) is compatible with σ.*

8 Linear mappings

Given U, V topological vector spaces, a function (mapping) $f : U \to V$, $A \subset U$ and $B \subset V$, we define:

$$f(A) = \{f(u) \mid u \in A\}, \tag{44}$$

and the inverse image of B, denoted $f^{-1}(B)$ as

$$f^{-1}(B) = \{u \in U \mid f(u) \in B\}. \tag{45}$$

Definição 8.1 (Linear Functions)**.** *A function $f : U \to V$ is said to be linear if*

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v), \forall u, v \in U,\ \alpha, \beta \in \mathbb{F}. \tag{46}$$

Definição 8.2 (Null Space and Range)**.** *Given $f : U \to V$, we define the null space and the range of f, denoted by $N(f)$ and $R(f)$ respectively, as*

$$N(f) = \{u \in U \mid f(u) = \theta\} \tag{47}$$

and

$$R(f) = \{v \in V \mid \exists u \in U \text{ such that } f(u) = v\}. \tag{48}$$

Note that if f is linear then $N(f)$ and $R(f)$ are subspaces of U and V respectively.

Proposição 8.3. *Let U, V be topological vector spaces. If $f : U \to V$ is linear and continuous at θ, then it is continuous everywhere.*

Proof. Since f is linear we have $f(\theta) = \theta$. Since f is continuous at θ, given $\mathcal{V} \subset V$ a neighborhood of zero, there exists $\mathcal{U} \subset U$ neighborhood of zero, such that

$$f(\mathcal{U}) \subset \mathcal{V}. \tag{49}$$

Thus

$$v - u \in \mathcal{U} \Rightarrow f(v - u) = f(v) - f(u) \in \mathcal{V}, \tag{50}$$

or

$$v \in u + \mathcal{U} \Rightarrow f(v) \in f(u) + \mathcal{V}, \tag{51}$$

which means that f is continuous at u. Since u is arbitrary, f is continuous everywhere.

9 Linearity and continuity

Definição 9.1 (Bounded Functions)**.** *A function $f : U \to V$ is said to be bounded if it maps bounded sets into bounded sets.*

Proposição 9.2. *A set E is bounded if and only if the following condition is satisfied: whenever $\{u_n\} \subset E$ and $\{\alpha_n\} \subset \mathbb{F}$ are such that $\alpha_n \to 0$ as $n \to \infty$ we have $\alpha_n u_n \to \theta$ as $n \to \infty$.*

Proof. Suppose E is bounded. Let $\mathcal{U}$ be a balanced neighborhood of θ in U, then $E \subset t\mathcal{U}$ for some t. For $\{u_n\} \subset E$, as $\alpha_n \to 0$, there exists N such that if $n > N$ then $t < \frac{1}{|\alpha_n|}$. Since $t^{-1}E \subset \mathcal{U}$ and $\mathcal{U}$ is balanced, we have that $\alpha_n u_n \in \mathcal{U}$, $\forall n > N$, and thus $\alpha_n u_n \to \theta$. Conversely, if E is not bounded, there is a neighborhood $\mathcal{V}$ of θ and $\{r_n\}$ such that $r_n \to \infty$ and E is not contained in $r_n\mathcal{V}$, that is, we can choose u_n such that $r_n^{-1}u_n$ is not in $\mathcal{V}$, $\forall n \in \mathbb{N}$, so that $\{r_n^{-1}u_n\}$ does not converge to θ.

Proposição 9.3. *Let $f : U \to V$ be a linear function. Consider the following statements*

1. *f is continuous,*
2. *f is bounded,*
3. *If $u_n \to \theta$ then $\{f(u_n)\}$ is bounded,*
4. *If $u_n \to \theta$ then $f(u_n) \to \theta$.*

Then,

- 1 *implies* 2,
- 2 *implies* 3,
- *if U is metrizable with invariant metric,then* 3 *implies* 4*, which implies* 1*.*

Proof.

1. 1 implies 2: Suppose f is continuous, for $\mathcal{W} \subset V$ neighborhood of zero, there exists a neighborhood of zero in U, denoted by $\mathcal{V}$, such that

$$f(\mathcal{V}) \subset \mathcal{W}. \tag{52}$$

If E is bounded, there exists $t_0 \in \mathbb{R}^+$ such that $E \subset t\mathcal{V}$, $\forall t \geq t_0$, so that

$$f(E) \subset f(t\mathcal{V}) = tf(\mathcal{V}) \subset t\mathcal{W}, \quad \forall t \geq t_0, \tag{53}$$

and thus f is bounded.

2. 2 implies 3: Suppose $u_n \to \theta$ and let $\mathcal{W}$ be a neighborhood of zero. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $u_n \in \mathcal{V} \subset \mathcal{W}$ where $\mathcal{V}$ is a balanced neighborhood of zero. On the other hand, for $n < N$, there exists K_n such that $u_n \in K_n\mathcal{V}$. Define $K = \max\{1, K_1, ..., K_n\}$. Then $u_n \in K\mathcal{V}, \forall n \in \mathbb{N}$ and hence $\{u_n\}$ is bounded. Finally from 2, we have that $\{f(u_n)\}$ is bounded.

3. 3 implies 4: Suppose U is metrizable with invariant metric and let $u_n \to \theta$. Given $K \in \mathbb{N}$, there exists $n_K \in \mathbb{N}$ such that if $n > n_K$ then $d(u_n, \theta) < \frac{1}{K^2}$. Define $\gamma_n = 1$ if $n < n_1$ and $\gamma_n = K$, if $n_K \leq n < n_{K+1}$ so that

$$d(\gamma_n u_n, \theta) = d(K u_n, \theta) \leq K d(u_n, \theta) < K^{-1}. \tag{54}$$

   Thus since 2 implies 3 we have that $\{f(\gamma_n u_n)\}$ is bounded so that, by Proposition 9.2 $f(u_n) = \gamma_n^{-1} f(\gamma_n u_n) \to \theta$ as $n \to \infty$.

4. 4 implies 1: suppose 1 fails. Thus there exists a neighborhood of zero $\mathcal{W} \subset V$ such that $f^{-1}(\mathcal{W})$ contains no neighborhood of zero in U. Particularly, we can select $\{u_n\}$ such that $u_n \in B_{1/n}(\theta)$ and $f(u_n)$ not in $\mathcal{W}$ so that $\{f(u_n)\}$ does not converge to zero. Thus 4 fails.

10 Continuity of operators in Banach spaces

Let U, V be Banach spaces. We call a function $A : U \to V$ an operator.

Proposição 10.1. *Let U, V be Banach spaces. A linear operator $A : U \to V$ is continuous if and only if there exists $K \in \mathbb{R}^+$ such that*

$$\|A(u)\|_V < K\|u\|_U, \forall u \in U.$$

Proof. Suppose A is linear and continuous. From Proposition 9.3,

$$\text{if } \{u_n\} \subset U \text{ is such that } u_n \to \theta \text{ then } A(u_n) \to \theta. \tag{55}$$

We claim that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u\|_U < \delta$ then $\|A(u)\|_V < \varepsilon$.

Suppose, to obtain contradiction that the claim is false.

Thus there exists $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there exists $u_n \in U$ such that $\|u_n\|_U \leq \frac{1}{n}$ and $\|A(u_n)\|_V \geq \varepsilon_0$.

Therefore $u_n \to \theta$ and $A(u_n)$ does not converge to θ, which contradicts (55).

Thus the claim holds.

In particular, for $\varepsilon = 1$ there exists $\delta > 0$ such that if $\|u\|_U < \delta$ then $\|A(u)\|_V < 1$. Thus given an arbitrary not relabeled $u \in U$, $u \neq \theta$, for

$$w = \frac{\delta u}{2\|u\|_U}$$

we have

$$\|A(w)\|_V = \frac{\delta\|A(u)\|_V}{2\|u\|_U} < 1,$$

that is

$$\|A(u)\|_V < \frac{2\|u\|_U}{\delta}, \forall u \in U.$$

Defining

$$K = \frac{2}{\delta}$$

the first part of the proof is complete. Reciprocally, suppose there exists $K > 0$ such that

$$\|A(u)\|_V < K\|u\|_U, \forall u \in U.$$

Hence $u_n \to \theta$ implies $\|A(u_n)\|_V \to \theta$, so that from Proposition 9.3, A is continuous.

The proof is complete.

11 Some classical results on Banach spaces

In this section we present some important results in Banach spaces. We start with the following theorem.

11.1 The Baire Category Theorem

Teorema 11.1. *Let U and V be Banach spaces and let $A : U \to V$ be a linear operator. Then A is bounded if and only if the set $C \subset U$ has at least one interior point, where*

$$C = A^{-1}[\{v \in V \mid \|v\|_V \leq 1\}].$$

Proof. Suppose there exists $u_0 \in U$ in the interior of C. Thus, there exists $r > 0$ such that

$$B_r(u_0) = \{u \in U \mid \|u - u_0\|_U < r\} \subset C.$$

Fix $u \in U$ such that $\|u\|_U < r$. Thus, we have

$$\|A(u)\|_V \leq \|A(u + u_0)\|_V + \|A(u_0)\|_V.$$

Observe also that

$$\|(u + u_0) - u_0\|_U < r,$$

so that $u + u_0 \in B_r(u_0) \subset C$ and thus

$$\|A(u + u_0)\|_V \leq 1$$

and hence

$$\|A(u)\|_V \leq 1 + \|A(u_0)\|_V, \tag{56}$$

$\forall u \in U$ such that $\|u\|_U < r$. Fix an arbitrary not relabeled $u \in U$ such that $u \neq \theta$. From (56)

$$w = \frac{u}{\|u\|_U}\frac{r}{2}$$

is such that

$$\|A(w)\|_V = \frac{\|A(u)\|_V}{\|u\|_U}\frac{r}{2} \leq 1 + \|A(u_0)\|_V,$$

so that

$$\|A(u)\|_V \leq (1 + \|A(u_0)\|_V)\|u\|_U \frac{2}{r}.$$

Since $u \in U$ is arbitrary, A is bounded.

Reciprocally, suppose A is bounded. Thus

$$\|A(u)\|_V \leq K\|u\|_U, \forall u \in U,$$

for some $K > 0$. In particular

$$D = \left\{ u \in U \mid \|u\|_U \leq \frac{1}{K} \right\} \subset C.$$

The proof is complete.

Definição 11.2. *A set S in a metric space U is said to be nowhere dense if $\overline{S}$ has an empty interior.*

Teorema 11.3 (Baire Category Theorem)**.** *A complete metric space is never the union of a countable number of nowhere dense sets.*

Proof. Suppose, to obtain contradiction, that U is a complete metric space and

$$U = \cup_{n=1}^{\infty} A_n$$

where each A_n is nowhere dense. Since A_1 is nowhere dense, there exist $u_1 \in U$ which is not in $\bar{A}_1$, otherwise we would have $U = \bar{A}_1$, which is not possible since U is open. Furthermore, $\bar{A}_1^c$ is open, so that we may obtain $u_1 \in A_1^c$ and $0 < r_1 < 1$ such that

$$B_1 = B_{r_1}(u_1)$$

satisfies

$$B_1 \cap A_1 = \emptyset.$$

Since A_2 is nowhere dense we have B_1 is not contained in $\bar{A}_2$. Therefore we may select $u_2 \in B_1 \setminus \bar{A}_2$ and since $B_1 \setminus \bar{A}_2$ is open, there exists $0 < r_2 < 1/2$ such that

$$\bar{B}_2 = \bar{B}_{r_2}(u_2) \subset B_1 \setminus \bar{A}_2,$$

that is

$$B_2 \cap A_2 = \emptyset.$$

Proceeding inductively in this fashion, for each $n \in \mathbb{N}$ we may obtain $u_n \in B_{n-1} \setminus \bar{A}_n$ such that we may choose an open ball $B_n = B_{r_n}(u_n)$ such that

$$\bar{B}_n \subset B_{n-1},$$

$$B_n \cap A_n = \emptyset$$

and

$$0 < r_n < 2^{1-n}.$$

Observe that $\{u_n\}$ is a Cauchy sequence, considering that if $m, n > N$ then $u_n, u_m \in B_N$, so that

$$d(u_n, u_m) < 2(2^{1-N}).$$

Define

$$u = \lim_{n \to \infty} u_n.$$

Since

$$u_n \in B_N, \forall n > N,$$

we get

$$u \in \bar{B}_N \subset B_{N-1}.$$

Therefore u is not in $A_{N-1}, \forall N > 1$, which means u is not in $\cup_{n=1}^{\infty} A_n = U$, a contradiction.

The proof is complete.

11.2 The Principle of Uniform Boundedness

Teorema 11.4 (The Principle of Uniform Boundedness)**.** *Let U be a Banach space. Let $\mathcal{F}$ be a family of linear bounded operators from U into a normed linear space V. Suppose for each $u \in U$ there exists a $K_u \in \mathbb{R}$ such that*

$$\|T(u)\|_V < K_u, \forall T \in \mathcal{F}.$$

Then, there exists $K \in \mathbb{R}$ such that

$$\|T\| < K, \forall T \in \mathcal{F}.$$

Proof. Define

$$B_n = \{u \in U \mid \|T(u)\|_V \leq n, \forall T \in \mathcal{F}\}.$$

By the hypotheses, given $u \in U$, $u \in B_n$ for all n sufficiently big. Thus,

$$U = \cup_{n=1}^{\infty} B_n.$$

Moreover each B_n is closed. By the Baire category theorem there exists $n_0 \in \mathbb{N}$ such that B_{n_0} has non-empty interior. That is, there exists $u_0 \in U$ and $r > 0$ such that

$$B_r(u_0) \subset B_{n_0}.$$

Thus, fixing an arbitrary $T \in \mathcal{F}$, we have

$$\|T(u)\|_V \leq n_0, \forall u \in B_r(u_0), .$$

Thus if $\|u\|_U < r$ then $\|(u + u_0) - u_0\|_U < r$, so that

$$\|T(u + u_0)\|_V \leq n_0,$$

that is

$$\|T(u)\|_V - \|T(u_0)\|_V \leq n_0.$$

Thus

$$\|T(u)\|_V \leq 2n_0, \text{ if } \|u\|_U < r. \tag{57}$$

For $u \in U$ arbitrary, $u \neq \theta$, define

$$w = \frac{ru}{2\|u\|_U},$$

from (57) we obtain

$$\|T(w)\|_V = \frac{r\|T(u)\|_V}{2\|u\|_U} \leq 2n_0,$$

so that

$$\|T(u)\|_V \leq \frac{4n_0\|u\|_U}{r}, \forall u \in U.$$

Hence

$$\|T\| \leq \frac{4n_0}{r}, \forall T \in \mathcal{F}.$$

The proof is complete.

11.3 The Open Mapping Theorem

Teorema 11.5 (The Open Mapping Theorem). *Let U and V be Banach spaces and let $A : U \rightarrow V$ be a bounded onto linear operator. Thus if $\mathcal{O} \subset U$ is open then $A(\mathcal{O})$ is open in V.*

Proof. First we will prove that given $r > 0$, there exists $r' > 0$ such that

$$A(B_r(\theta)) \supset B^V_{r'}(\theta). \tag{58}$$

Here $B^V_{r'}(\theta)$ denotes a ball in V of radius r' with center in θ. Since A is onto

$$V = \cup_{n=1}^{\infty} A(nB_1(\theta)).$$

By the Baire Category Theorem, there exists $n_0 \in \mathbb{N}$ such that the closure of $A(n_0B_1(\theta))$ has non-empty interior, so that $\overline{A(B_1(\theta))}$ has non-empty interior. We will show that there exists $r' > 0$ such that

$$B^V_{r'}(\theta) \subset \overline{A(B_1(\theta))}.$$

Observe that there exists $y_0 \in V$ and $r_1 > 0$ such that

$$B^V_{r_1}(y_0) \subset \overline{A(B_1(\theta))}. \tag{59}$$

Define $u_0 \in B_1(\theta)$ which satisfies $A(u_0) = y_0$. We claim that

$$\overline{A(B_{r_2}(\theta))} \supset B^V_{r_1}(\theta),$$

where $r_2 = 1 + \|u_0\|_U$. To prove the claim, pick

$$y \in A(B_1(\theta))$$

thus there exists $u \in U$ such that $\|u\|_U < 1$ and $A(u) = y$. Therefore

$$A(u) = A(u - u_0 + u_0) = A(u - u_0) + A(u_0).$$

But observe that

$$\begin{aligned} \|u - u_0\|_U &\leq \|u\|_U + \|u_0\|_U \\ &< 1 + \|u_0\|_U \\ &= r_2, \end{aligned} \tag{60}$$

so that

$$A(u - u_0) \in A(B_{r_2}(\theta)).$$

This means

$$y = A(u) \in A(u_0) + A(B_{r_2}(\theta)),$$

and hence

$$A(B_1(\theta)) \subset A(u_0) + A(B_{r_2}(\theta)).$$

That is, from this and (59), we obtain

$$A(u_0) + \overline{A(B_{r_2}(\theta))} \supset \overline{A(B_1(\theta))} \supset B^V_{r_1}(y_0) = A(u_0) + B^V_{r_1}(\theta),$$

and therefore

$$\overline{A(B_{r_2}(\theta))} \supset B^V_{r_1}(\theta).$$

Since

$$A(B_{r_2}(\theta)) = r_2 A(B_1(\theta)),$$

we have, for some not relabeled $r_1 > 0$ that

$$\overline{A(B_1(\theta))} \supset B^V_{r_1}(\theta).$$

Thus it suffices to show that

$$\overline{A(B_1(\theta))} \subset A(B_2(\theta)),$$

to prove (58). Let $y \in \overline{A(B_1(\theta))}$, since A is continuous we may select $u_1 \in B_1(\theta)$ such that

$$y - A(u_1) \in B^V_{r_1/2}(\theta) \subset \overline{A(B_{1/2}(\theta))}.$$

Now select $u_2 \in B_{1/2}(\theta)$ so that

$$y - A(u_1) - A(u_2) \in B^V_{r_1/4}(\theta).$$

By induction, we may obtain

$$u_n \in B_{2^{1-n}}(\theta),$$

such that

$$y - \sum_{j=1}^{n} A(u_j) \in B^V_{r_1/2^n}(\theta).$$

Define
$$u = \sum_{n=1}^{\infty} u_n,$$
we have that $u \in B_2(\theta)$, so that
$$y = \sum_{n=1}^{\infty} A(u_n) = A(u) \in A(B_2(\theta)).$$
Therefore
$$\overline{A(B_1(\theta))} \subset A(B_2(\theta)).$$
The proof of (58) is complete.

To finish the proof of this theorem, assume $\mathcal{O} \subset U$ is open. Let $v_0 \in A(\mathcal{O})$. Let $u_0 \in \mathcal{O}$ be such that $A(u_0) = v_0$. Thus there exists $r > 0$ such that
$$B_r(u_0) \subset \mathcal{O}.$$
From (58),
$$A(B_r(\theta)) \supset B^V_{r'}(\theta),$$
for some $r' > 0$. Thus
$$A(\mathcal{O}) \supset A(u_0) + A(B_r(\theta)) \supset v_0 + B^V_{r'}(\theta).$$
This means that v_0 is an interior point of $A(\mathcal{O})$. Since $v_0 \in A(\mathcal{O})$ is arbitrary, we may conclude that $A(\mathcal{O})$ is open.

The proof is complete.

Teorema 11.6 (The Inverse Mapping Theorem)**.** *A continuous linear bijection of one Banach space onto another has a continuous inverse.*

Proof. Let $A : U \to V$ satisfying the theorem hypotheses. Since A is open, A^{-1} is continuous.

11.4 The Closed Graph Theorem

Definição 11.7 (Graph of a Mapping)**.** *Let $A : U \to V$ be an operator, where U and V are normed linear spaces. The **graph** of A denoted by $\Gamma(A)$ is defined by*
$$\Gamma(A) = \{(u, v) \in U \times V \mid v = A(u)\}.$$

Teorema 11.8 (The Closed Graph Theorem)**.** *Let U and V be Banach spaces and let $A : U \to V$ be a linear operator. Then A is bounded if and only if its graph is closed.*

Proof. Suppose $\Gamma(A)$ is closed. Since A is linear $\Gamma(A)$ is a subspace of $U \oplus V$. Also, being $\Gamma(A)$ closed, it is a Banach space with the norm
$$\|(u, A(u))\| = \|u\|_U + \|A(u)\|_V.$$

Consider the continuous mappings

$$\Pi_1(u, A(u)) = u$$

and

$$\Pi_2(u, A(u)) = A(u).$$

Observe that Π_1 is a bijection, so that by the inverse mapping theorem Π_1^{-1} is continuous. As

$$A = \Pi_2 \circ \Pi_1^{-1},$$

it follows that A is continuous. The converse is immediate.

12 A note on finite dimensional normed spaces

We start this section with the following theorem.

Teorema 12.1. *Let V be a complex normed vector space (not necessarily of finite-dimension). Suppose $\{u_1, \ldots, u_n\} \subset V$ be a linearly independent set. Under such hypotheses, there exists $c > 0$ such that*

$$\|\alpha_1 u_1 + \cdots + \alpha_n u_n\|_V \geq c(|\alpha_1| + \cdots + |\alpha_n|), \ \forall \alpha_1, \ldots, \alpha_n \in \mathbb{C}. \tag{61}$$

Proof. For $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, let us denote

$$s = |\alpha_1| + \cdots + |\alpha_n|.$$

Thus, if $s = 0$, then $\alpha_1 = \ldots = \alpha_n = 0$ and (61) holds.
Suppose then $s > 0$
Denoting $\beta_j = \frac{\alpha_j}{s}$, $\forall j \in \{1, \ldots, n\}$ we have that (61) is equivalent to

$$\|\beta_1 u_1 + \cdots + \beta_n u_n\|_V \geq c, \ \forall \beta_1, \ldots, \beta_n \in \mathbb{C}, \text{ such that } \sum_{j=1}^n |\beta_j| = 1. \tag{62}$$

Suppose to obtain, contradiction, there is no $c > 0$ such that (62) holds.
Thus there exist sequences $\{v_m\} \subset V$ and $\{\beta_j^m\} \subset \mathbb{C}$ such that

$$v_m = \beta_1^m u_1 + \ldots + \beta_n^m u_n$$

such that

$$\sum_{j=1}^n |\beta_j^m| = 1, \ \forall m \in \mathbb{N}$$

and

$$\|v_m\|_V \to 0, \text{ as } m \to \infty.$$

In particular

$$|\beta_j^m| \leq 1, \ \forall m \in \mathbb{N}, \ j \in \{1, \ldots, n\}.$$

It is a well known result in elementary analysis that a bounded sequence $\mathbb{C}^n$ has a convergent subsequence.

Hence, there exists a subsequence $\{m_k\}$ of $\mathbb{N}$ and

$$\beta_j^0 \in \mathbb{C}, \ \forall j \in \{1, \ldots, n\}$$

such that

$$\beta_j^{m_k} \to \beta_j^0, \ \forall j \in \{1, \ldots, n\},$$

and

$$\sum_{j=1}^n |\beta_j^0| = 1.$$

Thus

$$v_{m_k} \to \sum_{j=1}^n \beta_j^0 u_j.$$

Since

$$\sum_{j=1}^n |\beta_j^0| = 1,$$

this contradicts

$$v_{m_k} \to \mathbf{0}.$$

Therefore, there exists $c > 0$ such that (61) holds.

The proof is complete.

□

Now we present the following result about the completeness of finite dimensional subspaces in a normed complete vector space.

Teorema 12.2. *Let V be a complex normed vector space and let M be finite dimensional subspace of V.*

Under such hypotheses, M is complete (closed). In particular, each normed finite dimensional vector space is complete.

Proof. Let $\{v_m\} \subset M$ be a Cauchy sequence. Let $n \in \mathbb{N}$ be the dimension of M. Let $\{u_1, \ldots, u_n\}$ be a basis for M.

Hence, there exists a sequence $\{\alpha_j^m\} \subset \mathbb{C}$ such that

$$v_m = \alpha_1^m u_1 + \cdots + \alpha_n^m u_n.$$

Let $\varepsilon > 0$. Since $\{v_m\}$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that, if $m, l > n_0$, then

$$\|v_m - v_l\|_V < \varepsilon.$$

Hence, from this and the last theorem, there exists $c > 0$ such that

$$\begin{aligned} \varepsilon &> \left\| \sum_{j=1}^{n} (\alpha_j^m - \alpha_j^l) u_j \right\|_V \\ &\geq c \sum_{i=1}^{n} |\alpha_j^m - \alpha_j^l| \\ &\geq c|\alpha_j^m - \alpha_j^l|, \ \forall m, l > n_0, \ \forall j \in \{1, \cdots, n\}. \end{aligned} \tag{63}$$

Thus $\{\alpha_j^m\} \subset \mathbb{C}$ is a Cauchy sequence.

Therefore, there exists a $\alpha_j^0 \in \mathbb{C}$ such that

$$\alpha_j^m \to \alpha_j^0, \ \forall j \in \{1, \ldots, n\}.$$

From this, denoting $v_0 = \sum_{j=1}^{n} \alpha_j^0 u_j \in M$, we get

$$\|v_m - v_0\|_V \leq \sum_{j=1}^{n} |\alpha_j^m - \alpha_j^0| \|u_j\|_V \to 0, \text{ as } m \to \infty.$$

From this last result, we may infer that M is complete. $\square$

Definição 12.3 (Equivalence between two norms)**.** *Let V be a vector space. Two norms*

$$\| \cdot \|_0, \| \cdot \|_1 : V \to \mathbb{R}^+$$

are said to be equivalent, if there exists $\alpha, \beta > 0$ such that

$$\alpha \|u\|_0 \leq \|u\|_1 \leq \beta \|u\|_0, \ \forall u \in V.$$

Teorema 12.4. *Let V be a finite dimensional vector space. Under such hypotheses, any two norms defined on V are equivalent.*

Proof. Assume the dimension of V is n.

Let $\{u_1, \ldots, u_n\} \subset V$ be a basis for V. Let $\| \cdot \|_0, \| \cdot \|_1 : V \to \mathbb{R}^+$ be two norms in V.

Let $u \in V$. Hence there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that

$$u = \sum_{j=1}^{n} \alpha_j u_j,$$

so that there exists $c > 0$ which does not depend on u, such that

$$\|u\|_1 \geq c(|\alpha_1| + \cdots + |\alpha_n|).$$

On the other hand

$$\|u\|_0 \leq \sum_{j=1}^{n} |\alpha_j| \|u_j\|_0 \leq K \sum_{j=1}^{n} |\alpha_j| \leq \frac{K}{c} \|u\|_1, \ \forall u \in V,$$

where $K = \max_{j \in \{1,\ldots n\}}\{\|u_j\|_0\}$.

Interchanging the roles of $\|\cdot\|_0$ and $\|\cdot\|_1$ we may obtain $K_1, c_1 > 0$ such that

$$\|u_1\|_1 \leq \frac{K_1}{c_1}\|u\|_0, \ \forall u \in V.$$

Denoting $\alpha = \frac{c}{K}$ and $\beta = \frac{K_1}{c_1}$, we have obtained,

$$\alpha\|u\|_0 \leq \|u\|_1 \leq \beta\|u\|_0, \ \forall u \in V.$$

The proof is complete. □

13 Hilbert Spaces

13.1 Introduction

At this point we introduce an important class of spaces namely, the Hilbert spaces, which are a special class of metric spaces.

14 The main definitions and results

Definição 14.1. *Let H be a vector space. We say that H is a real pre-Hilbert space if there exists a function $(\cdot,\cdot)_H : H \times H \to \mathbb{R}$ such that*

1. $(u, v)_H = (v, u)_H, \ \forall u, v \in H,$
2. $(u + v, w)_H = (u, w)_H + (v, w)_H, \ \forall u, v, w \in H,$
3. $(\alpha u, v)_H = \alpha(u, v)_H, \ \forall u, v \in H, \ \alpha \in \mathbb{R},$
4. $(u, u)_H \geq 0, \ \forall u \in H,$ *and* $(u, u)_H = 0,$ *if and only if* $u = \theta$.

Observação 14.2. *The function $(\cdot,\cdot)_H : H \times H \to \mathbb{R}$ is called an inner-product.*

Proposição 14.3 (Cauchy-Schwarz inequality)**.** *Let H be a pre-Hilbert space. Defining*

$$\|u\|_H = \sqrt{(u, u)_H}, \forall u \in H,$$

we have

$$|(u, v)_H| \leq \|u\|_H\|v\|_H, \forall u, v \in H.$$

Equality holds if and only if $u = \alpha v$ for some $\alpha \in \mathbb{R}$ or $v = \theta$.

Proof. If $v = \theta$ the inequality is immediate. Assume $v \neq \theta$. Given $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} 0 &\leq (u - \alpha v, u - \alpha v)_H \\ &= (u, u)_H + \alpha^2 (v, v)_H - 2\alpha(u, v)_H \\ &= \|u\|_H^2 + \alpha^2 \|v\|_H^2 - 2\alpha(u, v)_H. \end{aligned} \tag{64}$$

In particular for $\alpha = (u, v)_H / \|v\|_H^2$, we obtain

$$0 \leq \|u\|_H^2 - \frac{(u, v)_H^2}{\|v\|_H^2},$$

that is

$$|(u, v)_H| \leq \|u\|_H \|v\|_H.$$

The proof of the remaining conclusions is left as an exercise.

Proposição 14.4. *On a pre-Hilbert space H, the function*

$$\| \cdot \|_H : H \to \mathbb{R}$$

is a norm, where as above

$$\|u\|_H = \sqrt{(u, u)}.$$

Proof. The only non-trivial property to be verified, concerning the definition of norm, is the triangle inequality.

Observe that, given $u, v \in H$, from the Cauchy-Schwarz inequality we have,

$$\begin{aligned} \|u + v\|_H^2 &= (u + v, u + v)_H \\ &= (u, u)_H + (v, v)_H + 2(u, v)_H \\ &\leq (u, u)_H + (v, v)_H + 2|(u, v)_H| \\ &\leq \|u\|_H^2 + \|v\|_H^2 + 2\|u\|_H \|v\|_H \\ &= (\|u\|_H + \|v\|_H)^2. \end{aligned} \tag{65}$$

Therefore

$$\|u + v\|_H \leq \|u\|_H + \|v\|_H, \forall u, v \in H.$$

The proof is complete.

Definição 14.5. *A pre-Hilbert space H is to be a Hilbert space if it is complete, that is, if any cauchy sequence in H converges to an element of H.*

Definição 14.6 (Orthogonal Complement)**.** *Let H be a Hilbert space. Considering $M \subset H$ we define its orthogonal complement, denoted by $M^\perp$, by*

$$M^\perp = \{u \in H \mid (u, m)_H = 0, \ \forall m \in M\}.$$

Teorema 14.7. *Let H be a Hilbert space, M a closed subspace of H and suppose $u \in H$. Under such hypotheses There exists a unique $m_0 \in M$ such that*

$$\|u - m_0\|_H = \min_{m \in M}\{\|u - m\|_H\}.$$

Moreover $n_0 = u - m_0 \in M^\perp$ so that

$$u = m_0 + n_0,$$

where $m_0 \in M$ and $n_0 \in M^\perp$. Finally, such a representation through $M \oplus M^\perp$ is unique.

Proof. Define d by

$$d = \inf_{m \in M}\{\|u - m\|_H\}.$$

Let $\{m_i\} \subset M$ be a sequence such that

$$\|u - m_i\|_H \to d, \quad \text{as } i \to \infty.$$

Thus, from the parallelogram law we have

$$\begin{aligned}
\|m_i - m_j\|_H^2 &= \|m_i - u - (m_j - u)\|_H^2 \\
&= 2\|m_i - u\|_H^2 + 2\|m_j - u\|_H^2 \\
&\quad \|-2u + m_i + m_j\|_H^2 \\
&= 2\|m_i - u\|_H^2 + 2\|m_j - u\|_H^2 \\
&\quad -4\|-u + (m_i + m_j)/2\|_H^2 \\
&\leq 2\|m_i - u\|_H^2 + 2\|m_j - u\|_H^2 - 4d^2 \\
&\to 2d^2 + 2d^2 - 4d^2 = 0, \text{ as } i, j \to +\infty.
\end{aligned} \tag{66}$$

Thus $\{m_i\} \subset M$ is a Cauchy sequence. Since M is closed, there exists $m_0 \in M$ such that

$$m_i \to m_0, \text{ as } i \to +\infty,$$

so that

$$\|u - m_i\|_H \to \|u - m_0\|_H = d.$$

Define

$$n_0 = u - m_0.$$

We will prove that $n_0 \in M^\perp$.

Pick $m \in M$ and $t \in \mathbb{R}$, thus we have

$$\begin{aligned}
d^2 &\leq \|u - (m_0 - tm)\|_H^2 \\
&= \|n_0 + tm\|_H^2 \\
&= \|n_0\|_H^2 + 2(n_0, m)_H t + \|m\|_H^2 t^2.
\end{aligned} \tag{67}$$

Since

$$\|n_0\|_H^2 = \|u - m_0\|_H^2 = d^2,$$

we obtain

$$2(n_0, m)_H t + \|m\|_H^2 t^2 \geq 0, \forall t \in \mathbb{R}$$

so that

$$(n_0, m)_H = 0.$$

Being $m \in M$ arbitrary, we obtain

$$n_0 \in M^\perp.$$

It remains to prove the uniqueness. Let $m \in M$, thus

$$\begin{aligned} \|u - m\|_H^2 &= \|u - m_0 + m_0 - m\|_H^2 \\ &= \|u - m_0\|_H^2 + \|m - m_0\|_H^2, \end{aligned} \tag{68}$$

since

$$(u - m_0, m - m_0)_H = (n_0, m - m_0)_H = 0.$$

From (68) we obtain

$$\|u - m\|_H^2 > \|u - m_0\|_H^2 = d^2,$$

if $m \neq m_0$.

Therefore m_0 is unique.

Now suppose

$$u = m_1 + n_1,$$

where $m_1 \in M$ and $n_1 \in M^\perp$. As above, for $m \in M$

$$\begin{aligned} \|u - m\|_H^2 &= \|u - m_1 + m_1 - m\|_H^2 \\ &= \|u - m_1\|_H^2 + \|m - m_1\|_H^2, \\ &\geq \|u - m_1\|_H \end{aligned} \tag{69}$$

and thus since m_0 such that

$$d = \|u - m_0\|_H$$

is unique, we get

$$m_1 = m_0$$

and therefore

$$n_1 = u - m_0 = n_0.$$

The proof is complete.

Teorema 14.8 (The Riesz Lemma)**.** *Let H be a Hilbert space and let $f : H \to \mathbb{R}$ be a continuous linear functional. Then there exists a unique $u_0 \in H$ such that*

$$f(u) = (u, u_0)_H, \forall u \in H.$$

Moreover

$$\|f\|_{H^*} = \|u_0\|_H.$$

Proof. Define N by

$$N = \{u \in H \mid f(u) = 0\}.$$

Thus, as f is a continuous and linear N is a closed subspace of H. If $N = H$, then $f(u) = 0 = (u, \theta)_H, \forall u \in H$ and the proof would be complete. Thus assume $N \neq H$. By the last theorem there exists $v \neq \theta$ such that $v \in N^{\perp}$.

Define

$$u_0 = \frac{f(v)}{\|v\|_H^2} v.$$

Thus if $u \in N$ we have

$$f(u) = 0 = (u, u_0)_H = 0.$$

On the other hand if $u = \alpha v$ for some $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} f(u) &= \alpha f(v) \\ &= \frac{f(v)(\alpha v, v)_H}{\|v\|_H^2} \\ &= \left(\alpha v, \frac{f(v)v}{\|v\|_H^2}\right)_H \\ &= (\alpha v, u_0)_H. \end{aligned} \tag{70}$$

Therefore $f(u)$ equals $(u, u_0)_H$ in the space spanned by N and v. Now we show that this last space (then span of N and v) is in fact H. Just observe that given $u \in H$ we may write

$$u = \left(u - \frac{f(u)v}{f(v)}\right) + \frac{f(u)v}{f(v)}. \tag{71}$$

Since

$$u - \frac{f(u)v}{f(v)} \in N$$

we have finished the first part of the proof, that is, we have proven that

$$f(u) = (u, u_0)_H, \forall u \in H.$$

To finish the proof, assume $u_1 \in H$ is such that

$$f(u) = (u, u_1)_H, \forall u \in H.$$

Thus,

$$\begin{aligned} \|u_0 - u_1\|_H^2 &= (u_0 - u_1, u_0 - u_1)_H \\ &= (u_0 - u_1, u_0)_H - (u_0 - u_1, u_1)_H \\ &= f(u_0 - u_1) - f(u_0 - u_1) = 0. \end{aligned} \tag{72}$$

Hence $u_1 = u_0$.

Let us now prove that

$$\|f\|_{H^*} = \|u_0\|_H.$$

First observe that

$$\begin{aligned} \|f\|_{H^*} &= \sup\{f(u) \mid u \in H,\ \|u\|_H \leq 1\} \\ &= \sup\{|(u, u_0)_H| \mid u \in H,\ \|u\|_H \leq 1\} \\ &\leq \sup\{\|u\|_H \|u_0\|_H \mid u \in H,\ \|u\|_H \leq 1\} \\ &\leq \|u_0\|_H. \end{aligned} \tag{73}$$

On the other hand

$$\begin{aligned} \|f\|_{H^*} &= \sup\{f(u) \mid u \in H,\ \|u\|_H \leq 1\} \\ &\geq f\left(\frac{u_0}{\|u_0\|_H}\right) \\ &= \frac{(u_0, u_0)_H}{\|u_0\|_H} \\ &= \|u_0\|_H. \end{aligned} \tag{74}$$

From (73) and (74)

$$\|f\|_{H^*} = \|u_0\|_H.$$

The proof is complete.

Observação 14.9. *Similarly as above we may define a Hilbert space H over $\mathbb{C}$, that is, a complex one. In this case the complex inner product $(\cdot, \cdot)_H : H \times H \to \mathbb{C}$ is defined through the following properties:*

1. $(u, v)_H = \overline{(v, u)_H},\ \forall u, v \in H,$
2. $(u + v, w)_H = (u, w)_H + (v, w)_H,\ \forall u, v, w \in H,$
3. $(\alpha u, v)_H = \alpha(u, v)_H,\ \forall u, v \in H,\ \alpha \in \mathbb{C},$
4. $(u, u)_H \geq 0,\ \forall u \in H,$ *and* $(u, u) = 0,$ *if and only if* $u = \theta.$

Observe that in this case we have

$$(u, \alpha v)_H = \overline{\alpha}(u, v)_H,\ \forall u, v \in H,\ \alpha \in \mathbb{C},$$

where for $\alpha = a + bi \in \mathbb{C}$, we have $\overline{\alpha} = a - bi$. Finally, similar results as those proven above are valid for complex Hilbert spaces.

15 Orthonormal basis

In this section we study separable Hilbert spaces and the related orthonormal bases.

Definição 15.1. *Let H be a Hilbert space. A set $S \subset H$ is said to orthonormal if*

$$\|u\|_H = 1,$$

and

$$(u,v)_H = 0, \forall u,v \in S, \text{ such that } u \neq v.$$

If S is not properly contained in any other orthonormal set, it is said to be an orthonormal basis for H.

Teorema 15.2. *Let H be a Hilbert space and let $\{u_n\}_{n=1}^N$ be an orthonormal set. Then for all $u \in H$, we have*

$$\|u\|_H^2 = \sum_{n=1}^N |(u,u_n)_H|^2 + \left\| u - \sum_{n=1}^N (u,u_n)_H u_n \right\|_H^2.$$

Proof. Observe that

$$u = \sum_{n=1}^N (u,u_n)_H u_n + \left(u - \sum_{n=1}^N (u,u_n)_H u_n \right).$$

Furthermore, we may easily obtain that

$$\sum_{n=1}^N (u,u_n)_H u_n \text{ and } u - \sum_{n=1}^N (u,u_n)_H u_n$$

are orthogonal vectors so that

$$\begin{aligned}
\|u\|_H^2 &= (u,u)_H \\
&= \left\| \sum_{n=1}^N |(u,u_n)_H u_n \right\|_H^2 + \left\| u - \sum_{n=1}^N (u,u_n)_H u_n \right\|_H^2 \\
&= \sum_{n=1}^N |(u,u_n)_H|^2 + \left\| u - \sum_{n=1}^N (u,u_n)_H u_n \right\|_H^2. \qquad (75)
\end{aligned}$$

□

Corolário 15.3 (Bessel inequality)**.** *Let H be a Hilbert space and let $\{u_n\}_{n=1}^N$ be an orthonormal set. Then for all $u \in H$, we have*

$$\|u\|_H^2 \geq \sum_{n=1}^N |(u,u_n)_H|^2.$$

Teorema 15.4. *Each Hilbert space has an orthonormal basis.*

Proof. Define by C the collection of all orthonormal sets in H. Define an order in C by stating $S_1 \prec S_2$ if $S_1 \subset S_2$. Then C is partially ordered and obviously non-empty, since

$$v/\|v\|_H \in C, \forall v \in H, v \neq \theta.$$

Now let $\{S_\alpha\}_{\alpha\in L}$ be a linearly ordered subset of C. Clearly $\cup_{\alpha\in L} S_\alpha$ is an orthonormal set which is an upper bound for $\{S_\alpha\}_{\alpha\in L}$.

Therefore, every linearly ordered subset has an upper bound, so that by Zorn's lemma C has a maximal element, that is, an orthonormal set not properly contained in any other orthonormal set.

This completes the proof.

Teorema 15.5. *Let H be a Hilbert space and let $S = \{u_\alpha\}_{\alpha\in L}$ be an orthonormal basis. Then for each $v \in H$ we have*

$$v = \sum_{\alpha\in L} (u_\alpha, v)_H u_\alpha,$$

and

$$\|v\|_H^2 = \sum_{\alpha\in L} |(u_\alpha, v)_H|^2.$$

Proof. Let $L' \subset L$ a finite subset of L. From Bessel's inequality we have,

$$\sum_{\alpha\in L'} |(u_\alpha, v)_H| \leq \|v\|_H^2.$$

From this, we may infer that the set $A_n = \{\alpha \in L \mid |(u_\alpha, v)_H| > 1/n\}$ is finite, so that

$$A = \{\alpha \in L \mid |(u_\alpha, v)_H| > 0\} = \cup_{n=1}^\infty A_n$$

is at most countable.

Thus $(u_\alpha, v)_H \neq 0$ for at most countably many $\alpha's \in L$, which we order by $\{\alpha_n\}_{n\in\mathbb{N}}$. Since the sequence

$$s_N = \sum_{i=1}^N |(u_{\alpha_i}, v)_H|^2,$$

is monotone and bounded, it is converging to some real limit as $N \to \infty$. Define

$$v_n = \sum_{i=1}^n (u_{\alpha_i}, v)_H u_{\alpha_i},$$

so that for $n > m$ we have

$$\begin{aligned}\|v_n - v_m\|_H^2 &= \left\| \sum_{i=m+1}^n (u_{\alpha_i}, v)_H u_{\alpha_i} \right\|_H^2 \\ &= \sum_{i=m+1}^n |(u_{\alpha_i}, v)_H|^2 \\ &= |s_n - s_m|. \end{aligned} \tag{76}$$

Hence, $\{v_n\}$ is a Cauchy sequence which converges to some $v' \in H$.
Observe that

$$\begin{aligned}(v - v', u_{\alpha_l})_H &= \lim_{N\to\infty} (v - \sum_{i=1}^{N} (u_{\alpha_i}, v)_H u_{\alpha_i}, u_{\alpha_l})_H \\ &= (v, u_{\alpha_l})_H - (v, u_{\alpha_l})_H \\ &= 0. \end{aligned} \tag{77}$$

Also, if $\alpha \neq \alpha_l, \forall l \in \mathbb{N}$ then

$$(v - v', u_\alpha)_H = \lim_{N\to\infty} (v - \sum_{i=1}^{\infty} (u_{\alpha_i}, v)_H u_{\alpha_i}, u_\alpha)_H = 0.$$

Hence

$$v - v' \perp u_\alpha, \ \forall \alpha \in L.$$

If

$$v - v' \neq \theta,$$

then we could obtain an orthonormal set

$$\left\{ u_\alpha, \ \alpha \in L, \frac{v - v'}{\|v - v'\|_H} \right\}$$

which would properly contain the complete orthonormal set

$$\{u_\alpha, \ \alpha \in L\},$$

a contradiction.

Therefore $v - v' = \theta$, that is

$$v = \lim_{N\to\infty} \sum_{i=1}^{N} (u_{\alpha_i}, v)_H u_{\alpha_i}.$$

15.1 The Gram-Schmidt orthonormalization

Let H be a Hilbert space and and $\{u_n\} \subset H$ be a sequence of linearly independent vectors. Consider the procedure:

$$w_1 = u_1, \ \ v_1 = \frac{w_1}{\|w_1\|_H},$$

$$w_2 = u_2 - (v_1, u_2)_H v_1, \ \ v_2 = \frac{w_2}{\|w_2\|_H},$$

and inductively,

$$w_n = u_n - \sum_{k=1}^{n-1} (v_k, u_n)_H v_k, \ \ v_n = \frac{w_n}{\|w_n\|_H}, \forall n \in \mathbb{N}, n > 2.$$

Observe that clearly $\{v_n\}$ is an orthonormal set and for each $m \in \mathbb{N}$, $\{v_k\}_{k=1}^m$ and $\{u_k\}_{k=1}^m$ span the same vector subspace of H.

Such a process of obtaining the orthonormal set $\{v_n\}$ is known as the Gram-Schmidt orthonormalization.

We finish this section with the following theorem.

Teorema 15.6. *A Hilbert space H is separable if and only if has a countable orthonormal basis. If $dim(H) = N < \infty$, the H is isomorphic to $\mathbb{C}^N$. If $dim(H) = +\infty$ then H is isomorphic to l^2, where*

$$l^2 = \left\{ \{y_n\} \mid y_n \in \mathbb{C}, \forall n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} |y_n|^2 < +\infty \right\}.$$

Proof. Suppose H is separable and let $\{u_n\}$ be a countable dense set in H. To obtain an orthonormal basis it suffices to apply the Gram-Schmidt orthonormalization procedure to the greatest linearly independent subset of $\{u_n\}$.

Conversely, if $B = \{v_n\}$ is an orthonormal basis for H, the set of all finite linear combinations of elements of B with rational coefficients are dense in H, so that H is separable.

Moreover, if $dim(H) = +\infty$ consider the isomorphism $F : H \to l^2$ given by

$$F(u) = \{(u_n, u)_H\}_{n \in \mathbb{N}}.$$

Finally, if $dim(H) = N < +\infty$, consider the isomorphism $F : H \to \mathbb{C}^N$ given by

$$F(u) = \{(u_n, u)_H\}_{n=1}^N.$$

The proof is complete. □

16 Projection on a convex set

Teorema 16.1. *Let H be a Hilbert space and let $K \subset H$ be a non-empty, closed and convex set. Under such hypotheses, for each $f \in H$ there exists a unique $u \in K$ such that*

$$\|f - u\|_U = \min_{v \in K} \|f - v\|.$$

Moreover, $u \in K$ is such that

$$(f - u, v - u)_H \leq 0, \ \forall v \in K.$$

Proof. Define

$$d = \inf_{v \in K} \|f - v\|_H.$$

Hence, for each $n \in \mathbb{N}$ there exists $v_n \in K$ such that

$$d \leq \|f - v_n\|_H < d + 1/n.$$

Let $m, n \in \mathbb{N}$. Define $a = f - v_n$ and $b = f - v_m$. From the parallelogram law, we have

$$\|a+b\|_H^2 + \|a-b\|_H^2 = 2(\|a\|_H^2 + \|b\|_H^2),$$

that is,

$$\|2f - (v_n + v_m)\|_H^2 + \|v_n - v_m\|_H^2 = 2(\|f - v_n\|_H^2 + \|f - v_m\|_H^2),$$

so that

$$\begin{aligned} \|v_n - v_m\|_H^2 &= -4\left(\left\|f - \frac{v_n + v_m}{2}\right\|^2\right) + 2(\|f - v_n\|_H^2 + \|f - v_m\|_H^2) \\ &\leq -4d^2 + 2(d + 1/n)^2 + (2(d + 1/m)^2 \\ &\to -4d^2 + 2d^2 + 2d^2 \\ &= 0, \text{ as } m, n \to \infty. \end{aligned} \tag{78}$$

Hence $\{v_n\}$ is a Cauchy sequence so that there exists $u \in K$ such that

$$\|v_n - u\|_H \to 0, \text{ as } n \to \infty.$$

Therefore,

$$\|f - v_n\|_H \to \|f - u\|_H = d, \text{ as } n \to \infty.$$

Now let $v \in K$ and $t \in [0, 1]$. Define

$$w = (1-t)u + tv.$$

Observe that, since K is convex, $w \in K$, $\forall t \in [0, 1]$.

Hence,

$$\begin{aligned} \|f - u\|_H^2 &\leq \|f - w\|_H^2 \\ &= \|f - (1-t)u - tv\|_H^2 \\ &= \|(f - u) + t(u - v)\|_H^2 \\ &= \|f - u\|_H^2 + 2(f - u, u - v)_H t + t^2\|u - v\|_H^2. \end{aligned} \tag{79}$$

From this, we obtain

$$(f - u, v - u) \leq \frac{t}{2}\|u - v\|_H, \ \forall t \in [0, 1].$$

Letting $t \to 0^+$ we get

$$(f - u, v - u) \leq 0, \ \forall v \in K.$$

The proof is complete. □

Corolário 16.2. *In the context of the last theorem, assume*

$$(f - u, v - u) \leq 0, \forall v \in K.$$

Under such hypotheses,

$$\|f - u\|_H = \min_{v \in K} \|f - v\|_H.$$

Finally, such a $u \in K$ is unique.

Proof. Let $v \in K$. Thus,

$$\begin{aligned} \|f-u\|_H^2 - \|f-v\|_H^2 &= \|f\|^2 - 2(f,u)_H + \|u\|_H^2 \\ &\quad -\|f\|_H^2 + 2(f,v)_H - \|v\|_H^2 \\ &\quad -\|u-v\|_H^2 + \|u\|_H^2 - 2(u,v)_H + \|v\|_H^2 \\ &= 2(u,u)_H - 2(u,v)_H + 2(f,v) - 2(f,u) - \|u-v\|_H^2 \\ &= 2(f-u, v-u)_H - \|u-v\|_H^2 \\ &\leq 0. \end{aligned} \tag{80}$$

Summarizing,

$$\|f-v\|_H \geq \|f-u\|_H, \ \forall v \in K.$$

Suppose now that $u_1, u_2 \in K$ be such that

$$(f-u_1, v-u_1)_H \leq 0, \ \forall v \in K,$$

$$(f-u_2, v-u_2)_H \leq 0, \ \forall v \in K.$$

With $v = u_2$ in the last first inequality and $v = u_1$ in the last second one, we get

$$(f-u_1, u_2-u_1)_H \leq 0$$

and

$$(f-u_2, u_1-u_2)_H \leq 0.$$

Adding these two last inequalities, we obtain

$$(f, u_2-u_1)_H + (f, u_1-u_2)_H - (u_1, u_2-u_1)_H + (u_2, u_2-u_1)_H \leq 0,$$

that is,

$$\|u_2-u_1\|_H^2 \leq 0,$$

so that

$$\|u_1-u_2\|_H = 0,$$

and therefore,

$$u_1 = u_2.$$

Hence, the $u \in K$ in question is unique.

The proof is complete. □

Proposição 16.3. *Let H be a Hilbert space and $K \subset H$ a non-empty, closed and convex set. Let $f \in H$. Define $P_K(f) = u$ where $u \in K$ is such that*

$$\|f-u\|_H = \min_{v \in K} \|f-v\|_H.$$

Under such hypotheses,

$$\|P_K f_1 - P_K f_2\|_H \leq \|f_1 - f_2\|_H, \ \forall f_1, f_2 \in H.$$

Proof. Let $f_1, f_2 \in H$ and $u_1 = P_K f_1$ and $u_2 = P_K f_2$.

From the last proposition,

$$(f_1 - u_1, v - u_1)_H \leq 0, \ \forall v \in K$$

and

$$(f_2 - u_2, v - u_2)_H \leq 0, \ \forall v \in K.$$

With $v = u_2$ in the first last inequality and $v = u_1$ in the last second one, we obtain

$$(f_1 - u_1, u_2 - u_1)_H \leq 0,$$

$$(f_2 - u_2, u_1 - u_2)_H \leq 0.$$

Adding these two last inequalities, we get

$$(f_1, u_2 - u_1)_H - (f_2, u_2 - u_1)_H + (u_2 - u_1, u_2 - u_1)_H \leq 0,$$

that is,

$$\begin{aligned} \|u_2 - u_1\|_H^2 &\leq (f_2 - f_1, u_2 - u_1)_H \\ &\leq \|f_2 - f_1\|_H \|u_2 - u_1\|_H, \end{aligned} \tag{81}$$

so that

$$\|u_2 - u_1\|_H \leq \|f_2 - f_1\|_H.$$

This completes the proof. □

Corolário 16.4. *Let H be a Hilbert space and let $M \subset H$ be a closed vector subspace of H.*

Let $f \in H$. Thus, $u = P_M(f)$ is such that $u \in M$ and

$$(f - u, v)_H = 0, \forall v \in M.$$

Proof. In the previous results, we have got,

$$(f - u, v - u) \leq 0, \ \forall v \in M.$$

Let $v \in M$ be such that $v \neq \mathbf{0}$.

Thus,

$$(f - u, tv - u)_H \leq 0, \ \forall t \in \mathbb{R}.$$

Hence,

$$t(f - u, v)_H \leq (f - u, u), \ \forall t \in \mathbb{R}.$$

From this we obtain

$$(f - u, v) = 0, \ \forall v \in M.$$

□

Observação 16.5. *Reciprocally, if*

$$(f - u, v) = 0, \ \forall v \in M,$$

then,

$$(f - u, v - u)_H = 0, \ \forall v \in M,$$

so that

$$u = P_M(f).$$

17 The theorems of Stampacchia and Lax-Milgram

In this section we present the statement and proof of two well known results, namely, the Stampacchia and Lax-Milgram theorems.

Definição 17.1. *Let $a : H \times H \to \mathbb{R}$ be a bilinear form.*

1. *We say that a is bounded if there exists $c > 0$ such that*

$$|a(u, v)| \leq c\|u\|_H\|v\|_H, \ \forall u, v \in H.$$

2. *We say that a is coercive if there exists $\alpha > 0$ such that*

$$|a(v, v)| \geq \alpha\|v\|_H^2, \ \forall v \in H.$$

Teorema 17.2 (Stampacchia)**.** *Let H be a Hilbert space and let $a : H \times H \to \mathbb{R}$ be a bounded and coercive bilinear form.*

Let $K \subset H$ be a non-empty, closed and convex set. Under such hypotheses, for each $f \in H$ there exists a unique $u \in K$ such that

$$a(u, v - u) \geq (f, v - u)_H, \ \forall v \in K. \tag{82}$$

Moreover, if a is symmetric, that is, $a(u, v) = a(v, u)$, $\forall u, v \in H$, such $u \in K$ in question is also such that

$$\frac{1}{2}a(u, u) - (f, u)_H = \min_{v \in K}\left\{\frac{a(v, v)}{2} - (f, v)_H\right\}. \tag{83}$$

Proof. Fix $u \in H$. The function

$$v \mapsto a(u, v), \ \forall v \in H,$$

is continuous and linear.

Hence from the Riesz representation theorem, there exists a unique vector denoted by $A(u) \in H$ such that

$$(A(u), v)_H = a(u, v), \ \forall v \in H.$$

Clear such an operator A is linear, and

$$|(A(u), v)_H| = |a(u, v)| \leq c\|u\|_H\|v\|_H,$$

for some $c > 0$, so that

$$\|A(u)\|_H = \sup_{v \in H}\{|(A(u),v)_H| \; : \; \|v||_H \leq 1\} \leq c\|u\|_H^{\cdot}$$

Moreover,

$$(Av, v)_H = a(v, v) \geq \alpha \|v\|_H^2, \; \forall v \in H,$$

for some $\alpha > 0$.

Let $\rho > 0$ to be specified.

Define $T(v) = P_K(\rho f - \rho A(v) + v)$.

Observe that

$$\begin{aligned} \|T(v_1) - T(v_2)\|_H^2 &= \|P_K(\rho f - \rho A v_1 + v_1) - P_K(\rho f - \rho v_2 + v_2)\|_H^2 \\ &\leq \|\rho A(v_1) - \rho A(v_2) + (v_1 - v_2)\|_H^2 \\ &= \|v_1 - v_2\|_H^2 - 2\rho(A(v_1 - v_2), v_1 - v_2)_H + \rho^2 \|A(v_1 - v_2)\|_H^2 \\ &\leq (1 - 2\alpha\rho + c\rho^2)\|v_1 - v_2\|_H^2. \end{aligned} \tag{84}$$

Let $F(\rho) = 1 - 2\alpha\rho + c\rho^2$.

Thus, if $F'(\rho_0) = 0$ then

$$-2\alpha + 2\rho_0 c = 0,$$

that is,

$$\rho_0 = \frac{\alpha}{c}.$$

Therefore

$$F(\rho_0) = 1 - 2\frac{\alpha^2}{c} + c\frac{\alpha^2}{c^2} = 1 - \frac{\alpha^2}{c}.$$

Observe that we may redefine a larger $c > 0$ such that

$$0 < 1 - \frac{\alpha^2}{c} < 1.$$

Hence,

$$\|T(v_1) - T(v_2)\|_H \leq \lambda \|v_1 - v_2\|_H,$$

where

$$\lambda = \sqrt{1 - \frac{\alpha^2}{c}} < 1.$$

From this and Banach fixed point theorem, there exists $u \in K$ such that

$$T(u) = u,$$

that is,

$$P_K(\rho f - \rho A u + u) = u.$$

From this and Theorem 16.1, we obtain

$$(\rho f - \rho A(u) + u - u, v - u)_H \leq 0, \; \forall v \in K.$$

Thus,

$$a(u, v-u) = (A(u), v-u)_H \geq (f, v-u)_H, \ \forall v \in K.$$

Assume now that $a(u, v)$ is also symmetric. Thus, $a(u, v)$ define a inner product in H, inducing a norm

$$\sqrt{a(u,u)}$$

which is equivalent to $\|u\|_H$, since

$$\sqrt{c}\|u\|_H \geq \sqrt{a(u,u)} \geq \sqrt{\alpha}\|u\|_H, \ \forall u \in H.$$

From the Riesz representation theorem, there exists a unique $g \in H$ such that

$$(f, v)_H = a(g, v), \ \forall v \in H.$$

Similarly to the indicated above we may obtain $u \in K$ such that

$$u = P_K(g)$$

so that

$$a(g-u, v-u) \leq 0, \ \forall v \in K.$$

Hence,

$$a(g, v-u) - a(u, v-u) \leq 0,$$

that is

$$a(u, v-u) \geq a(g, v-u) = (f, v-u), \ \forall v \in K.$$

Moreover from $u = P_K(g)$ we obtain

$$a(g-u, g-u) = \min_{v \in K} a(g-v, g-v).$$

Therefore

$$a(g,g) - 2a(g,u) + a(u,u) \leq a(g,g) - 2a(g,v) + a(v,v),$$

that is,

$$\frac{a(u,u)}{2} - a(g,u) \leq \frac{a(v,v)}{2} - a(g,v),$$

so that

$$\frac{a(u,u)}{2} - (f,u)_H \leq \frac{a(v,v)}{2} - (f,v)_H, \ \forall v \in K.$$

The proof is complete. □

Corolário 17.3 (Lax-Milgram theorem)**.** *Assume $a(u, v)$ is a bounded, coercive and symmetric bilinear form on H. Under such hypotheses, there exists a unique $u \in H$ such that*

$$a(u, v) = (f, v), \ \forall v \in H.$$

Moreover, such a $u \in H$ is such that

$$\frac{1}{2}a(u,u) - (f,u)_H = \min_{v \in H} \frac{1}{2}a(v,v) - (f,v)_H.$$

Proof. The proof follows from the Stampacchia theorem with $K = H$. □

18 The Hahn-Banach Theorems and the Weak Topologies

19 Introduction

In this chapter we present the Hahn-Banach theorems and some important applications. Also, a study on weak topologies is developed in details.

20 The Hahn-Banach theorems

In this chapter U always denotes a Banach space.

Teorema 20.1 (The Hahn-Banach theorem)**.** *Consider a functional $p : U \to \mathbb{R}$ such that*

$$p(\lambda u) = \lambda p(u), \forall u \in U, \lambda > 0, \tag{85}$$

and

$$p(u+v) \leq p(u) + p(v), \forall u, v \in U. \tag{86}$$

Let $V \subset U$ be a proper subspace of U and let $g : V \to \mathbb{R}$ be a linear functional such that

$$g(u) \leq p(u), \forall u \in V. \tag{87}$$

Under such hypotheses, there exists a linear functional $f : U \to \mathbb{R}$ such that

$$g(u) = f(u), \forall u \in V, \tag{88}$$

and

$$f(u) \leq p(u), \forall u \in U. \tag{89}$$

Proof. Choose $z \in U \setminus V$. Denote by $\tilde{V}$ the space spanned by V and z, that is,

$$\tilde{V} = \{v + \alpha z \mid v \in V \text{ e } \alpha \in \mathbb{R}\}. \tag{90}$$

We may define an extension of g from V to $\tilde{V}$, denoted by $\tilde{g}$, by

$$\tilde{g}(\alpha z + v) = \alpha \tilde{g}(z) + g(v), \tag{91}$$

where $\tilde{g}(z)$ will be properly specified in the next lines.

Let $v_1, v_2 \in V$, $\alpha > 0$, $\beta > 0$. Thus,

$$\begin{aligned} \beta g(v_1) + \alpha g(v_2) &= g(\beta v_1 + \alpha v_2) \\ &= (\alpha+\beta) g\left(\frac{\beta}{\alpha+\beta} v_1 + \frac{\alpha}{\alpha+\beta} v_2\right) \\ &\leq (\alpha+\beta) p\left(\frac{\beta}{\alpha+\beta}(v_1 - \alpha z) + \frac{\alpha}{\alpha+\beta}(v_2 + \beta z)\right) \\ &\leq \beta p(v_1 - \alpha z) + \alpha p(v_2 + \beta z) \end{aligned} \tag{92}$$

and therefore

$$\frac{1}{\alpha}[-p(v_1-\alpha z)+g(v_1)] \leq \frac{1}{\beta}[p(v_2+\beta z)-g(v_2)],$$

$\forall v_1, v_2 \in V,\ \alpha, \beta > 0$. Thus, there exists $a \in \mathbb{R}$ such that

$$\sup_{v \in V, \alpha > 0}\left[\frac{1}{\alpha}(-p(v-\alpha z)+g(v))\right] \leq a \leq \inf_{v \in V, \alpha > 0}\left[\frac{1}{\alpha}(p(v+\alpha z)-g(v))\right]. \tag{93}$$

We shall define $\tilde{g}(z) = a$. Therefore, if $\alpha > 0$, then

$$\begin{aligned}\tilde{g}(\alpha z + v) &= a\alpha + g(v) \\ &\leq \left[\frac{1}{\alpha}(p(v+\alpha z)-g(v))\right]\alpha + g(v) \\ &= p(v+\alpha z). \end{aligned} \tag{94}$$

On the other hand, if $\alpha < 0$, then $-\alpha > 0$. Thus,

$$a \geq \frac{1}{-\alpha}(-p(v-(-\alpha)z)+g(v)),$$

so that

$$\begin{aligned}\tilde{g}(\alpha z + v) &= a\alpha + g(v) \\ &\leq \left[\frac{1}{-\alpha}(-p(v+\alpha z)+g(v))\right]\alpha + g(v) \\ &= p(v+\alpha z) \end{aligned} \tag{95}$$

and hence

$$\tilde{g}(u) \leq p(u), \forall u \in \tilde{V}.$$

Define now by $\mathcal{E}$ the set of all extensions e of g, which satisfy $e(u) \leq p(u)$ on the domain of e, where such a domain is always a subspace of U. We shall also define a partial order for $\mathcal{E}$ denoting $e_1 \prec e_2$ as the domain of e_2 contains the domain of e_1 and $e_1 = e_2$ on the domain of e_1. Let $\{e_\alpha\}_{\alpha \in A}$ be an ordered subset of $\mathcal{E}$. Let V_α be the domain of e_α, $\forall \alpha \in A$. Define e on $\cup_{\alpha \in A} V_\alpha$ by setting $e = e_\alpha$ on V_α. Clearly $e_\alpha \prec e$, $\forall \alpha \in A$ so that each ordered subset of $\mathcal{E}$ has an upper bound. From this and Zorn Lemma, $\mathcal{E}$ has a maximal element f defined on some subspace $\tilde{U} \subset U$ such that $f(u) \leq p(u), \forall u \in \tilde{U}$. Suppose, to obtain contradiction, that $\tilde{U} \neq U$ and let $z_1 \in U \setminus \tilde{U}$. As above indicated, we may obtain an extension f_1 from $\tilde{U}$ to the subspace spanned by z_1 and $\tilde{U}$, which contradicts the maximality of f.

The proof is complete.

Definição 20.2 (Topological dual spaces). *Let U be a Banach space. We shall define its dual topological space, as the set of all linear continuous functionals defined on U. We suppose such a dual space of U, may be represented by another vector space U^*, through a bilinear form $\langle \cdot, \cdot \rangle_U : U \times U^* \to \mathbb{R}$ (here we are referring to standard representations of dual spaces of Sobolev and Lebesgue spaces, to* *be addressed in the subsequent chapters). Thus, given $f : U \to \mathbb{R}$ linear and continuous, we assume the existence of a unique $u^* \in U^*$ such that*

$$f(u) = \langle u, u^* \rangle_U, \forall u \in U. \tag{96}$$

The norm of f , denoted by $\|f\|_{U^*}$, is defined as

$$\|f\|_{U^*} = \sup_{u \in U}\{|\langle u, u^* \rangle_U| \ : \ \|u\|_U \leq 1\} = \|u^*\|_{U^*}. \tag{97}$$

Corolário 20.3. *Let $V \subset U$ be a proper subspace of U and let $g : V \to \mathbb{R}$ be a linear and continuous functional with norm*

$$\|g\|_{V^*} = \sup_{u \in V}\{|g(u)| \mid \|u\|_U \leq 1\}. \tag{98}$$

Under such hypotheses, there exists u^* in U^* such that

$$\langle u, u^* \rangle_U = g(u), \forall u \in V, \tag{99}$$

and

$$\|u^*\|_{U^*} = \|g\|_{V^*}. \tag{100}$$

Proof. It suffices to apply Theorem 20.1 with $p(u) = \|g\|_{V^*}\|u\|_V$. Indeed, from such a theorem, there exists a linear functional $f : U \to \mathbb{R}$ such that

$$f(u) = g(u), \ \forall u \in V$$

and

$$f(u) \leq p(u) = \|g\|_{V^*}\|u\|_U,$$

that is,

$$|f(u)| \leq p(u) = \|g\|_{V^*}\|u\|_U, \forall u \in U.$$

Therefore,

$$\|f\|_{U^*} = \sup_{u \in U}\{|f(u)| \ : \ \|u\|_U \leq 1\} \leq \|g\|_{V^*}.$$

On the other hand,

$$\|f\|_{U^*} \geq \sup_{u \in V}\{|f(u)| \ : \ \|u\|_U \leq 1\} = \|g\|_{V^*}.$$

Thus,

$$\|f\|_{U^*} = \|g\|_{V^*}.$$

Finally, since f linear and continuous, there exists $u^* \in U^*$ such that

$$f(u) = \langle u, u^* \rangle_U, \ \forall u \in U,$$

and hence

$$\langle u, u^* \rangle_U = f(u) = g(u), \ \forall u \in V.$$

Moreover,

$$\|u^*\|_{U^*} = \|f\|_{U^*} = \|g\|_{V^*}.$$

The proof is complete.

Corolário 20.4. *Let $u_0 \in U$. Under such hypotheses, there exists $u_0^* \in U^*$ such that*

$$\|u_0^*\|_{U^*} = \|u_0\|_U \text{ and } \langle u_0, u_0^* \rangle_U = \|u_0\|_U^2. \tag{101}$$

Proof. It suffices to apply the Corollary 20.3 with $V = \{\alpha u_0 \mid \alpha \in \mathbb{R}\}$ and $g(tu_0) = t\|u_0\|_U^2$ so that $\|g\|_{V^*} = \|u_0\|_U$.

Indeed, from the last corollary, there exists $u_0^* \in U^*$ such that

$$\langle tu_0, u_0^* \rangle_U = g(tu_0), \ \forall t \in \mathbb{R},$$

and

$$\|u_0^*\|_{U^*} = \|g\|_{V^*},$$

where,

$$\|g\|_{V^*} = \sup_{t \in \mathbb{R}} \{t\|u_0\|_U^2 \ : \ \|tu_0\|_U \leq 1\} = \|u_0\|_U.$$

Moreover, also from the last corollary,

$$\|u_0^*\|_{U^*} = \|g\|_{V^*} = \|u_0\|_U.$$

Finally,

$$\langle tu_0, u_0^* \rangle_U = g(tu_0) = t\|u_0\|_U^2, \ \forall t \in \mathbb{R},$$

so that

$$\langle u_0, u_0^* \rangle_U = \|u_0\|_U^2.$$

This completes the proof.

Corolário 20.5. *Let $u \in U$. Under such hypotheses*

$$\|u\|_U = \sup_{u^* \in U^*} \{|\langle u, u^* \rangle_U| \mid \|u^*\|_{U^*} \leq 1\}. \tag{102}$$

Proof. Suppose $u \neq \mathbf{0}$, otherwise the result is immediate. Since

$$|\langle u, u^* \rangle_U| \leq \|u\|_U \|u^*\|_{U^*}, \forall u \in U, u^* \in U^*$$

we have

$$\sup_{u^* \in U^*} \{|\langle u, u^* \rangle_U| \mid \|u^*\|_{U^*} \leq 1\} \leq \|u\|_U. \tag{103}$$

However, from the last corollary, there exists $u_0^* \in U^*$ such that $\|u_0^*\|_{U^*} = \|u\|_U$ and $\langle u, u_0^* \rangle_U = \|u\|_U^2$. Define $u_1^* = \|u\|_U^{-1} u_0^*$. Thus, $\|u_1^*\|_U = 1$ and $\langle u, u_1^* \rangle_U = \|u\|_U$.

The proof is complete.

Definição 20.6 (Affine hyperplane)**.** *Let U be a Banach space. An affine hyperplane H is a set defined by*

$$H = \{u \in U \mid \langle u, u^* \rangle_U = \alpha\} \tag{104}$$

for some $u^* \in U^*$ and $\alpha \in \mathbb{R}$.

Proposição 20.7. *An affine hyperplane H defined as above indicated is closed.*

Proof. The result follows directly from the continuity of $\langle u, u^* \rangle_U$ as functional on U.

Definição 20.8 (Separation)**.** *Let $A, B \subset U$. We say that a hyperplane H, as above indicated separates A and B, as there exist $\alpha \in \mathbb{R}$ and $u^* \in U^*$ such that*

$$\langle u, u^* \rangle_U \leq \alpha, \forall u \in A, \text{ and } \langle u, u^* \rangle_U \geq \alpha, \forall u \in B. \tag{105}$$

We say that H separates A and B strictly if there exists $\varepsilon > 0$ such that

$$\langle u, u^* \rangle_U \leq \alpha - \varepsilon, \forall u \in A, \text{ and } \langle u, u^* \rangle_U \geq \alpha + \varepsilon, \forall u \in B, \tag{106}$$

Teorema 20.9 (The Hahn-Banach theorem, the geometric form)**.** *Let $A, B \subset U$ be two non-empty, convex sets such that $A \cap B = \emptyset$ and A is open. Under such hypotheses, there exists a closed hyperplane which separates A and B, that is, there exist $\alpha \in \mathbb{R}$ and $u^* \in U^*$ such that*

$$\langle u, u^* \rangle_U \leq \alpha \leq \langle v, u^* \rangle_U, \ \forall u \in A, \ v \in B.$$

To prove such a theorem, we need two lemmas.

Lema 20.10. *Let $C \subset U$ be a convex set such that $\mathbf{0} \in C$. For each $u \in U$ define*

$$p(u) = \inf\{\alpha > 0, \ \alpha^{-1} u \in C\}. \tag{107}$$

Under such hypotheses, p is such that there exists $M \in \mathbb{R}^+$ such that

$$0 \leq p(u) \leq M \|u\|_U, \forall u \in U, \tag{108}$$

and

$$C = \{u \in U \mid p(u) < 1\}. \tag{109}$$

Moreover,

$$p(u + v) \leq p(u) + p(v), \forall u, v \in U.$$

Proof. Let $r > 0$ be such that $B(\mathbf{0}, r) \subset C$. Let $u \in U$ such that $u \neq \mathbf{0}$. Thus,

$$\frac{u}{\|u\|_U} r \in \overline{B(\mathbf{0}, r)} \subset \overline{C},$$

and therefore

$$p(u) \leq \frac{\|u\|_U}{r}, \forall u \in U \tag{110}$$

which proves (108). Suppose now $u \in C$. Since C is open there exists $\varepsilon > 0$ sufficiently small such that $(1 + \varepsilon) u \in C$. Thus, $p(u) \leq \frac{1}{1+\varepsilon} < 1$. Reciprocally, if $p(u) < 1$, there exists $0 < \alpha < 1$ such that $\alpha^{-1} u \in C$ and hence, since C is convex, we get $u = \alpha(\alpha^{-1} u) + (1 - \alpha)\mathbf{0} \in C$.

Finally, let $u, v \in C$ and $\varepsilon > 0$. Thus, $\frac{u}{p(u)+\varepsilon} \in C$ and $\frac{v}{p(v)+\varepsilon} \in C$ so that $\frac{tu}{p(u)+\varepsilon} + \frac{(1-t)v}{p(v)+\varepsilon} \in C, \forall t \in [0, 1]$. Particularly, for $t = \frac{p(u)+\varepsilon}{p(u)+p(v)+2\varepsilon}$ we obtain $\frac{u+v}{p(u)+p(v)+2\varepsilon} \in C$, and thus,

$$p(u + v) \leq p(u) + p(v) + 2\varepsilon, \forall \varepsilon > 0.$$

The proof of this lemma is complete.

Lema 20.11. *Let $C \subset U$ be an non-empty, open and convex set and let $u_0 \in U$ such that $u_0 \notin C$. Under such hypotheses, there exists $u^* \in U^*$ such that $\langle u, u^* \rangle_U < \langle u_0, u^* \rangle_U, \forall u \in C$*

Proof. By translation, if necessary, there is no loos in generality in assuming $\mathbf{0} \in C$. Consider the functional p defined in the last lema. Define $V = \{\alpha u_0 \mid \alpha \in \mathbb{R}\}$. Define also g on V, by

$$g(tu_0) = t, \ \forall t \in \mathbb{R}. \tag{111}$$

Let $t \in \mathbb{R}$ be such that $t \neq 0$. Since

$$\frac{tu_0}{t} = u_0 \notin C,$$

we have

$$g(tu_0) = t \leq p(tu_0)$$

and therefore

$$g(u) \leq p(u), \forall u \in V.$$

From the Hahn-Banach theorem, there exists a linear functional f defined on U which extends g such that

$$f(u) \leq p(u) \leq M\|u\|_U. \tag{112}$$

Here, we have applied the Lemma 20.10. In particular, $f(u_0) = g(u_0) = g(1u_0) = 1$, also from the last lemma, $f(u) < 1, \forall u \in C$. The existence of u^* satisfying this lemma conclusion follows from the continuity of f, indicated in (112).

Proof of Theorem 20.9. Define $C = A + (-B)$ so that $C$is convex and $\mathbf{0} \notin C$. From Lemma 20.11, there exists $u^* \in U^*$ such that $\langle w, u^* \rangle_U < 0, \forall w \in C$, and thus,

$$\langle u, u^* \rangle_U < \langle v, u^* \rangle_U, \forall u \in A, \quad v \in B. \tag{113}$$

Therefore, there exists $\alpha \in \mathbb{R}$ such that

$$\sup_{u \in A} \langle u, u^* \rangle_U \leq \alpha \leq \inf_{v \in B} \langle v, u^* \rangle_U, \tag{114}$$

which completes the proof.

Proposição 20.12. *Let U be a Banach space and let $A, B \subset U$ be such that A is compact, B is closed and $A \cap B = \emptyset$.*

Under such hypotheses, there exists $\varepsilon_1 > 0$ such that

$$[A + B_{\varepsilon_1}(\mathbf{0})] \cap [B + B_{\varepsilon_1}(\mathbf{0})] = \emptyset.$$

Proof. Suppose, to obtain contradiction, the proposition conclusion is false.

Thus, for each $n \in \mathbb{N}$ there exists $u_n \in U$ such that $d(u_n, A) < \frac{1}{n}$ and $d(u_n, B) < \frac{1}{n}$.

Therefore, there exist $v_n \in A$ and $w_n \in B$ such that

$$\|u_n - v_n\|_U < \frac{1}{n} \tag{115}$$

e

$$\|u_n - w_n\|_U < \frac{1}{n}, \ \forall n \in \mathbb{N}. \tag{116}$$

Since $\{v_n\} \subset A$ and A is compact, there exist a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ and $v_0 \in A$, such that

$$\|v_{n_j} - v_0\|_U \to 0, \text{ as } j \to \infty.$$

Thus, from this, (115) and (116) we obatin,

$$\|u_{n_j} - v_0\|_U \to 0, \text{ quando } j \to \infty,$$

e

$$\|w_{n_j} - v_0\|_U \to 0, \text{ quando } j \to \infty.$$

Since A and B are closed we may infer that

$$v_0 \in A \cap B,$$

which contradicts $A \cap B = \emptyset$.

The proof is complete.

Teorema 20.13 (The Hahn-Banach theorem, the second geometric form)**.** *Let $A, B \subset U$ be two non-empty, convex sets such that $A \cap B = \emptyset$. Suppose A is compact and B is closed. Under such hypotheses, there exists an hyperplane which separates A and B strictly.*

Proof. Observe that, from the last proposition, there exists $\varepsilon > 0$ sufficiently small such that $A_\varepsilon = A + B(0, \varepsilon)$ and $B_\varepsilon = B + B(0, \varepsilon)$ are disjoint and convex sets. From Theorem 20.9, there exists $u^* \in U^*$ such that $u^* \neq \mathbf{0}$ and

$$\langle u + \varepsilon w_1, u^* \rangle_U \leq \langle u + \varepsilon w_2, u^* \rangle_U, \forall u \in A, \ v \in B, \ w_1, w_2 \in B(0, 1). \tag{117}$$

Thus, there exists $\alpha \in \mathbb{R}$ such that

$$\langle u, u^* \rangle_U + \varepsilon \|u^*\|_{U^*} \leq \alpha \leq \langle v, u^* \rangle_U - \varepsilon \|u^*\|_{U^*}, \forall u \in A, \ v \in B. \tag{118}$$

The proof is complete.

Corolário 20.14. *Suppose $V \subset U$ is a vector subspace such that $\overline{V} \neq U$. Under such hypotheses, there exists $u^* \in U^*$ such that $u^* \neq \mathbf{0}$ and*

$$\langle u, u^* \rangle_U = 0, \forall u \in V. \tag{119}$$

Proof. Let $u_0 \in U$ be such that $u_0 \notin \overline{V}$. Applying Theorem 20.9 to $A = \overline{V}$ and $B = \{u_0\}$ we obtain $u^* \in U^*$ and $\alpha \in \mathbb{R}$ such that $u^* \neq \mathbf{0}$ e

$$\langle u, u^* \rangle_U < \alpha < \langle u_0, u^* \rangle_U, \forall u \in V. \tag{120}$$

Since V is a subspace, we must have $\langle u, u^* \rangle_U = 0, \forall u \in V$.

21 The weak topologies

Definição 21.1 (Weak neighborhoods)**.** *Let U be a Banach space and let $u_0 \in U$. We define a weak neighborhood of u_0, denoted by $\mathcal{V}_w(u_0)$, as*

$$\mathcal{V}_w(u_0) = \{u \in U \mid |\langle u - u_0, u_i^* \rangle_U| < \varepsilon_i, \forall i \in \{1, ..., m\}\}, \tag{121}$$

for some $m \in \mathbb{N}$, $\varepsilon_i > 0$, and $u_i^* \in U^*$, $\forall i \in \{1, ..., m\}$.

Let $A \subset U$. We say that $u_0 \in A$ is weakly interior to A, as there exists a weak neighborhood $V_w(u_0)$ of u_0 contained in A.

If all points of A are weakly interior, we say that A is weakly open.

Finally, we define the weak topology $\sigma(U, U^*)$ for U, as the set of all subsets weakly open of U.

Proposição 21.2. *A Banach space U is Hausddorff as endowed with the weak topology $\sigma(U, U^*)$.*

Proof. Choose $u_1, u_2 \in U$ such that $u_1 \neq u_2$. From the Hahn-Banach theorem, second geometric form, there exists an hyperplane separating $\{u_1\}$ e $\{u_2\}$ strictly, thats is, there exist $u^* \in U^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle u_1, u^* \rangle_U < \alpha < \langle u_2, u^* \rangle_U. \tag{122}$$

Define

$$\mathcal{V}_{w1}(u_1) = \{u \in U \mid |\langle u - u_1, u^* \rangle| < \alpha - \langle u_1, u^* \rangle_U\}, \tag{123}$$

and

$$\mathcal{V}_{w2}(u_2) = \{u \in U \mid |\langle u - u_2, u^* \rangle_U| < \langle u_2, u^* \rangle_U - \alpha\}. \tag{124}$$

We claim that

$$V_{w_1}(u_1) \cap V_{w_2}(u_2) = \emptyset.$$

Suppose, to obtain contradiction, there exists $u \in V_{w_1}(u_1) \cap V_{w_2}(u_2)$.

Thus,

$$\langle u - u_1, u^* \rangle_U < \alpha - \langle u_1, u^* \rangle_U,$$

and therefore

$$\langle u, u^* \rangle_U < \alpha.$$

Also

$$-\langle u - u_2, u^* \rangle_U < \langle u_2, u^* \rangle_U - \alpha,$$

and hence

$$\langle u, u^* \rangle_U > \alpha.$$

We have got

$$\langle u, u^* \rangle_U < \alpha < \langle u, u^* \rangle_U,$$

a contradiction.

Summarizing, we have obtained $u_1 \in \mathcal{V}_{w_1}(u_1)$, $u_2 \in \mathcal{V}_{w_2}(u_2)$ and $\mathcal{V}_{w_1}(u_1) \cap \mathcal{V}_{w_2}(u_2) = \emptyset$.

The proof is complete.

Observação 21.3. *If* $\{u_n\} \in U$ *is such that* u_n *converges to* u *in* $\sigma(U, U^*)$*, then we write* $u_n \rightharpoonup u$*, weakly.*

Proposição 21.4. *Let* U *be a Banach space. For a sequence* $\{u_n\} \subset U$*, we have*

1. $u_n \rightharpoonup u$*, for* $\sigma(U, U^*) \Leftrightarrow \langle u_n, u^* \rangle_U \to \langle u, u^* \rangle_U, \forall u^* \in U^*$,
2. *If* $u_n \to u$ *strongly (in norm), then* $u_n \rightharpoonup u$ *weakly,*
3. *If* $u_n \rightharpoonup u$ *weakly, then* $\{\|u_n\|_U\}$ *is bounded and* $\|u\|_U \leq \liminf_{n\to\infty} \|u_n\|_U$*,*
4. *If* $u_n \rightharpoonup u$ *weakly and* $u_n^* \to u^*$ *weakly in* U^**, then* $\langle u_n, u_n^* \rangle_U \to \langle u, u^* \rangle_U$*.*

Proof. 1. The result follows from the definition of $\sigma(U, U^*)$.

Indeed, suppose that $\{u_n\} \subset U$ and $u_n \rightharpoonup u$, weakly.

Let $u^* \in U^*$ and let $\varepsilon > 0$.

Define

$$V_w(u) = \{v \in U \ : \ |\langle v - u, u^* \rangle_U| < \varepsilon\}.$$

From the hypotheses, there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$, then

$$u_n \in V_w(u).$$

That is,

$$|\langle u_n - u, u^* \rangle_U| < \varepsilon,$$

if $n > n_0$.

Therefore,

$$\langle u_n, u^* \rangle_U \to \langle u, u^* \rangle_U, \text{ as } n \to \infty$$

$\forall u^* \in U^*$.

Reciprocally, suppose that

$$\langle u_n, u^* \rangle_U \to \langle u, u^* \rangle_U, \text{ as } n \to \infty$$

$\forall u^* \in U^*$.

Let $V(u) \in \sigma(U, U^*)$ be a set which contains $\{u\}$.

Thus, there exists a weak neighborhood a $V_w(u)$ such that $u \in V_w(u) \subset V(u)$, where there exist $m \in \mathbb{N}$, $\varepsilon_i > 0$ and $u_i^* \in U^*$ such that

$$V_w(u) = \{v \in U \ : \ |\langle v - u, u_i^* \rangle_U| < \varepsilon_i, \ \forall i \in \{1, \cdots, m\}\}.$$

From the hypotheses, for each $i \in \{1, \cdots m\}$, there exists $n_i \in \mathbb{N}$ such that if $n > n_i$, then

$$|\langle u_n - u, u_i^* \rangle_U| < \varepsilon_i.$$

Define $n_0 = \max\{n_1, \cdots, n_m\}$.

Thus

$$u_n \in V_w(u) \subset V(u), \quad \text{if } n > n_0.$$

From this we may infer that $u_n \rightharpoonup u$ for $\sigma(U, U^*)$.

2. This follows from the inequality

$$|\langle u_n, u^* \rangle_U - \langle u, u^* \rangle_U| \leq \|u^*\|_{U^*} \|u_n - u\|_U. \tag{125}$$

3. For each $u^* \in U^*$ the sequence $\{\langle u_n, u^* \rangle_U\}$ is convergent for some bounded sequence. From this and the Uniform Boundedness Principle, there exists $M > 0$ such that $\|u_n\|_U \leq M, \forall n \in \mathbb{N}$. Moreover, for $u^* \in U^*$, we have

$$|\langle u_n, u^* \rangle_U| \leq \|u^*\|_{U^*} \|u_n\|_U, \tag{126}$$

and letting $n \to \infty$, we obtain

$$|\langle u, u^* \rangle_U| \leq \liminf_{n \to \infty} \|u^*\|_{U^*} \|u_n\|_U. \tag{127}$$

Thus,

$$\|u\|_U = \sup_{u^* \in U^*} \{|\langle u, u^* \rangle_U| \ : \ \|u\|_{U^*} \leq 1\} \leq \liminf_{n \to \infty} \|u_n\|_U. \tag{128}$$

4. Just observe that

$$\begin{aligned} |\langle u_n, u_n^* \rangle_U - \langle u, u^* \rangle_U| &\leq |\langle u_n, u_n^* - u^* \rangle_U| \\ &\quad + |\langle u - u_n, u^* \rangle_U| \\ &\leq \|u_n^* - u^*\|_{U^*} \|u_n\|_U \\ &\quad + |\langle u_n - u, u^* \rangle_U| \\ &\leq M \|u_n^* - u^*\|_{U^*} \\ &\quad + |\langle u_n - u, u^* \rangle_U| \\ &\to 0, \text{ quando } n \to \infty. \end{aligned} \tag{129}$$

Teorema 21.5. *Let U be a Banach space and let $A \subset U$ be a non-empty convex set. Under such hypotheses, A is closed for the topology $\sigma(U, U^*)$ if, and only if, A is closed for the the topology induced by $\| \cdot \|_U$.*

Proof. If $A = U$ the result is immediate. Thus, assume $A \neq U$. Suppose that A is strongly closed. Let $u_0 \notin A$. From the Hahn-Banach theorem there exists a closed hyperplane which separates u_0 and A strictly, that is, there exist $\alpha \in \mathbb{R}$ and $u^* \in U^*$ such that

$$\langle u_0, u^* \rangle_U < \alpha < \langle v, u^* \rangle_U, \forall v \in A. \tag{130}$$

Define

$$\mathcal{V} = \{u \in U \mid \langle u, u^* \rangle_U < \alpha\}, \tag{131}$$

so that $u_0 \in \mathcal{V}$, $\mathcal{V} \subset U \setminus A$.

Let

$$V_w(u_0) = \{v \in U \ : \ |\langle v - u_0, u^* \rangle_U| < \alpha - \langle u_0, u^* \rangle_U.$$

Let $v \in V_w(u_0)$.

Thus,

$$\begin{aligned} \langle v, u^* \rangle_U &= \langle v - u_0 + u_0, u^* \rangle_U \\ &= \langle v - u_0, u^* \rangle_U + \langle u_0, u^* \rangle_U \\ &\leq |\langle v - u_0, u^* \rangle_U| + \langle u_0, u^* \rangle_U \\ &< \alpha - \langle u_0, u^* \rangle_U + \langle u_0, u^* \rangle_U \\ &= \alpha. \end{aligned} \tag{132}$$

From this we may infer that $V_w(u_0) \subset \mathcal{V} \subset U \setminus A$, that is, u_0 is an interior point for $\sigma(U, U^*)$ of $U \setminus A$, $\forall u_0 \in U \setminus A$

Therefore, $\mathcal{V}$ is weakly open.

Summarizing, $U \setminus A$ is open in $\sigma(U, U^*)$ and thus A is closed for $\sigma(U, U^*)$ (weakly closed).

Finally, the reciprocal is immediate.

Teorema 21.6. *Let (Z, σ) be a topological space and let U be a Banach space. Let $\phi : Z \to U$ be a function, considering U with the weak topology $\sigma(U, U^*)$.*

Under such hypotheses, ϕ is continuous if, and only if, $f_{u^*} : Z \to \mathbb{R}$, where

$$f_{u^*}(z) = \langle \phi(z), u^* \rangle_U$$

is continuous, $\forall u^* \in U^*$.

Proof. Assume ϕ is continuous. Let $z_0 \in Z$ and let $\{z_\alpha\}_{\alpha \in I}$ be a net such that

$$z_\alpha \to z_0.$$

From the hypotheses,

$$\phi(z_\alpha) \rightharpoonup \phi(z_0), \text{ in } \sigma(U, U^*).$$

Therefore,

$$\langle \phi(z_\alpha), u^* \rangle_U \to \langle \phi(z_0), u^* \rangle_U, \ \forall u^* \in U^*.$$

Thus, f_{u^*} is continuous at z_0, $\forall u^* \in U^*$, $\forall z_0 \in Z$.

Reciprocally, assume $f_{u^*} : Z \to \mathbb{R}$, where

$$f_{u^*}(z) = \langle \phi(z), u^* \rangle_U$$

is continuous, $\forall u^* \in U^*$.

Suppose, to obtain contradiction, that ϕ is not continuous.

Thus, there exists $z_0 \in Z$ such that ϕ is not continuous at z_0.

In particular, there exists a net $\{z_\alpha\}_{\alpha \in I}$ such that $z_\alpha \to z_0$ and we do not have

$$\phi(z_\alpha) \rightharpoonup \phi(z_0), \text{ em } \sigma(U, U^*).$$

Hence, there exists $u^* \in U^*$ such that we do not have

$$\langle \phi(z_\alpha), u^* \rangle_U \to \langle \phi(z_0), u^* \rangle_U,$$

and thus f_{u^*} is not continuous at z_0, a contradiction.

Therefore, ϕ is continuous.

The proof is complete. □

22 The weak-star topology

Definição 22.1 (Reflexive spaces)**.** *Let U be a Banach space. We say that U is reflexive, if the canonical injection*

$$J : U \to U^{**}$$

is onto, where

$$\langle u, u^* \rangle_U = \langle u^*, J(u) \rangle_{U^*}, \ \forall u \in U, \ u^* \in U^*.$$

Thus, if U is reflexive, we may identify the bi-dual space of U, U^{**}, with U.

The weak topology for U^* may be defined similarly to $\sigma(U, U^*)$ and it is denoted by $\sigma(U^*, U^{**})$.

We define as well, the weak-star topology for U^*, denoted by $\sigma(U^*, U)$, as it follows.

Firstly, we define weak-star neighborhoods.

Let $u_0^* \in U^*$. We define a weak-star neighborhood for u_0^*, denoted by $V_w(u_0^*)$, as

$$V_w(u_0^*) = \{u^* \in U^* \ : \ |\langle u_i, u^* - u_0^* \rangle_U| < \varepsilon_i, \ \forall i \in \{1, \cdots, m\}\},$$

where $m \in \mathbb{N}$, $\varepsilon_i > 0$ e $u_i \in U$, $\forall i \in \{1, \cdots, m\}$.

Let $A \subset U^*$. We say that $u_0^* \in A$ is weakly-star interior to A, as there exists a weak-star neighborhood $V_w(u_0^*)$ contained in A.

If all point of A are weakly-star interior, we say that A weakly-star open.

Finally, we define the weak-star topology $\sigma(U^*, U)$ for U^*, as the set of all subsets weakly-star open of U^*.

Observe that $\sigma(U^*, U^{**})$ and $\sigma(U^*, U)$ coincide if U is reflexive.

23 Weak-star compactness

Teorema 23.1 (Banach and Alaoglu)**.** *Let U be a Banach space. Denote*

$$B_{U^*} = \{u^* \in U^* \ : \ \|u^*\|_{U^*} \leq 1\}.$$

Under such hypotheses, B_{U^*} is compact for U^* with the weak-star topology $\sigma(U^*, U)$.

Proof. For each $u \in U$, we shall associate a real number ω_u and denote

$$\omega = \prod_{u \in U} \omega_u \in \mathbb{R}^U,$$

and consider the projections

$$P_u : \mathbb{R}^U \to \mathbb{R}$$

where

$$P_u(\omega) = \omega_u, \ \forall \omega \in \mathbb{R}^U, \ u \in U.$$

We shall define a topology for $\mathbb{R}^U$, which is induced by the weak neighborhoods specified in the next lines.

Let $\tilde{\omega} \in \mathbb{R}^U$. We define a weak neighborhood $\tilde{V}(\tilde{\omega})$ of $\tilde{\omega}$ as

$$\tilde{V}(\tilde{w}) = \{\omega \in \mathbb{R}^U \ : \ |P_{u_i}(\omega) - P_{u_i}(\tilde{\omega})| < \varepsilon_i, \ \forall \in \{1, \cdots, m\}\},$$

where $m \in \mathbb{N}$, $\varepsilon_i > 0$ and $u_i \in U$, $\forall i \in \{1, \cdots, m\}$.

Let $A \subset \mathbb{R}^U$. We say that $\tilde{\omega} \in A$ is interior to A, as there exists a neighborhood $\tilde{V}_w(\tilde{\omega})$ contained in A.

If all points of A are interior, we say that A is weakly open.

Finally, we define the weak topology σ para $\mathbb{R}^U$, as the set of all subset weakly open of $\mathbb{R}^U$.

Now consider U^* with the topology $\sigma(U^*, U)$ and let $\phi : U^* \to \mathbb{R}^U$ where

$$\phi(u^*) = \prod_{u \in U} \langle u, u^* \rangle_U.$$

We shall show that ϕ is continuous. Suppose, to obtain contradiction, that ϕ is not continuous. Thus, there exists $u^* \in U^*$ such that ϕ is not continuous at u^*.

Hence there exist a net $\{u_\alpha^*\}_{\alpha \in I}$ such that

$$u_\alpha^* \to u^* \text{ in } \sigma(U^*, U),$$

but we do not have

$$\phi(u_\alpha^*) \to \phi(u^*) \text{ in } \sigma.$$

Therefore, there exists a weak neighborhood $\tilde{V}(\phi(u^*))$ such that for each $\beta \in I$ there exists $\alpha_\beta \in I$ such that $\alpha_\beta \succ \beta$ and

$$\phi(u_{\alpha_\beta}^*) \not\in \tilde{V}(\phi(u^*)),$$

with with no loss of generality, we may assume

$$\tilde{V}(\phi(u^*)) = \{\omega \in \mathbb{R}^U \ : \ |P_{u_i}(w) - P_{u_i}(\phi(u^*))| < \varepsilon_i, \ \forall i \in \{1, \cdots, m\}\},$$

where $m \in \mathbb{N}$, $\varepsilon_i > 0$ and $u_i \in U$, $\forall i \in \{1, \cdots, m\}$.

From this, we get $j \in \{1, \cdots, m\}$ and a sub-net $\{u_{\alpha_\beta}^*\}$ also denoted by $\{u_{\alpha_\beta}^*\}$ such that

$$|P_{u_j}(\phi(u_{\alpha_\beta}^*)) - P_{u_j}(\phi(u^*))| \geq \varepsilon_j, \forall \alpha_\beta \in I.$$

Thus,

$$\begin{aligned} |P_{u_j}(\phi(u^*_{\alpha_\beta})) - P_{u_j}(\phi(u^*))| &= |\langle u_j, u^*_{\alpha_\beta} - u^*\rangle_U| \\ &\geq \varepsilon_j, \ \forall \alpha_\beta \in I. \end{aligned} \tag{133}$$

Therefore, we do not have,

$$\langle u_j, u^*_{\alpha_\beta}\rangle_U \to \langle u_j, u^*\rangle_U,$$

that is, we do not have,

$$u^*_\alpha \to u^*, \text{ em } \sigma(U^*, U),$$

a contradiction.

Hence, ϕ is continuous with $\mathbb{R}^U$ with the topology σ above specified.

We shall prove now that

$$\phi^{-1} : \phi(U^*) \to U^*$$

is also continuous.

This follows from a little adaptation with of the last proposition, considering that

$$f_u(\omega) = \langle u, \phi^{-1}(w)\rangle_U = \omega_u = P_u(\omega),$$

on $\phi(U^*)$ so that f_u, is continuous on $\phi(U^*)$, for all $u \in U$.

Thus, from the last proposition, ϕ^{-1} is continuous.

On the other hand, observe that

$$\phi(B_{U^*}) = K$$

where

$$\begin{aligned} K &= \{\omega \in \mathbb{R}^U \ : \ |\omega_u| \leq \|u\|_U, \omega_{u+v} = \omega_u + \omega_v, \\ & \omega_{\lambda u} = \lambda\omega_u, \forall u, v \in U, \ \lambda \in \mathbb{R}\}. \end{aligned} \tag{134}$$

To finish this proof, it suffices, from the continuity of ϕ^{-1}, to show that $K \subset \mathbb{R}^U$ is compact with $\mathbb{R}^U$ with the topology σ.

Observe that $K = K_1 \cap K_2$ where

$$K_1 = \{\omega \in \mathbb{R}^U \ : \ |\omega_u| \leq \|u\|_U, \forall u \in U\}, \tag{135}$$

and

$$K_2 = \{\omega \in \mathbb{R}^U \ : \ \omega_{u+v} = \omega_u + \omega_v, \ \omega_{\lambda u} = \lambda\omega_u, \forall u, v \in U, \ \lambda \in \mathbb{R}\}. \tag{136}$$

The set $K_3 = \prod_{u \in U}[-\|u\|_U, \|u\|_U]$ is compact as a Cartesian product of compact real intervals.

Since $K_1 \subset K_3$ and K_1 is closed, we have that K_1 is compact concerning the topology in question.

On the other hand, K_2 is closed, since defining the closed sets $A_{u,v}$ e $B_{\lambda,u}$ (these seta are closed from the continuity of projections P_u com $\mathbb{R}^U$ for the topology σ, as inverse images of closed sets in $\mathbb{R}$) by

$$A_{u,v} = \{\omega \in \mathbb{R}^U \ : \ \omega_{u+v} - \omega_u - \omega_v = 0\}, \tag{137}$$

and

$$B_{\lambda,u} = \{\omega \in \mathbb{R}^U \ : \ \omega_{\lambda u} - \lambda\omega_u = 0\} \tag{138}$$

we have

$$K_2 = (\cap_{u,v \in U} A_{u,v}) \cap (\cap_{(\lambda,u) \in \mathbb{R}\times U} B_{\lambda,u}). \tag{139}$$

Recall that K_2 is closed as an intersection of closed sets.

Finally, we have that $K_1 \cap K_2 \subset K_1$ is compact.

This completes the proof.

Teorema 23.2 (Kakutani)**.** *Let U be a Banach space. Then U is reflexive if and only if*

$$B_U = \{u \in U \mid \|u\|_U \leq 1\} \tag{140}$$

is compact for the weak topology $\sigma(U, U^*)$.

Proof. Suppose U is reflexive, then $J(B_U) = B_{U^{**}}$. From the last theorem $B_{U^{**}}$ is compact for the topology $\sigma(U^{**}, U^*)$. Therefore it suffices to verify that $J^{-1} : U^{**} \to U$ is continuous from U^{**} with the topology $\sigma(U^{**}, U^*)$ to U, with the topology $\sigma(U, U^*)$.

From Proposition **??** it is sufficient to show that the function $u \mapsto \langle f, J^{-1}u\rangle_U$ is continuous for the topology $\sigma(U^{**}, U^*)$, for each $f \in U^*$. Since $\langle f, J^{-1}u\rangle_U = \langle u, f\rangle_{U^*}$ we have completed the first part of the proof. For the second we need two lemmas.

Lema 23.3 (Helly)**.** *Let U be a Banach space, $f_1, ..., f_n \in U^*$ and $\alpha_1, ..., \alpha_n \in \mathbb{R}$, then* 1 *and* 2 *are equivalent, where:*

1.

$$\text{Given } \varepsilon > 0, \text{ there exists } u_\varepsilon \in U \text{ such that } \|u_\varepsilon\|_U \leq 1 \text{ and}$$
$$|\langle u_\varepsilon, f_i\rangle_U - \alpha_i| < \varepsilon, \forall i \in \{1, ..., n\}.$$

2.

$$\left|\sum_{i=1}^n \beta_i\alpha_i\right| \leq \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*}, \forall \beta_1, ..., \beta_n \in \mathbb{R}. \tag{141}$$

Proof. 1 $\Rightarrow$ 2*: Fix $\beta_1, ..., \beta_n \in \mathbb{R}$, $\varepsilon > 0$ and define $S = \sum_{i=1}^n |\beta_i|$. From* 1*, we have*

$$\left|\sum_{i=1}^n \beta_i\langle u_\varepsilon, f_i\rangle_U - \sum_{i=1}^n \beta_i\alpha_i\right| < \varepsilon S \tag{142}$$

and therefore

$$\left|\sum_{i=1}^n \beta_i\alpha_i\right| - \left|\sum_{i=1}^n \beta_i\langle u_\varepsilon, f_i\rangle_U\right| < \varepsilon S \tag{143}$$

or

$$\left|\sum_{i=1}^n \beta_i \alpha_i\right| < \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*} \|u_\varepsilon\|_U + \varepsilon S \leq \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*} + \varepsilon S \tag{144}$$

so that

$$\left|\sum_{i=1}^n \beta_i \alpha_i\right| \leq \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*} \tag{145}$$

since ε is arbitrary.

Now let us show that $2 \Rightarrow 1$. Define $\vec{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ and consider the function $\varphi(u) = (\langle f_1, u\rangle_U, ..., \langle f_n, u\rangle_U)$. Item 1 is equivalent to $\vec{\alpha}$ belongs to the closure of $\varphi(B_U)$. Let us suppose that $\vec{\alpha}$ does not belong to the closure of $\varphi(B_U)$ and obtain a contradiction. Thus we can separate $\vec{\alpha}$ and the closure of $\varphi(B_U)$ strictly, that is there exists $\vec{\beta} = (\beta_1, ..., \beta_n) \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that

$$\varphi(u) \cdot \vec{\beta} < \gamma < \vec{\alpha} \cdot \vec{\beta}, \forall u \in B_U \tag{146}$$

Taking the supremum in u we contradict 2. □

Also we need the lemma.

Lema 23.4. *Let U be a Banach space. Then $J(B_U)$ is dense in $B_{U^{**}}$ for the topology $\sigma(U^{**}, U^*)$.*

Proof. Let $u^{**} \in B_{U^{**}}$ and consider $\mathcal{V}_{u^{**}}$ a neighborhood of u^{**} for the topology $\sigma(U^{**}, U^*)$. It suffices to show that $J(B_U) \cap \mathcal{V}_{u^{**}} \neq \emptyset$. As $\mathcal{V}_{u^{**}}$ is a weak neighborhood, there exists $f_1, ..., f_n \in U^*$ and $\varepsilon > 0$ such that

$$\mathcal{V}_{u^{**}} = \{\eta \in U^{**} \mid \langle f_i, \eta - u^{**}\rangle_{U^*}| < \varepsilon, \forall i \in \{1, ..., n\}\}. \tag{147}$$

Define $\alpha_i = \langle f_i, u^{**}\rangle_{U^*}$ and thus for any given $\beta_1, ..., \beta_n \in \mathbb{R}$ we have

$$\left|\sum_{i=1}^n \beta_i \alpha_i\right| = \left|\langle\sum_{i=1}^n \beta_i f_i, u^{**}\rangle_{U^*}\right| \leq \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*}, \tag{148}$$

so that from Helly lemma, there exists $u_\varepsilon \in U$ such that $\|u_\varepsilon\|_U \leq 1$ and

$$|\langle u_\varepsilon, f_i\rangle_U - \alpha_i| < \varepsilon, \forall i \in \{1, ..., n\} \tag{149}$$

or,

$$|\langle f_i, J(u_\varepsilon) - u^{**}\rangle_{U^*}| < \varepsilon, \forall i \in \{1, ..., n\} \tag{150}$$

and hence

$$J(u_\varepsilon) \in \mathcal{V}_{u^{**}}. \tag{151}$$

□

Now we will complete the proof of Kakutani Theorem. Suppose B_U is weakly compact (that is, compact for the topology $\sigma(U, U^*)$). Observe that $J : U \to U^{**}$ is weakly continuous, that is, it is continuous with U endowed with the topology $\sigma(U, U^*)$ and U^{**} endowed with the topology $\sigma(U^{**}, U^*)$. Thus as B_U is weakly compact, we have that $J(B_U)$ is compact for the topology $\sigma(U^{**}, U^*)$. From the last lemma, $J(B_U)$ is dense $B_{U^{**}}$ for the topology $\sigma(U^{**}, U^*)$. Hence $J(B_U) = B_{U^{**}}$, or $J(U) = U^{**}$, which completes the proof. □

Proposição 23.5. *Let U be a reflexive Banach space. Let $K \subset U$ be a convex closed bounded set. Then K is weakly compact.*

Proof. From Theorem 21.5, K is weakly closed (closed for the topology $\sigma(U, U^*)$). Since K is bounded, there exists $\alpha \in \mathbb{R}^+$ such that $K \subset \alpha B_U$. Since K is weakly closed and $K = K \cap \alpha B_U$, we have that it is weakly compact. □

Proposição 23.6. *Let U be a reflexive Banach space and $M \subset U$ a closed subspace. Then M with the norm induced by U is reflexive.*

Proof. We can identify two weak topologies in M, namely:

$$\sigma(M, M^*) \text{ and the trace of } \sigma(U, U^*). \tag{152}$$

It can be easily verified that these two topologies coincide (through restrictions and extensions of linear forms). From theorem 2.4.2, it suffices to show that B_M is compact for the topology $\sigma(M, M^*)$. But B_U is compact for $\sigma(U, U^*)$ and $M \subset U$ is closed (strongly) and convex so that it is weakly closed, thus from last proposition, B_M is compact for the topology $\sigma(U, U^*)$, and therefore it is compact for $\sigma(M, M^*)$. □

24 Separable sets

Definição 24.1 (Separable spaces)**.** *A metric space U is said to be separable if there exist a set $K \subset U$ such that K is countable and dense in U.*

The next Proposition is proved in [3].

Proposição 24.2. *Let U be a separable metric space. If $V \subset U$ then V is separable.*

Teorema 24.3. *Let U be a Banach space such that U^* is separable. Then U is separable.*

Proof. Consider $\{u_n^*\}$ a countable dense set in U^*. Observe that

$$\|u_n^*\|_{U^*} \quad = \quad \sup\{|\langle u_n^*, u\rangle_U| \mid u \in U \text{ and } \|u\|_U = 1\} \tag{153}$$

so that for each $n \in \mathbb{N}$, there exists $u_n \in U$ such that $\|u_n\|_U = 1$ and $\langle u_n^*, u_n\rangle_U \geq \frac{1}{2}\|u_n^*\|_{U^*}$.

Define U_0 as the vector space on $\mathbb{Q}$ spanned by $\{u_n\}$, and U_1 as the vector space on $\mathbb{R}$ spanned by $\{u_n\}$. It is clear that U_0 is dense in U_1 and we will show that U_1 is dense in U, so that U_0 is a dense set in U. For, suppose u^* is such that $\langle u, u^* \rangle_U = 0, \forall u \in U_1$. Since $\{u_n^*\}$ is dense in U^*, given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\|u_n^* - u^*\|_{U^*} < \varepsilon$, so that

$$\begin{aligned} \frac{1}{2}\|u_n^*\|_{U^*} \leq \langle u_n, u_n^* \rangle_U &= \langle u_n, u_n^* - u^* \rangle_U + \langle u_n, u^* \rangle_U \\ &\leq \|u_n^* - u^*\|_{U^*}\|u_n\|_U + 0 \\ &< \varepsilon \end{aligned} \tag{154}$$

or

$$\|u^*\|_{U^*} \leq \|u_n^* - u^*\|_{U^*} + \|u_n^*\|_{U^*} < \varepsilon + 2\varepsilon = 3\varepsilon. \tag{155}$$

Therefore, since ε is arbitrary, $\|u^*\|_{U^*} = 0$, that is $u^* = \theta$. By Corollary 20.14 this completes the proof. □

Proposição 24.4. *U is reflexive if and only if U^* is reflexive.*

Proof. Suppose U is reflexive, as B_{U^*} is compact for $\sigma(U^*, U)$ and $\sigma(U^*, U) = \sigma(U^*, U^{**})$ we have that B_{U^*} is compact for $\sigma(U^*, U^{**})$, which means that U^* is reflexive.

Suppose U^* is reflexive, from above U^{**} is reflexive. Since $J(U)$ is a closed subspace of U^{**}, from Proposition 23.6, J(U) is reflexive. From the Kakutani Theorem $J(B_U)$ is weakly compact. At this point we shall prove that $J^{-1} : J(U) \to V$ is continuous from $J(U)$ with the topology $\sigma(U^{**}, U^*)$ to U with the topology $\sigma(U, U^*)$.

Let $\{u_\alpha^{**}\}_{\alpha \in I} \subset J(B_U)$ be a net such that

$$u_\alpha^{**} \rightharpoonup u_0^{**}$$

weakly in $\sigma(U^{**}, U^*)$.

Let $u^* \in U^*$. Thus,

$$\langle u^*, u_\alpha^{**} \rangle_{U^*} \to \langle u^*, u_0^{**} \rangle_{U^*}.$$

From this

$$\langle u^*, J(J^{-1}(u_\alpha^{**})) \rangle_{U^*} \to \langle u^*, J(J^{-1}(u_0^{**})) \rangle_{U^*},$$

so that

$$\langle (J^{-1}(u_\alpha^{**})), u^* \rangle_U \to \langle (J^{-1}(u_0^{**})), u^* \rangle_U.$$

Since the net is question and $u^* \in U^*$ have been arbitrary, we may infer that J^{-1} is weakly continuous for the concerning topology.

Hence $J^{-1}(J(B_U))$ is also weakly compact so that from this, from the fact that B_U is weakly closed and

$$B_U \subset J^{-1}J(B_U),$$

it follows that B_U is compact for the topology $\sigma(U, U^*)$.

From such a result and from the Kakutani Theorem we may infer that U is reflexive.

The proof is complete. □

Proposição 24.5. *Let U be a Banach space. Then U is reflexive and separable if and only if U^* is reflexive and separable.*

Our final result in this section refers to the metrizability of B_{U^*}.

Teorema 24.6. *Let U be separable Banach space. Under such hypotheses B_{U^*} is metrizable with respect to the weak-star topology $\sigma(U^*, U)$. Conversely, if B_{U^*} is mertizable in $\sigma(U^*, U)$ then U is separable.*

Proof. Let $\{u_n\}$ be a dense countable set in B_U. For each $u^* \in U^*$ define

$$\|u^*\|_w = \sum_{n=1}^{\infty} \frac{1}{2^n} | \langle u_n, u^* \rangle_U |.$$

It may be easily verified that $\| \cdot \|_w$ is a norm in U^* and

$$\|u^*\|_w \leq \|u^*\|_U.$$

So, we may define a metric in U^* by

$$d(u^*, v^*) = \|u^* - v^*\|_w.$$

Now we shall prove that the topology induced by d coincides with $\sigma(U^*, U)$ in U^*.
For, let $u_0^* \in B_{U^*}$ and let V be neighborhood of u_0^* in $\sigma(U^*, U)$.
We need to prove that there exists $r > 0$ such that

$$V_w = \{u^* \in B_{U^*} \mid d(u_0^*, u^*) < r\} \subset V.$$

Observe that for V we may assume the general format

$$V = \{u^* \in U^* \mid |\langle v_i, u^* - u_0^* \rangle_U| < \varepsilon,$$

for some $\varepsilon > 0$ and $v_1, ..., v_k \in U$.
There is no loss in generality in assuming

$$\|v_i\|_U \leq 1, \forall i \in \{1, ..., k\}.$$

Since $\{u_n\}$ is dense in U, for each $i \in \{1, ..., k\}$ there exists $n_i \in \mathbb{N}$ such that

$$\|u_{n_i} - v_i\|_U < \frac{\varepsilon}{4}.$$

Choose $r > 0$ small enough such that

$$2^{n_i} r < \frac{\varepsilon}{2}, \forall i \in \{1, ..., k\}.$$

We are going to show that $V_w \subset V$, where

$$V_w = \{u^* \in B_{U^*} \mid d(u_0^*, u^*) < r\} \subset V.$$

Observe that, if $u^* \in V_w$ then

$$d(u_0^*, u^*) < r,$$

so that

$$\frac{1}{2^{n_i}} |\langle u_{n_i}, u^* - u_0^* \rangle_U| < r, \forall i \in \{1, ..., k\},$$

so that

$$\begin{aligned} |\langle v_i, u^* - u_0^* \rangle_U| &\leq |\langle v_i - u_{n_i}, u^* - u_0^* \rangle_U| + |\langle u_{n_i}, u^* - u_0^* \rangle_U| \\ &\leq (\|u^*\|_{U^*} + \|u_0^*\|_{U^*})\|v_i - u_{n_i}\|_U + |\langle u_{n_i}, u^* - u_0^* \rangle_U| \\ &< 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{156}$$

Therefore $u^* \in V$, so that $V_w \subset V$.

Now let $u_0 \in B_{U^*}$ and fix $r > 0$. We have to obtain a neighborhood $V \in \sigma(U^*U)$ such that

$$V \subset V_w = \{u^* \in B_{U^*} \mid d(u_0^*, u^*) < r\}.$$

We shall define $k \in \mathbb{N}$ and $\varepsilon > 0$ in the next lines so that $V \subset V_w$, where

$$V = \{u^* \in B_{U^*} \mid |\langle u_i, u^* - u_0^* \rangle_U| < \varepsilon, \forall i \in \{1, ..., k\}\}.$$

For $u^* \in V_w$ we have

$$\begin{aligned} d(u^*, u_0^*) &= \sum_{n=1}^{k} \frac{1}{2^n} |\langle u_n, u^* - u_0^* \rangle_U| \\ &\quad + \sum_{n=k+1}^{\infty} \frac{1}{2^n} |\langle u_n, u^* - u_0^* \rangle_U| \\ &< \varepsilon + 2 \sum_{n=k+1}^{\infty} \frac{1}{2^n} \\ &= \varepsilon + \frac{1}{2^{k-1}}. \end{aligned} \tag{157}$$

Hence, it suffices to take $\varepsilon = r/2$, and k sufficiently big such that

$$\frac{1}{2^{k-1}} < r/2.$$

The first part of the proof is finished.

Conversely, assume B_U is metrizable in $\sigma(U^*, U)$. We are going to show that U is separable.

Define,

$$\tilde{V}_n = \left\{ u^* \in B_{U^*} \mid d(u^*, \theta) < \frac{1}{n} \right\}.$$

From the first part, we may find V_n a neighborhood of zero in $\sigma(U^*, U)$ such that

$$V_n \subset \tilde{V}_n.$$

Moreover, we may assume that V_n has the form

$$V_n = \{u^* \in B_{U^*} \mid |\langle u, u^* - \theta\rangle_U| < \varepsilon_n, \forall u \in C_n\},$$

where C_n is a finite set.

Define

$$D = \cup_{i=1}^{\infty} C_n.$$

Thus D is countable and we are going to prove that such a set is dense in U.

For, suppose $u^* \in U^*$ is such that

$$\langle u, u^*\rangle_U = 0, \forall u \in D.$$

Hence,

$$u^* \in V_n \subset \tilde{V}_n, \forall n \in \mathbb{N},$$

so that $u^* = \theta$.

The proof is complete. □

25 Uniformly convex spaces

Definição 25.1 (Uniformly convex spaces). *A Banach space U is said to be uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that:*

If $u, v \in U$, $\|u\|_U \leq 1$, $\|v\|_U \leq 1$, and $\|u - v\|_U > \varepsilon$ then $\frac{\|u+v\|_U}{2} < 1 - \delta$.

Teorema 25.2 (Milman Pettis). *Every uniformly convex Banach space is reflexive.*

Proof. Let $\eta \in U^{**}$ be such that $\|\eta\|_{U^{**}} = 1$. It suffices to show that $\eta \in J(B_U)$. Since $J(B_U)$ is closed in U^{**}, we have only to show that for each $\varepsilon > 0$ there exists $u \in U$ such that $\|\eta - J(u)\|_{U^{**}} < \varepsilon$.

Thus, suppose given $\varepsilon > 0$. Let $\delta > 0$ be the corresponding constant relating the uniformly convex property.

Choose $f \in U^*$ such that $\|f\|_{U^*} = 1$ and

$$\langle f, \eta\rangle_{U^*} > 1 - \frac{\delta}{2}. \tag{158}$$

Define

$$V = \left\{\zeta \in U^{**} \mid |\langle f, \zeta - \eta\rangle_{U^*}| < \frac{\delta}{2}\right\}.$$

Observe that V is neighborhood of η in $\sigma(U^{**}, U^*)$. Since $J(B_U)$ is dense in $B_{U^{**}}$ concerning the topology $\sigma(U^{**}, U^*)$, we have that $V \cap J(B_U) \neq \emptyset$ and thus there exists $u \in B_U$ such that $J(u) \in V$. Suppose, to obtain contradiction, that

$$\|\eta - J(u)\|_{U^{**}} > \varepsilon.$$

Therefore, defining

$$W = (J(u) + \varepsilon B_{U^{**}})^c,$$

we have that $\eta \in W$, where W is also a weak neighborhood of η in $\sigma(U^{**}, U^*)$, since $B_{U^{**}}$ is closed in $\sigma(U^{**}, U^*)$.

Hence $V \cap W \cap J(B_U) \neq \emptyset$, so that there exists some $v \in B_U$ such that $J(v) \in V \cap W$. Thus, $J(u) \in V$ and $J(v) \in V$, so that

$$|\langle u, f\rangle_U - \langle f, \eta\rangle_{U^*}| < \frac{\delta}{2},$$

and

$$|\langle v, f\rangle_U - \langle f, \eta\rangle_{U^*}| < \frac{\delta}{2}.$$

Hence,

$$\begin{aligned} 2\langle f, \eta\rangle_{U^*} &< \langle u+v, f\rangle_U + \delta \\ &\leq \|u+v\|_U + \delta. \end{aligned} \tag{159}$$

From this and (158) we obtain

$$\frac{\|u+v\|_U}{2} > 1-\delta,$$

and thus from the definition of uniform convexity, we obtain

$$\|u-v\|_U \leq \varepsilon. \tag{160}$$

On the other hand, since $J(v) \in W$, we have

$$\|J(u) - J(v)\|_{U^{**}} = \|u-v\|_U > \varepsilon,$$

which contradicts (160). The proof is complete. □

26 Topics on Linear Operators

The main references for this chapter are Reed and Simon [5] and Bachman and Narici [1].

26.1 Topologies for bounded operators

Let U, Y be Banach spaces. First we recall that the set of all bounded linear operators $A : U \to Y$, denoted by $\mathcal{L}(U, Y)$, is a Banach space with the norm

$$\|A\| = \sup\{\|Au\|_Y \mid \|u\|_U \leq 1\}.$$

The topology related to the metric induced by this norm is called the uniform operator topology.

Let us introduce now the strong operator topology, which is defined as the weakest topology for which the functions

$$E_u : \mathcal{L}(U, Y) \to Y$$

are continuous where

$$E_u(A) = Au, \forall A \in \mathcal{L}(U, Y).$$

For such a topology a base at origin is given by sets of the form

$$\{A \mid A \in \mathcal{L}(U,Y),\ \|Au_i\|_Y < \varepsilon, \forall i \in \{1,,...,n\}\},$$

where $u_1,...,u_n \in U$ and $\varepsilon > 0$.

Observe that a sequence $\{A_n\} \subset \mathcal{L}(U,Y)$ converges to A concerning this last topology if

$$\|A_n u - Au\|_Y \to 0, \text{ as } n \to \infty, \forall u \in U.$$

In the next lines we describe the weak operator topology in $\mathcal{L}(U,Y)$. Such a topology is weakest one such that the functions

$$E_{u,v} : \mathcal{L}(U,Y) \to \mathbb{C}$$

are continuous, where

$$E_{u,v}(A) = \langle Au, v\rangle_Y, \forall A \in \mathcal{L}(U,Y), u \in U,\ v \in Y^*.$$

For such a topology, a base at origin is given by sets of the form

$$\{A \in \mathcal{L}(U,Y) \mid |\langle Au_i, v_j\rangle_Y| < \varepsilon, \forall i \in \{1,...,n\},\ j \in \{1,...,m\}\}.$$

where $\varepsilon > 0$, $u_1,...,u_n \in U$, $v_1,...,v_m \in Y^*$.

A sequence $\{A_n\} \subset \mathcal{L}(U,Y)$ converges to $A \in \mathcal{L}(U,Y)$ if

$$|\langle A_n u, v\rangle_Y - \langle Au, v\rangle_Y| \to 0,$$

as $n \to \infty$, $\forall u \in U,\ v \in Y^*$.

27 Adjoint operators

We start this section recalling the definition of adjoint operator.

Definição 27.1. *Let U, Y be Banach spaces. Given a bounded linear operator $A : U \to Y$ and $v^* \in Y^*$, we have that $T(u) = \langle Au, v^*\rangle_Y$ is such that*

$$|T(u)| \leq \|Au\|_Y \cdot \|v^*\| \leq \|A\| \|v^*\|_{Y^*} \|u\|_U.$$

Hence $T(u)$ is a continuous linear functional on U and considering our fundamental representation hypothesis, there exists $u^* \in U^*$ such that

$$T(u) = \langle u, u^*\rangle_U, \forall u \in U.$$

We define A^* by setting $u^* = A^* v^*$, so that

$$T(u) = \langle u, u^*\rangle_U = \langle u, A^* v^*\rangle_U$$

that is,

$$\langle u, A^* v^*\rangle_U = \langle Au, v^*\rangle_Y, \forall u \in U,\ v^* \in Y^*.$$

We call $A^* : Y^* \to U^*$ the adjoint operator relating $A : U \to Y$.

Teorema 27.2. *Let U, Y be Banach spaces and let $A : U \to Y$ be a bounded linear operator. Then*

$$\|A\| = \|A^*\|.$$

Proof. Observe that

$$\begin{aligned}
\|A\| &= \sup_{u \in U}\{\|Au\| \mid \|u\|_U = 1\} \\
&= \sup_{u \in U}\{\sup_{v^* \in Y^*}\{\langle Au, v^*\rangle_Y \mid \|v^*\|_{Y^*} = 1\}, \|u\|_U = 1\} \\
&= \sup_{(u,v^*) \in U \times Y^*}\{\langle Au, v^*\rangle_Y \mid \|v^*\|_{Y^*} = 1, \|u\|_U = 1\} \\
&= \sup_{(u,v^*) \in U \times Y^*}\{\langle u, A^*v^*\rangle_U \mid \|v^*\|_{Y^*} = 1, \|u\|_U = 1\} \\
&= \sup_{v^* \in Y^*}\{\sup_{u \in U}\{\langle u, A^*v^*\rangle_U \mid \|u\|_U = 1\}, \|v^*\|_{Y^*} = 1\} \\
&= \sup_{v^* \in Y^*}\{\|A^*v^*\|, \|v^*\|_{Y^*} = 1\} \\
&= \|A^*\|.
\end{aligned} \tag{161}$$

□

In particular, if $U = Y = H$ where H is Hilbert space, we have

Teorema 27.3. *Given the bounded linear operators $A, B : H \to H$ we have*

1. $(AB)^* = B^*A^*$,
2. $(A^*)^* = A$,
3. *If A has a bounded inverse A^{-1} then A^* has a bounded inverse and*

$$(A^*)^{-1} = (A^{-1})^*.$$

4. $\|AA^*\| = \|A\|^2$.

Proof. 1. Observe that

$$(ABu, v)_H = (Bu, A^*v)_H = (u, B^*A^*v)_H, \forall u, v \in H.$$

2. Observe that

$$(u, Av)_H = (A^*u, v)_H = (u, A^{**}v)_H, \forall u, v \in H.$$

3. We have that

$$I = AA^{-1} = A^{-1}A,$$

so that

$$I = I^* = (AA^{-1})^* = (A^{-1})^*A^* = (A^{-1}A)^* = A^*(A^{-1})^*.$$

4. Observe that

$$\|A^*A\| \leq \|A\|\|A^*\| = \|A\|^2,$$

and

$$\begin{aligned} \|A^*A\| &\geq \sup_{u\in U}\{(u, A^*Au)_H \mid \|u\|_U = 1\} \\ &= \sup_{u\in U}\{(Au, Au)_H \mid \|u\|_U = 1\} \\ &= \sup_{u\in U}\{\|Au\|_H^2 \mid \|u\|_U = 1\} = \|A\|^2, \end{aligned} \tag{162}$$

and hence

$$\|A^*A\| = \|A\|^2.$$

□

Definição 27.4. *Given $A \in \mathcal{L}(H)$ we say that A is self-adjoint if*

$$A = A^*.$$

Teorema 27.5. *Let U and Y be Banach spaces and let $A : U \to Y$ be a bounded linear operator. Then*

$$[R(A)]^\perp = N(A^*),$$

where

$$[R(A)]^\perp = \{v^* \in Y^* \mid \langle Au, v^*\rangle_Y = 0, \ \forall u \in U\}.$$

Proof. Let $v^* \in N(A^*)$. Choose $v \in R(A)$. Thus there exists u in U such that $Au = v$ so that

$$\langle v, v^*\rangle_Y = \langle Au, v^*\rangle_Y = \langle u, A^*v^*\rangle_U = 0.$$

Since $v \in R(A)$ is arbitrary we have obtained

$$N(A^*) \subset [R(A)]^\perp.$$

Suppose $v^* \in [R(A)]^\perp$. Choose $u \in U$. Thus,

$$\langle Au, v^*\rangle_Y = 0,$$

so that

$$\langle u, A^*v^*\rangle_U, \forall u \in U.$$

Therefore $A^*v^* = \theta$, that is, $v^* \in N(A^*)$. Since $v^* \in [R(A)]^\perp$ is arbitrary, we get

$$[R(A)]^\perp \subset N(A^*).$$

This completes the proof. □

The next result is relevant for subsequent developments.

Lema 27.6. *Let U, Y be Banach spaces and let $A : U \to Y$ be a bounded linear operator. Suppose also that $R(A) = \{A(u) \; : \; u \in U\}$ is closed. Under such hypotheses, there exists $K > 0$ such that for each $v \in R(A)$ there exists $u_0 \in U$ such that*

$$A(u_0) = v$$

and

$$\|u_0\|_U \leq K\|v\|_Y.$$

Proof. Define $L = N(A) = \{u \in U \; : \; A(u) = \theta\}$ (the null space of A). Consider the space U/L, where

$$U/L = \{\overline{u} \; : \; u \in U\},$$

where

$$\overline{u} = \{u + w \; : \; w \in L\}.$$

Define $\overline{A} : U/L \to R(A)$, by

$$\overline{A}(\overline{u}) = A(u).$$

Observe that $\overline{A}$ is one-to-one, linear, onto and bounded. Moreover $R(A)$ is closed so that it is a Banach space. Hence by the inverse mapping theorem we have that $\overline{A}$ has a continuous inverse. Thus, for any $v \in R(A)$ there exists $\overline{u} \in U/L$ such that

$$\overline{A}(\overline{u}) = v$$

so that

$$\overline{u} = \overline{A}^{-1}(v),$$

and therefore

$$\|\overline{u}\| \leq \|\overline{A}^{-1}\|\|v\|_Y.$$

Recalling that

$$\|\overline{u}\| = \inf_{w \in L}\{\|u + w\|_U\},$$

we may find $u_0 \in \overline{u}$ such that

$$\|u_0\|_U \leq 2\|\overline{u}\| \leq 2\|\overline{A}^{-1}\|\|v\|_Y,$$

and so that

$$A(u_0) = \overline{A}(\overline{u_0}) = \overline{A}(\overline{u}) = v.$$

Taking $K = 2\|\overline{A}^{-1}\|$ we have completed the proof. $\square$

Teorema 2. *Let U, Y be Banach spaces and let $A : U \to Y$ be a bound linear operator. Assume $R(A)$ is closed. Under such hypotheses*

$$R(A^*) = [N(A)]^{\perp}.$$

Proof. Let $u^* \in R(A^*)$. Thus there exists $v^* \in Y^*$ such that

$$u^* = A^*(v^*).$$

Let $u \in N(A)$. Hence,

$$\langle u, u^* \rangle_U = \langle u, A^*(v^*) \rangle_U = \langle A(u), v^* \rangle_Y = 0.$$

Since $u \in N(A)$ is arbitrary, we get $u^* \in [N(A)]^\perp$, so that

$$R(A^*) \subset [N(A)]^\perp.$$

Now suppose $u^* \in [N(A)]^\perp$. Thus

$$\langle u, u^* \rangle_U = 0, \ \forall u \in N(A).$$

Fix $v \in R(A)$. From the Lemma 27.6, there exists $K > 0$ (which does not depend on v) and $u_v \in U$ such that

$$A(u_v) = v$$

and

$$\|u_v\|_U \leq K \|v\|_Y.$$

Define $f : R(A) \to \mathbb{R}$ by

$$f(v) = \langle u_v, u^* \rangle_U.$$

Observe that

$$|f(v)| \leq \|u_v\|_U \|u^*\|_{U^*} \leq K \|v\|_Y \|u^*\|_{U^*},$$

so that f is a bounded linear functional. Hence by a Hahn-Banach theorem corollary, there exists $v^* \in Y^*$ such that

$$f(v) = \langle v, v^* \rangle_Y \equiv F(v), \ \forall v \in R(A),$$

that is, F is an extension of f from $R(A)$ to Y.

In particular

$$f(v) = \langle u_v, u^* \rangle_U = \langle v, v^* \rangle_Y = \langle A(u_v), v^* \rangle_Y \ \forall v \in R(A),$$

where $A(u_v) = v$, so that

$$\langle u_v, u^* \rangle_U = \langle A(u_v), v^* \rangle_Y \ \forall v \in R(A).$$

Now let $u \in U$ and define $A(u) = v_0$. Observe that

$$u = (u - u_{v_0}) + u_{v_0},$$

and

$$A(u - u_{v_0}) = A(u) - A(u_{v_0}) = v_0 - v_0 = \theta.$$

Since $u^* \in [N(A)]^\perp$ we get

$$\langle u - u_{v_0}, u^* \rangle_U = 0$$

so that

$$\begin{aligned}\langle u, u^* \rangle_U &= \langle (u - u_{v_0}) + u_{v_0}, u^* \rangle_U \\ &= \langle u_{v_0}, u^* \rangle_U \\ &= \langle A(u_{v_0}), v^* \rangle_Y \\ &= \langle A(u - u_{v_0}) + A(u_{v_0}), v^* \rangle_Y \\ &= \langle A(u), v^* \rangle_Y. \end{aligned} \tag{163}$$

Hence,

$$\langle u, u^* \rangle_U = \langle A(u), v^* \rangle_Y, \ \forall u \in U.$$

We may conclude that $u^* = A^*(v^*) \in R(A^*)$. Since $u^* \in [N(A)]^\perp$ is arbitrary we obtain

$$[N(A)]^\perp \subset R(A^*).$$

The proof is complete. □

We finish this section with the following result.

Definição 27.7. *Let U be a Banach space and $S \subset U$. We define the positive conjugate cone of S, denoted by $S^\oplus$ by*

$$S^\oplus = \{u^* \in U^* \ : \ \langle u, u^* \rangle_U \geq 0, \ \forall u \in S\}.$$

Similarly, we define the the negative cone of S, denoted by denoted by $S^\ominus$ by

$$S^\ominus = \{u^* \in U^* \ : \ \langle u, u^* \rangle_U \leq 0, \ \forall u \in S\}.$$

Teorema 27.8. *Let U, Y be Banach spaces and $A : U \to Y$ be a bounded linear operator. Let $S \subset U$. Then*

$$[A(S)]^\oplus = (A^*)^{-1}(S^\oplus),$$

where

$$(A^*)^{-1} = \{v^* \in Y^* \ : \ A^* v^* \in S^\oplus\}.$$

Proof. Let $v^* \in [A(S)]^\oplus$ and $u \in S$. Thus,

$$\langle A(u), v^* \rangle_Y \geq 0,$$

so that

$$\langle u, A^*(v^*) \rangle_U \geq 0.$$

Since $u \in S$ is arbitrary, we get

$$v^* \in (A^*)^{-1}(S^\oplus).$$

From this

$$[A(S)]^\oplus = (A^*)^{-1}(S^\oplus).$$

Reciprocally, let $v^* \in (A^*)^{-1}(S^\oplus)$. Hence $A^*(v^*) \in S^\oplus$ so that, for $u \in S$ we obtain

$$\langle u, A^*(v^*) \rangle_U \geq 0,$$

and therefore

$$\langle A(u), v^* \rangle_Y \geq 0.$$

Since $u \in S$ is arbitrary, we get $v^* \in [A(S)]^{\oplus}$, that is,

$$(A^*)^{-1}(S^{\oplus}) \subset [A(S)]^{\oplus}.$$

The proof is complete. □

28 Compact operators

We start this section defining compact operators.

Definição 28.1. *Let U and Y be Banach spaces. An operator $A \in \mathcal{L}(U, Y)$ (linear and bounded) is said to compact if A takes bounded sets into pre-compact sets. Summarizing, A is compact if for each bounded sequence $\{u_n\} \subset U$, $\{Au_n\}$ has a convergent subsequence in Y.*

Teorema 28.2. *A compact operator maps weakly convergent sequences into norm convergent sequences.*

Proof. Let $A : U \to Y$ be a compact operator. Suppose

$$u_n \rightharpoonup u \text{ weakly in U.}$$

By the uniform boundedness theorem, $\{\|u_n\|\}$ is bounded. Thus, given $v^* \in Y^*$ we have

$$\begin{aligned} \langle Au_n, v^* \rangle_Y &= \langle u_n, A^* v^* \rangle_U \\ &\to \langle u, A^* v^* \rangle_U \\ &= \langle Au, v^* \rangle_Y. \end{aligned} \tag{164}$$

Being $v^* \in Y^*$ arbitrary, we get that

$$Au_n \rightharpoonup Au \text{ weakly in } Y. \tag{165}$$

Suppose Au_n does not converge in norm to Au. Thus there exists $\varepsilon > 0$ and a subsequence $\{Au_{n_k}\}$ such that

$$\|Au_{n_k} - Au\|_Y \geq \varepsilon, \forall k \in \mathbb{N}.$$

As $\{u_{n_k}\}$ is bounded and A is compact, $\{Au_{n_k}\}$ has a subsequence converging para $\tilde{v} \neq Au$. But then such a sequence converges weakly to $\tilde{v} \neq Au$, which contradicts (165). The proof is complete. □

Teorema 28.3. *Let H be a separable Hilbert space. Thus each compact operator in $\mathcal{L}(H)$ is the limit in norm of a sequence of finite rank operators.*

Proof. Let A be a compact operator in H. Let $\{\phi_j\}$ an orthonormal basis in H. For each $n \in \mathbb{N}$ define

$$\lambda_n = \sup\{\|A\psi\|_H \mid \psi \in [\phi_1, ..., \phi_n]^\perp \text{ and } \|\psi\|_H = 1\}.$$

It is clear that $\{\lambda_n\}$ is a non-increasing sequence that converges to a limit $\lambda \geq 0$. We will show that $\lambda = 0$. Choose a sequence $\{\psi_n\}$ such that

$$\psi_n \in [\phi_1, ..., \phi_n]^\perp,$$

$\|\psi_n\|_H = 1$ and $\|A\psi_n\|_H \geq \lambda/2$. Now we will show that

$$\psi_n \rightharpoonup \theta, \text{ weakly in } H.$$

Let $\psi^* \in H^* = H$, thus there exists a sequence $\{a_j\} \subset \mathbb{C}$ such that

$$\psi^* = \sum_{j=1}^{\infty} a_j \phi_j.$$

Suppose given $\varepsilon > 0$. We may find $n_0 \in \mathbb{N}$ such that

$$\sum_{j=n_0}^{\infty} |a_j|^2 < \varepsilon.$$

Choose $n > n_0$. Hence there exists $\{b_j\}_{j>n}$ such that

$$\psi_n = \sum_{j=n+1}^{\infty} b_j \phi_j,$$

and

$$\sum_{j=n+1}^{\infty} |b_j|^2 = 1.$$

Therefore

$$\begin{aligned} |(\psi_n, \psi^*)_H| &= \left| \sum_{j=n+1}^{\infty} (\phi_j, \phi_j)_H a_j \cdot b_j \right| \\ &= \left| \sum_{j=n+1}^{\infty} a_j \cdot b_j \right| \\ &\leq \sqrt{\sum_{j=n+1}^{\infty} |a_j|^2} \sqrt{\sum_{j=n+1}^{\infty} |b_j|^2} \\ &\leq \sqrt{\varepsilon}, \end{aligned} \tag{166}$$

if $n > n_0$. Since $\varepsilon > 0$ is arbitrary,

$$(\psi_n, \psi^*)_H \to 0, \text{ as } n \to \infty.$$

Since $\psi^* \in H$ is arbitrary, we get

$$\psi_n \rightharpoonup \theta, \text{ weakly in } H.$$

Hence, as A is compact, we have

$$A\psi_n \to \theta \text{ in norm },$$

so that $\lambda = 0$. Finally, we may define $\{A_n\}$ by

$$A_n(u) = A\left(\sum_{j=1}^{n}(u, \phi_j)_H \phi_j\right) = \sum_{j=1}^{n}(u, \phi_j)_H A\phi_j,$$

for each $u \in H$. Thus

$$\|A - A_n\| = \lambda_n \to 0, \text{ as } n \to \infty.$$

The proof is complete. □

29 The square root of a positive operator

Definição 29.1. *Let H be a Hilbert space. A mapping $E : H \to H$ is said to be a projection on $M \subset H$ if for each $z \in H$ we have*

$$Ez = x$$

where $z = x + y$, $x \in M$ and $y \in M^\perp$.

Observe that

1. E is linear,
2. E is idempotent, that is $E^2 = E$,
3. $R(E) = M$,
4. $N(E) = M^\perp$.

Also observe that from

$$Ez = x$$

we have

$$\|Ez\|_H^2 = \|x\|_H^2 \leq \|x\|_H^2 + \|y\|_H^2 = \|z\|_H^2,$$

so that

$$\|E\| \leq 1.$$

Definição 29.2. *Let $A, B \in \mathcal{L}(H)$. We write*

$$A \geq \theta$$

if

$$(Au, u)_H \geq 0, \forall u \in H,$$

and in this case we say that A is positive. Finally, we denote

$$A \geq B$$

if

$$A - B \geq \theta.$$

Teorema 29.3. *Let A and B be bounded self-adjoint operators such that $A \geq \theta$ and $B \geq \theta$. If $AB = BA$ then*

$$AB \geq \theta.$$

Proof. If $A = \theta$ the result is obvious. Assume $A \neq \theta$ and define the sequence

$$A_1 = \frac{A}{\|A\|}, \quad A_{n+1} = A_n - A_n^2, \forall n \in \mathbb{N}.$$

We claim that

$$\theta \leq A_n \leq I, \forall n \in \mathbb{N}.$$

We prove the claim by induction.

For $n = 1$, it is clear that $A_1 \geq \theta$. And since $\|A_1\| = 1$, we get

$$(A_1 u, u)_H \leq \|A_1\| \|u\|_H \|u\|_H = (Iu, u)_H, \forall u \in H,$$

so that

$$A_1 \leq I.$$

Thus

$$\theta \leq A_1 \leq I.$$

Now suppose $\theta \leq A_n \leq I$. Since A_n is self adjoint we have,

$$\begin{aligned} (A_n^2(I - A_n)u, u)_H &= ((I - A_n)A_n u, A_n u)_H \\ &= ((I - A_n)v, v)_H \geq 0, \forall u \in H \end{aligned} \quad (167)$$

where $v = A_n u$. Therefore

$$A_n^2(I - A_n) \geq \theta.$$

Similarly, we may obtain

$$A_n(I - A_n)^2 \geq \theta,$$

so that

$$\theta \leq A_n^2(I - A_n) + A_n(I - A_n)^2 = A_n - A_n^2 = A_{n+1}.$$

So, also we have,

$$\theta \leq I - A_n + A_n^2 = I - A_{n+1},$$

that is,

$$\theta \leq A_{n+1} \leq I,$$

so that

$$\theta \leq A_n \leq I, \ \forall n \in \mathbb{N}.$$

Observe that,

$$\begin{aligned} A_1 &= A_1^2 + A_2 \\ &= A_1^2 + A_2^2 + A_3 \\ ... &\ .... \\ &= A_1^2 + ... + A_n^2 + A_{n+1}. \end{aligned} \tag{168}$$

Since $A_{n+1} \geq \theta$, we obtain

$$A_1^2 + A_2^2 + ... + A_n^2 = A_1 - A_{n+1} \leq A_1. \tag{169}$$

From this, for a fixed $u \in H$, we have

$$\begin{aligned} \sum_{j=1}^{n} \|A_j u\|^2 &= \sum_{j=1}^{n} (A_j u, A_j u)_H \\ &= \sum_{j=1}^{n} (A_j^2 u, u)_H \\ &\leq (A_1 u, u)_H. \end{aligned} \tag{170}$$

Since $n \in \mathbb{N}$ is arbitrary, we get,

$$\sum_{j=1}^{\infty} \|A_j u\|^2$$

is a converging series, so that

$$\|A_n u\| \to 0,$$

that is,

$$A_n u \to \theta, \text{ as } n \to \infty.$$

From this and (169), we get

$$\sum_{j=1}^{n} A_j^2 u = (A_1 - A_{n+1})u \to A_1 u, \text{ as } n \to \infty.$$

Finally, we may write,

$$\begin{aligned}(ABu,u)_H &= \|A\|(A_1Bu,u)_H \\ &= \|A\|(BA_1u,u)_H \\ &= \|A\|(B\lim_{n\to\infty}\sum_{j=1}^n A_j^2u,u)_H \\ &= \|A\|\lim_{n\to\infty}\sum_{j=1}^n(BA_j^2u,u)_H \\ &= \|A\|\lim_{n\to\infty}\sum_{j=1}^n(BA_ju,A_ju)_H \\ &\geq 0. \end{aligned} \tag{171}$$

Hence

$$(ABu,u)_H \geq 0, \forall u \in H.$$

The proof is complete. □

Teorema 29.4. *Let $\{A_n\}$ be a sequence of self-adjoint commuting operators in $\mathcal{L}(H)$. Let $B \in \mathcal{L}(H)$ be a self adjoint operator such that*

$$A_iB = BA_i, \forall i \in \mathbb{N}.$$

Suppose also that

$$A_1 \leq A_2 \leq A_3 \leq ... \leq A_n \leq ... \leq B.$$

Under such hypotheses there exists a self adjoint, bounded, linear operator A such that

$$A_n \to A \text{ in norm },$$

and

$$A \leq B.$$

Proof. Consider the sequence $\{C_n\}$ where

$$C_n = B - A_n \geq 0, \forall n \in \mathbb{N}.$$

Fix $u \in H$. First, we show that $\{C_nu\}$ converges. Observe that

$$C_iC_j = C_jC_i, \forall i,j \in \mathbb{N}.$$

Also, if $n > m$ then

$$A_n - A_m \geq \theta$$

so that

$$C_m = B - A_m \geq B - A_n = C_n.$$

Therefore from $C_m \geq \theta$ and $C_m - C_n \geq \theta$ we obtain

$$(C_m - C_n)C_m \geq \theta, \text{ if } n > m$$

and also

$$C_n(C_m - C_n) \geq \theta.$$

Thus,

$$(C_m^2 u, u)_H \geq (C_n C_m u, u)_H \geq (C_n^2 u, u)_H,$$

and we may conclude that

$$(C_n^2 u, u)_H$$

is a monotone non-increasing sequence of real numbers, bounded below by 0, so that there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{n \to \infty} (C_n^2 u, u)_H = \alpha.$$

Since each C_n is self adjoint we obtain

$$\begin{aligned} \|(C_n - C_m)u\|_H^2 &= ((C_n - C_m)u, (C_n - C_m)u)_H \\ &= ((C_n - C_m)(C_n - C_m)u, u)_H \\ &= (C_n^2 u, u)_H - 2(C_n C_m u, u) + (C_m^2 u, u)_H \\ &\to \alpha - 2\alpha + \alpha = 0, \end{aligned} \tag{172}$$

as

$$m, n \to \infty.$$

Therefore $\{C_n u\}$ is a Cauchy sequence in norm, so that there exists the limit

$$\lim_{n \to \infty} C_n u = \lim_{n \to \infty} (B - A_n)u,$$

and hence there exists

$$\lim_{n \to \infty} A_n u, \forall u \in H.$$

Now define A by

$$Au = \lim_{n \to \infty} A_n u.$$

Since the limit

$$\lim_{n \to \infty} A_n u, \forall u \in H$$

exists we have that

$$\sup_{n \in \mathbb{N}} \{\|A_n u\|_H\}$$

is finite for all $u \in H$. By the principle of uniform boundedness

$$\sup_{n \in \mathbb{N}} \{\|A_n\|\} < \infty$$

so that there exists $K > 0$ such that

$$\|A_n\| \leq K, \forall n \in \mathbb{N}.$$

Therefore

$$\|A_n u\|_H \leq K\|u\|_H,$$

so that

$$\|Au\| = \lim_{n\to\infty} \{\|A_n u\|_H\} \leq K\|u\|_H, \forall u \in H$$

which means that A is bounded. Fixing $u, v \in H$, we have

$$(Au, v)_H = \lim_{n\to\infty} (A_n u, v)_H = \lim_{n\to\infty} (u, A_n v)_H = (u, Av)_H,$$

and thus A is self adjoint. Finally

$$(A_n u, u)_H \leq (Bu, u)_H, \forall n \in \mathbb{N},$$

so that

$$(Au, u) = \lim_{n\to\infty} (A_n u, u)_H \leq (Bu, u)_H, \forall u \in H.$$

Hence $A \leq B$.

The proof is complete. □

Definição 29.5. *Let $A \in \mathcal{L}(A)$ be a positive operator. The self adjoint operator $B \in \mathcal{L}(H)$ such that*

$$B^2 = A$$

is called the square root of A. If $B \geq \theta$ we denote

$$B = \sqrt{A}.$$

Teorema 29.6. *Suppose $A \in \mathcal{L}(H)$ is positive. Then there exists $B \geq \theta$ such that*

$$B^2 = A.$$

Furthermore B commutes with any $C \in \mathcal{L}(H)$ such that commutes with A.

Proof. There is no loss of generality in considering

$$\|A\| \leq 1,$$

which means $\theta \leq A \leq I$, because we may replace A by

$$\frac{A}{\|A\|}$$

so that if

$$C^2 = \frac{A}{\|A\|}$$

then

$$B = \|A\|^{1/2} C.$$

Let

$$B_0 = \theta,$$

and consider the sequence of operators given by

$$B_{n+1} = B_n + \frac{1}{2}(A - B_n^2), \forall n \in \mathbb{N} \cup \{0\}.$$

Since each B_n is polynomial in A, we have that B_n is self adjoint and commute with any operator with commutes with A. In particular

$$B_i B_j = B_j B_i, \forall i, j \in \mathbb{N}.$$

First we show that

$$B_n \leq I, \forall n \in \mathbb{N} \cup \{0\}.$$

Since $B_0 = \theta$, and $B_1 = \frac{1}{2}A$, the statement holds for $n = 1$. Suppose $B_n \leq I$. Thus

$$\begin{aligned} I - B_{n+1} &= I - B_n - \frac{1}{2}A + \frac{1}{2}B_n^2 \\ &= \frac{1}{2}(I - B_n)^2 + \frac{1}{2}(I - A) \geq \theta \end{aligned} \tag{173}$$

so that

$$B_{n+1} \leq I.$$

The induction is complete, that is,

$$B_n \leq I, \forall n \in \mathbb{N}.$$

Now we prove the monotonicity also by induction. Observe that

$$B_0 \leq B_1,$$

and supposing

$$B_{n-1} \leq B_n,$$

we have

$$\begin{aligned} B_{n+1} - B_n &= B_n + \frac{1}{2}(A - B_n^2) - B_{n-1} - \frac{1}{2}(A - B_{n-1}^2) \\ &= B_n - B_{n-1} - \frac{1}{2}(B_n^2 - B_{n-1}^2) \\ &= B_n - B_{n-1} - \frac{1}{2}(B_n + B_{n-1})(B_n - B_{n-1}) \\ &= (I - \frac{1}{2}(B_n + B_{n-1}))(B_n - B_{n-1}) \\ &= \frac{1}{2}((I - B_{n-1}) + (I - B_n))(B_n - B_{n-1}) \geq \theta. \end{aligned}$$

The induction is complete, that is

$$\theta = B_0 \leq B_1 \leq B_2 \leq ... \leq B_n \leq ... \leq I.$$

By the last theorem there exists a self adjoint operator B such that

$$B_n \to B \text{ in norm.}$$

Fixing $u \in H$ we have

$$B_{n+1}u = B_n u + \frac{1}{2}(A - B_n^2)u,$$

so that taking the limit in norm as $n \to \infty$, we get

$$\theta = (A - B^2)u.$$

Being $u \in H$ arbitrary we obtain

$$A = B^2.$$

It is also clear that

$$B \geq \theta$$

The proof is complete. $\square$

30 Spectral Analysis, a General Approach in Normed spaces

31 Introduction

In this section we present some results about the spectrum and resolvent sets for a bounded operator defined on a normed space.

We start with the following definition.

Definição 31.1. *Let V be a complex normed vector space and let $A : D \subset V \to V$ be a linear operator, where D is dense on V. We say that $A^{-1} : R(A) \to D$ is the inverse operator related to A, as A is a bijection from D to $R(A)$ and*

$$A^{-1}y = u \text{ if, and only if, } Au = y, \; \forall u \in D, y \in R(A),$$

where $R(A) = \{Au : \; u \in D\}$, is the range of A.

In such case we have,

$$A^{-1}Au = u, \; \forall u \in D$$

and

$$AA^{-1}y = y, \; \forall y \in R(A).$$

Let $\lambda \in \mathbb{C}$.

1. *If $R(\lambda I - A)$ is dense in V and $\lambda I - A$ has a bounded inverse, we write $\lambda \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A.*

$(\lambda I - A)^{-1}$	Boundedness of $(\lambda I - A)^{-1}$	$R(\lambda I - A)$	Set
exists	bounded	dense in V	$\rho(A)$
exists	unbounded	dense in V	$C\sigma(A)$
exists	bounded or not	not dense in V	$R\sigma(A)$
not exists	—	dense or not in V	$P\sigma(A)$

Table 1: About the spectrum and resolvent sets of A

2. *If $R(\lambda I - A)$ is dense in V and $(\lambda I - A)^{-1}$ exists but it is not bounded, we write $\lambda \in C\sigma(A)$, where $C\sigma(A)$ denotes the continuous spectrum of A.*

3. *If $R(\lambda I - A)$ is not dense in V and $\lambda I - A$ has an inverse either bounded or unbounded, we write $\lambda \in R\sigma(A)$, where $R\sigma(A)$ denotes the residual spectrum of A.*

4. *If $(\lambda I - A)^{-1}$ does not exist, we write $\lambda \in P\sigma(A)$ where $P\sigma(A)$ denotes the point spectrum of A.*

Table 1 summarizes such results.

Observação 31.2. *Observe that*

$$\mathbb{C} = \rho(A) \cup C\sigma(A) \cup R\sigma(A) \cup P\sigma(A)$$

and such union is disjoint.

The spectrum of A, denoted by $\sigma(A)$, is defined by

$$\sigma(A) = C\sigma(A) \cup R\sigma(A) \cup P\sigma(A).$$

Teorema 31.3 (Riesz)**.** *Let V be a normed vector space and let $0 < \alpha < 1$. Let M be a proper closed vector subspace of V.*

Under such hypotheses, there exists $u_\alpha \in V$ such that

$$\|u_\alpha\|_V = 1$$

and

$$\|u - u_\alpha\|_V \geq \alpha, \; \forall u \in M.$$

Proof. Since $M \subset V$ is a proper closed subspace of V, we may select a $v \in V \setminus M$.

Define

$$d = \inf_{u \in M} \|u - v\|_V.$$

Observe that, since M is closed, we have $d > 0$, otherwise if we had $d = 0$ we had $v \in \overline{M} = M$, which contradicts $v \notin M$.

Also,

$$d/\alpha > d.$$

Hence, there exists $u_0 \in M$ such that

$$0 < d \leq \|u_0 - v\|_V < d/\alpha.$$

Define

$$u_\alpha = \frac{v - u_0}{\|v - u_0\|_V}.$$

Thus, $\|u_\alpha\|_V = 1$ and also for $u \in M$ we have

$$\begin{aligned} \|u - u_\alpha\|_V &= \left\| u - \frac{v}{\|v - u_0\|_V} + \frac{u_0}{\|v - u_0\|_V} \right\|_V \\ &= \| u(\|v - u_0\|_V) + u_0 - v\|_V \frac{1}{\|v - u_0\|_V}. \end{aligned} \tag{174}$$

From this, since $u\|v - u_0\|_V + u_0 \in M$, we have

$$\|u - u_\alpha\|_V \geq \frac{d}{\|v - u_0\|_V} > \alpha, \ \forall u \in M.$$

The proof is complete. □

Teorema 31.4. *Let V be a complex normed vector space and let $A : D \subset V \to V$ be a linear compact operator.*

Under such hypotheses, $P\sigma(A)$ is countable and 0 is its unique possible limit point.

Proof. Let $\varepsilon > 0$. We shall prove that there exists at most a finite number of points in P_ε where

$$P_\varepsilon = \{\lambda \in P\sigma(A) \ : \ |\lambda| \geq \varepsilon\}.$$

Observe that in such a case

$$P\sigma(A) \setminus \{0\} = \cup_{n=1}^{\infty} P_{1/n}$$

and such a set is countable and has 0 as the unique possible limit point.

Suppose, to obtain contradiction, there exists $\varepsilon > 0$ such that P_ε has infinite points. Hence, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset P_\varepsilon$ and a sequence of linearly independent eigenvectors $\{u_n\}$ such that

$$Au_n = \lambda_n u_n, \ \forall n \in \mathbb{N}.$$

Define

$$M_n = \text{Span}\{u_1, \ldots, u_n\},$$

so that

$$M_{n-1} \subset M_n$$

and M_n is finite dimensional, $\forall n \in \mathbb{N}$. Observe that $M_{n-1} \subset M_n$ properly.

From the Riesz theorem, there exists $y_n \in M_n$ such that $\|y_n\|_V = 1$ and

$$\|y_n - u\| \geq 1/2, \ \forall u \in M_{n-1}, \ \forall n > 1.$$

Let

$$u = \sum_{i=1}^{n} \alpha_i u_i \in M_n.$$

Thus,

$$Au = \sum_{i=1}^{n} \alpha_i A u_i = \sum_{i=1}^{n} \alpha_i \lambda_i u_i.$$

Therefore,

$$\begin{aligned} (\lambda_n - A)u &= \lambda_n u - Au \\ &= \sum_{i=1}^{n-1} \alpha_i(\lambda_n - \lambda_i)u_i \in M_{n-1}. \end{aligned} \tag{175}$$

Therefore

$$(\lambda_n - A)(M_n) \subset M_{n-1}$$

and from this

$$A(M_n) \subset M_n, \forall n \in \mathbb{N}.$$

Let $1 < m < n$. Thus,

$$w = (\lambda_n - A)y_n + Ay_m \in M_{n-1},$$

so that

$$Ay_n - Ay_m = \lambda_n y_n - w = \lambda_n(y_n - \lambda_n^{-1} w).$$

Since, $\lambda_n^{-1} w \in M_{n-1}$ we get

$$||Ay_n - Ay_m\| = |\lambda_n|\|y_n - \lambda_n^{-1} w\| \geq \frac{|\lambda_n|}{2} \geq \frac{\varepsilon}{2},$$

$\forall 1 \leq m < n \in \mathbb{N}$.

Therefore, $\{y_n\}$ is a bounded sequence and such that $\{Ay_n\}$ has no Cauchy subsequence, that is, $\{Ay_n\}$ has no convergent subsequence, which contradicts A be compact.

The proof is complete. □

Definição 31.5. *Let V be a normed vector space and let $A : D \subset V \to V$ be a linear operator, where D is dense in V.*

We say that $\lambda \in \mathbb{C}$ is an proper approximate value of A if for each $\varepsilon > 0$ there exists $u \in D$ such that

$$\|u\|_V = 1 \text{ and } \|(\lambda I - A)u\|_V < \varepsilon.$$

In such a case we denote $\lambda \in \pi(A)$, where $\pi(A)$ is the approximate spectrum of A.

Teorema 31.6. *Considering the statements of the last definition, we have that $\lambda \in \pi(A)$ if, and only if, $\lambda I - A$ has no a bounded inverse.*

Proof. Suppose $\lambda \in \pi(A)$. Thus, for each $n \in \mathbb{N}$, there exists $u_n \in D$ such that $\|u_n\|_V = 1$ and

$$\|(\lambda I - A)u_n\|_V < 1/n. \tag{176}$$

Suppose, to obtain contradiction, there exists $K > 0$ such that

$$\|(\lambda I - A)u\|_V \geq K\|u\|_V, \ \forall u \in D.$$

In particular we have

$$\|(\lambda I - A)u_n\|_V \geq 1, \ \forall n \in \mathbb{N},$$

which contradicts (176).

Reciprocally, suppose $\lambda I - A$ has no bounded inverse.

Thus, there is no $K > 0$ such that

$$\|(\lambda I - A)u\|_V \geq K\|u\|_V, \ \forall u \in D.$$

Hence, for each $\varepsilon > 0$ we may find $u \in D$ such that

$$\|(\lambda I - A)u\|_V < \varepsilon\|u\|_V.$$

From this, for each $\varepsilon > 0$ we may find $u \in D$ such that

$$\|u\|_V = 1 \text{ and } \|(\lambda I - A)u\|_V < \varepsilon.$$

Therefore $\lambda \in \pi(A)$. ☐

32 Sesquilinear functionals

Definição 32.1. *Let V be a complex vector space. A functional $f : V \times V \to \mathbb{C}$ is said to be a sesquilinear functional, as*

1. $f(u_1 + u_2, v) = f(u_1, v) + f(u_2, v), \ \forall u_1, u_2, v \in V.$
2. $f(\alpha u, v) = \alpha f(u, v), \ \forall u, v \in V, \ \forall \alpha \in \mathbf{C}.$
3. $f(u, v_1 + v_2) = f(u, v_1) + f(u, v_2), \ \forall u, v_1, v_2 \in V.$
4. $f(u, \alpha v) = \overline{\alpha} f(u, v), \ \forall u, v \in V, \ \alpha \in \mathbb{C}.$

Observação 32.2. *Let H be a complex Hilbert space and let $A : H \to H$ be a linear operator. Hence $f : H \times H \to \mathbb{C}$ defined by*

$$f(u, v) = (Au, v)_H, \forall u, v \in H$$

is a sequilinear functional.

Observação 32.3. *Given a sesquilinear functional*

$$f : H \times H \to \mathbb{C}$$

we shall define $\hat{f} : H \to \mathbb{C}$ *by*

$$\hat{f}(u) = f(u,u), \; \forall u \in H.$$

Exercício 32.4. *In the context of the last definitions, prove that*

$$\begin{aligned} f(u,v) &= \hat{f}\left(\frac{1}{2}(u+v)\right) - \hat{f}\left(\frac{1}{2}(u-v)\right) \\ &+ i\hat{f}\left(\frac{1}{2}(u+iv)\right) - i\hat{f}\left(\frac{1}{2}(u-iv)\right), \; \forall u,v \in H. \end{aligned} \tag{177}$$

Conclude that if $\hat{f}_1 = \hat{f}_2$*, then* $f_1 = f_2$*.*

Teorema 32.5. *Let* V *be a complex vector space and let* $f : V \times V \to \mathbb{C}$ *be a symmetric sesquilinear functional, that is, assume* f *is such that* $f(u,v) = \overline{f(v,u)}, \; \forall u,v \in V$*, where* $\overline{f(v,u)}$ *denotes the complex conjugate of* $f(v,u)$*.*

Under such hypotheses,

$$\hat{f}(u) \in \mathbb{R}, \; \forall u \in V.$$

Proof. Suppose

$$g(u,v) = \overline{f(v,u)}, \; \forall u,v \in V.$$

Thus,

$$\hat{f}(u) = f(u,u) = \overline{f(u,u)} = \overline{\hat{f}(u)},$$

so that

$$\hat{f}(u) \in \mathbb{R}, \forall u \in V.$$

Reciprocally, suppose $\hat{f}$ is real.
Define

$$g(u,v) = \overline{f(v,u)}, \; \forall u,v \in V.$$

Hence,

$$\hat{g}(u) = g(u,u) = \overline{f(u,u)} = f(u,u) = \hat{f}(u), \; \forall u \in V.$$

From this and the last exercise, we obtain $g = f$, so that

$$f(u,v) = \overline{f(v,u)}, \; \forall u,v \in V.$$

□

Observação 32.6. *Let* $A : D \subset V \to V$ *be a symmetric operator, that is, such that*

$$(Au,v)_V = (u,Av)_V, \; \forall u,v \in V,$$

where V *is a space with inner product.*

Thus,

$$\begin{aligned} f(u,v) &= (Au,v)_V \\ &= (u,Av)_V \\ &= \overline{(Av,u)_V} \\ &= \overline{f(v,u)},\ \forall u,v \in V, \end{aligned} \tag{178}$$

so that f is symmetric.

Definição 32.7. *Let V be a normed vector space. A sesquilinear functional $f : V \times V \to \mathbb{C}$ is said to be bounded if there exists $K > 0$ such that*

$$|f(u,v)| \leq K\|u\|_V\|v\|_V,\ \forall u,v \in V. \tag{179}$$

Defining

$$B = \{K > 0 \text{ such that } (179) \text{ is satisfied } \}$$

we also define the norm of f, denoted by $\|f\|$ as

$$\|f\| = \inf\{K \ :\ K \in B\}.$$

Moreover, defining

$$C = \{K > 0 \ :\ \text{ such that } |\hat{f}(u)| \leq K\|u||_V,\ \forall u \in V\},$$

we define also the norm of $\hat{f}$, denoted by $\|\hat{f}\|$, as

$$\|\hat{f}\| = \inf\{K > 0 \ :\ K \in C\}.$$

Proposição 32.8. *Considering the context of the last definition,*

$$\|f\| = \sup_{(u,v)\in V\times V}\{|f(u,v)| \ :\ \|u\|_V = \|v\|_V = 1\}$$

and

$$\|\hat{f}\| = \sup_{u\in V}\{|\hat{f}(u)| \ :\ \|u\|_V = 1\}.$$

Proof. We firstly denote

$$\alpha = \sup_{(u,v)\in V\times V}\{|f(u,v)| \ :\ \|u\|_V = \|v\|_V = 1\}.$$

Observe that

$$|f(u,v)| \leq \|f\|\|u\|_V\|v\|_V,\ \forall u,v \in V,$$

so that

$$\alpha \leq \|f\|. \tag{180}$$

On the other hand,

$$\begin{aligned}
|f(u,v)| &= \left| f\left(u\frac{\|u\|_V}{\|u\|_V}, v\frac{\|v\|_V}{\|v\|_V}\right)\right| \\
&= \left| f\left(\frac{u}{\|u\|_V}, \frac{v}{\|v\|_V}\right)\right| \|u\|_V\|v\|_V \\
&\leq \alpha\|u\|_V\|v\|_V, \; \forall u \neq \mathbf{0}, v \neq \mathbf{0}. \qquad (181)
\end{aligned}$$

Hence $\alpha \in B$ so that

$$\alpha \geq \inf B = \|f\|. \qquad (182)$$

From (180) and (182) we may infer that

$$\alpha = \|f\|.$$

Similarly, the second result may be proven.
The proof is complete. □

Teorema 32.9. *Let V be complex normed vector space e let $F : V \times V \to \mathbb{C}$ be a sesquilinear, bounded and symmetric functional. Under such hypotheses,*

$$\|f\| = \|\hat{f}\|.$$

Proof. Observe that

$$\begin{aligned}
f(u,v) &= \hat{f}\left(\frac{1}{2}(u+v)\right) - \hat{f}\left(\frac{1}{2}(u-v)\right) \\
&\quad + i\hat{f}\left(\frac{1}{2}(u+iv)\right) - i\hat{f}\left(\frac{1}{2}(u-iv)\right), \; \forall u,v \in H. \qquad (183)
\end{aligned}$$

Since $\hat{f}(u) \in \mathbb{R}, \; \forall u \in V$, we have that

$$\begin{aligned}
|Re[f(u,v)]| &= \leq \left|\hat{f}\left(\frac{1}{2}(u+v)\right)\right| + \left|\hat{f}\left(\frac{1}{2}(u-v)\right)\right| \\
&\leq \frac{1}{4}\|\hat{f}\|\|u+v\|_V^2 + \frac{1}{4}\|\hat{f}\|\|u-v\|_V^2 \\
&= \frac{1}{4}\|\hat{f}\|\left(2\|u\|_V^2 + 2\|v\|_V^2\right), \; \forall u,v \in V. \qquad (184)
\end{aligned}$$

Thus, if $\|u\|_V = \|v\|_V = 1$, we get

$$|Re[f(u,v)]| \leq \|\hat{f}\|.$$

Observe that in its polar form, we have

$$f(u,v) = re^{i\theta}.$$

Denoting $\alpha = e^{-i\theta}$, we obtain

$$f(\alpha u, v) = \alpha f(u, v) = r = |f(u, v)| = |Re[f(\alpha u, v)]| \leq \|\hat{f}\|.$$

Thus,

$$\|f\| = \sup\{|f(u, v)| \ : \ \|u\|_V = \|v\|_V = 1\} \leq \|\hat{f}\|.$$

However, from the definitions, $\|f\| \geq \|\hat{f}\|$.
From these last two lines, we may infer that

$$\|f\| = \|\hat{f}\|.$$

The proof is complete. □

Definição 32.10 (Normal operator)**.** *Let H be a complex Hilbert space. We say that a bounded linear operator $A : H \to H$ is normal as*

$$A^*A = AA^*.$$

Teorema 32.11. *Let H be a complex Hilbert space. and let $A : H \to H$ be a bounded linear operator. Under such hypotheses, A is normal if, and only if,*

$$\|A^*u\|_H = \|Au\|_H, \ \forall u \in H.$$

Proof. Suppose A is normal. Thus,

$$(A^*Au, u)_H = (AA^*u, u)_H$$

so that

$$\|Au\|_H^2 = (Au, Au)_H = (A^*, A^*u)_H = \|A^*u\|_H^2,$$

that is,

$$\|Au\|_H = \|A^*u\|_H, \ \forall u \in H.$$

Reciprocally, suppose that

$$\|Au\|_H = \|A^*u\|_H, \ \forall u \in H.$$

Hence

$$(Au, Au)_H = (A^*u, A^*u)_H$$

so that

$$(A^*Au, u)_H = (A^{**}A^*u, u)_H = (AA^*u, u)_H, \ \forall u \in H.$$

From this, denoting

$$f_1(u, v) = (A^*Au, v)_H$$

and

$$f_2(u, v) = (AA^*u, v)_H$$

we obtain

$$\hat{f}_1(u) = \hat{f}_2(u), \ \forall u \in H.$$

Thus, $f_1 = f_2$, so that

$$(A^*Au, v)_H = (AA^*u, v)_H, \ \forall u, v \in H.$$

From this, we may infer that

$$AA^* = AA^*.$$

□

Teorema 32.12. *Let H be a complex Hilbert space and let $A \in L(H)$. Under such hypotheses the following proprieties are equivalent.*

1. *There exists $\lambda \in \pi(A)$ such that $|\lambda| = \|A\|$.*
2. $$\|A\| = \sup_{u \in H}\{|(Au, u)_H| \ : \ \|u\|_H = 1\}.$$

Proof.

- 1 implies 2: Suppose $\lambda \in \pi(A)$ is such that

$$|\lambda| = \|A\|.$$

We shall prove that

$$\lambda \in \overline{\{(Au, u)_H \ : \ u \in H, \ \|u\|_H = 1\}}.$$

From this we may obtain

$$\begin{aligned} \|A\| &= |\lambda| \\ &\leq \sup_{u \in H}\{|(Au, u)_H| \ : \ \|u\|_H = 1\} \\ &\leq \sup_{(u,v) \in H \times H}\{|(Au, v)_H| \ : \ \|u\|_H = 1, \ \|v\|_H = 1\} \\ &= \|A\|. \end{aligned} \tag{185}$$

which would complete the first part of the proof.

From $\lambda \in \pi(A)$, there exists $\{u_n\}_{n \in \mathbb{N}} \subset H$ such that

$$\|u_n\|_H = 1$$

and

$$\|Au_n - \lambda u_n\|_H \to 0.$$

Thus,

$$\begin{aligned} |(Au_n, u_n)_H - \lambda| &= |(Au_n, u_n)_H - \lambda(u_n, u_n)_H| \\ &= |(Au_n - \lambda u_n, u_n)_H| \\ &\leq \|Au_n - \lambda u_n\|_H \|u_n\|_H \\ &\to 0, \ \text{as } n \to \infty. \end{aligned} \tag{186}$$

Thus,

$$\lambda \in \overline{\{(Au, u)_H \ : \ u \in H, \ \|u\|_H = 1\}}.$$

The first part of the proof is complete.

- 2 implies 1:

  Reciprocally, suppose
$$\|A\| = \sup_{u \in H}\{|(Au,u)_H| \;:\; \|u\|_H = 1\}.$$
  Hence, there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\|_H = 1$, $\forall n \in \mathbb{N}$, and
$$|(Au_n, u_n)_H| \to \|A\|_H.$$
  Thus $\{(Au_n,u_n)_H\} \subset \mathbb{C}$ is a bounded sequence. From this there exists a subsequence $\{(Au_{n_k}, u_{n_k})_H\}$ of $\{(Au_n,u_n)_H\}$ and $\lambda \in \mathbb{C}$ such that
$$(Au_{n_k}, u_{n_k})_H \to \lambda, \text{ as } k \to \infty.$$
  Therefore,
$$\begin{aligned} \|Au_{n_k} - \lambda u_{n_k}\|_H^2 &= \|Au_{n_k}\|_H^2 - \overline{\lambda}(Au_{n_k}, u_{n_k})_H - \lambda(u_{n_k}, Au_{n_k})_H + |\lambda|^2 \\ &\leq \|A\|^2\|u_{n_k}\|_H^2 - \overline{\lambda}(Au_{n_k}, u_{n_k})_H - \lambda\overline{(Au_{n_k}, u_{n_k})_H} + |\lambda|^2 \\ &\to |\lambda|^2 - \overline{\lambda}\lambda - \lambda\overline{\lambda} + |\lambda|^2 = 0. \end{aligned} \tag{187}$$
  Summarizing,
$$\|Au_{n_k} - \lambda u_{n_k}\|_H \to 0, \text{ as } k \to \infty,$$
  so that $\lambda \in \pi(A)$.

  The proof is complete.

□

Teorema 32.13. *Let H be a Hilbert space and let $A \in L(H)$ be an self-adjoint operator. Define*
$$M = \sup_{u \in H}\{(Au,u)_H \;:\; \|u\|_H = 1\}$$
and
$$m = \inf_{u \in H}\{(Au,u)_H \;:\; \|u\|_H = 1\}.$$
Under such hypotheses, $m \in \sigma(A)$ and $M \in \sigma(A)$.

Proof. Choose $\alpha \in \mathbb{R}$ such that
$$M - \alpha \geq m - \alpha > 0.$$
Define$\hat{M} = M - \alpha$ and $\hat{A} = A - \alpha I$. Since $\hat{A}$ is self-adjoint, we have that
$$\|\hat{A}\| = \hat{M}.$$
Thus there exists a subsequence $\{u_n\} \subset H$ such that $\|u_n\|_H = 1$ and
$$(\hat{A}u_n, u_n)_H \to \hat{M}, \text{ as } n \to \infty.$$

Thus,

$$(\hat{A}u_n - \hat{M}u_n, u_n)_H \to 0, \text{ as } n \to \infty.$$

Hence,

$$\begin{aligned}
\|\hat{A}u_n - \hat{M}u_n\|_H^2 &= (\hat{A}u_n - \hat{M}u_n, \hat{A}u_n - \hat{M}u_n)_H \\
&= \|\hat{A}u_n\|_H^2 - 2\hat{M}(\hat{A}u_n, u_n)_H + \hat{M}^2 \\
&\leq \|\hat{A}\|_H^2\|u_n\|_H^2 - 2\hat{M}(\hat{A}u_n, u_n)_H + \hat{M}^2 \\
&\to \hat{M}^2 - 2\hat{M}^2 + \hat{M}^2 \\
&= 0. \qquad (188)
\end{aligned}$$

Summarizing,

$$\|\hat{A}u_n - \hat{M}u_n\|_H \to 0, \text{ as } n \to \infty,$$

so that

$$\|Au_n - Mu_n\|_H \to 0, \text{ as } n \to \infty.$$

From this we may infer that

$$M \in \pi(A) \subset \sigma(A).$$

Similarly, select $\beta \in \mathbb{R}$ such that

$$-m + \beta \geq -M + \beta > 0.$$

Define $\hat{A} = -A + \beta I$. The remaining parts of the proof are similar to those of the previous case. This completes the proof. □

Teorema 32.14. *Let H be a complex Hilbert space. Let $U : H \to H$ be a linear bounded operator. Under such hypotheses, U is a isometry if, and only if,*

$$U^*U = I,$$

where I denotes the identity operator.

Proof. Observe that

$$(Uu, Uv)_H = (u, v)_H, \ \forall u, v \in H,$$

if, and only if,

$$(U^*Uu, v)_H = (u, v), \ \forall u, v \in H,$$

if, and only if,

$$U^*U = I.$$

The proof is complete. □

Teorema 32.15. *Let H be a complex Hilbert space. Under such hypothesis, $U : H \to H$ is a bijective isometry if, and only if,*

$$U^*U = UU^* = I, \text{ in } H.$$

Proof. Assume $U : H \to H$ is a bijective isometry.

From the last theorem $U^*U = I$, and U is a bijection, we obtain $U^{-1} = U^*$ so that $UU^* = I$ in H.

On the other hand, if $U^*U = UU^* = I$ in H, we have that $U^* = U^{-1}$ and the domain of $U^* = U^{-1}$ is H, so that the range of U is also H.

From this U is a bijection.

Moreover, $(u, v)_H = (U^*Uu, v)_H = (Uu, Uv)_H, \forall u, v \in H$, and thus U is a bijective isometry. □

Teorema 32.16. *Let H be a complex Hilbert space. Suppose $U : H \to H$ is a linear operator such that*

$$\|Uu\|_H = \|u\|_H, \ \forall u \in H$$

(in such a case we say that U is unitary).

Under such hypotheses, U is a isometry.

Proof. From the hypotheses,

$$(Uu, Uu)_H = (u, u)_H, \ \forall u \in H.$$

Thus,

$$(U^*Uu, u)_H = (u, u)_H, \ \forall u \in H,$$

so that

$$((U^*U - I)u, u)_H = 0, \ \forall u \in H.$$

Since $U^*U - I$ is self adjoint, it follows that

$$\|U^*U - I\| = \sup_{u \in H}\{|((U^*U - I)u, u)_H| \ : \ \|u\|_H = 1\} = 0,$$

so that $U^*U = I$.

From this and Theorem 32.14, we have that U is a isometry.

□

Corolário 32.17. *Let $U : H \to H$ be a unitary operator.*

Under such hypotheses, if $\lambda \in \mathbb{C}$ is an eigenvalue of U then

$$|\lambda| = 1.$$

Proof. Suppose $Uu = \lambda u$ and $\|u\|_H = 1$. Thus,

$$1 = (u, u)_H = (Uu, Uu)_H = (\lambda u, \lambda u)_H = \lambda\overline{\lambda}(u, u)_H = |\lambda|^2.$$

The proof is complete. □

Teorema 32.18. *Let H be a complex Hilbert space. Let $A : D_A \subset H \to H$ be a linear self-adjoint operator but not necessarily bounded, where D_A is dense in H, that is, $\overline{D_A} = H$. Let U be the Cayley transform of A, that is $U = (A - i)(A + i)^{-1}$.*

Under such hypotheses, U is unitary.

Proof. Firstly, we shall prove that $A \pm i$ is injective, so that its inverse is well defined on $R(A \pm i)$.

Since A is self-adjoint, we have that

$$\begin{aligned}
\|(A \pm i)u\|_H^2 &= ((A \pm i)u, (A \pm i))_H \\
&= (Au, Au)_H \pm (Au, iu)_H \pm (iu, Au)_H + (iu, iu)_H \\
&= \|Au\|_H^2 + (\pm i \mp i)(u, Au)_H + \|u\|_H^2 \\
&= \|Au\|_H^2 + \|u\|_H^2 \geq \|u\|_H^2. \qquad (189)
\end{aligned}$$

Thus, if $(A \pm i)u = 0$ then $u = 0$, so that $(A \pm i)$ is injective.

Now, we are going to show that

$$R(A \pm i) = H.$$

Let $z \perp R(A + i)$. Hence,

$$((A + i)u, z)_H = (Au, z)_H + i(u, z)_H = 0, \forall u \in D_A.$$

From this

$$(Au, z)_H = ((u, iz), \ \forall u \in H,$$

so that

$$A^* z = Az = iz,$$

that is,

$$(A - i)z = 0.$$

Thus, $z = 0$.

Summarizing these last results, if $z \perp R(A + i)$ then $z = 0$, so that

$$\overline{R(A + i)} = H.$$

Now we are going to show that $\overline{R(A + i)} = R(A + i) = H$. Let $v \in H$. Thus there exists a sequence $\{v_n\} \subset R(A + i)$ such that

$$v_n \to v, \text{ in norm, as } n \to \infty.$$

Therefore there exists a sequence $\{u_n\} \subset D_A$ such that

$$Au_n + iu_n = v_n \to v, \text{ as } n \to \infty.$$

Similarly as above, we may obtain,

$$\|v_n - v_m\|_H^2 = \|Au_n + iu_n - Au_m - iu_m\|_H^2 = \|A(u_n - u_m)\|_H^2 + \|u_n - u_m\|_H^2, \ \forall m, n \in \mathbb{N}.$$

From this, since $\{v_n\}$ is a Cauchy sequence, we may infer that $\{u_n\}$ and $\{Au_n\}$ are Cauchy sequences, so that there exists $u \in H$, and $w \in H$ such that

$$Au_n \to w$$

and

$$u_n \to u, \text{ as } n \to \infty.$$

Since $A = A^*$ is closed, we may infer that $w = Au$ and

$$Au_n + iu_n \to Au + iu.$$

Again, since A is closed we get $(u, Au) \in Gr(A)$ where $Gr(A)$ denotes the graph of A, so that $Au + iu = v \in R(A + i)$.

Summarizing, if $v \in H$ then $v \in R(A + i)$, so that $R(A + i) = H = \overline{R(A + i)}$.

A similar result we may obtain for $A - i$, that is

$$R(A - i) = H.$$

Observe that

$$U = (A - i)(A + i)^{-1}$$

and

$$R((A + i)^{-1}) = D_A = D_{(A+i)},$$

and $R(A - i) = H$.

Thus $R(U) = H$, that is $U : H \to H$ is linear and onto (recalling that $D_{(A+i)^{-1}} = H$).

At this point, we shall prove that U is unitary.

Let $v \in H$. Since $R(A + i) = H$, there exists $u \in D_A$ such that

$$(A + i)u = v.$$

Hence

$$Uv = (A - i)(A + i)^{-1}v = (A - i)u,$$

so that

$$\begin{aligned} \|Uv\|_H^2 &= \|(A - i)u\|_H^2 \\ &= \|Au\|_H^2 + \|u\|_H^2 \\ &= \|(A + i)u\|_H^2 \\ &= \|v\|_H^2, \ \forall v \in H. \end{aligned} \tag{190}$$

Summarizing,

$$\|Uv\|_H = \|v\|_H, \ \forall v \in H,$$

so that U is unitary.

The proof is complete. □

Observação 32.19. *Let $v \in H$. Since $R(A + i) = H$, we may obtain $u \in H$ such that $v = (A + i)u$.*

From this we have

$$Uv = (A - i)(A + i)^{-1}v = (A - i)u,$$

so that

$$(I + U)v = 2Au$$

and

$$(I-U)v = 2iu.$$

Thus, if $(I-U)v = 0$, *then* $u = 0$ *so that* $v = (A+i)u = 0$.

Therefore, $I - U$ *is injective and its inverse exists on* $R(I-U)$.

Moreover, for $u \in R(I-U)$ *as above, we have,*

$$(I+U)[(I-U)^{-1}](2iu) = (I+U)[(I-U)^{-1}](I-U)v = (I+U)v = 2Au,$$

so that

$$Au = i(I+U)(I-U)^{-1}u, \; \forall u \in R(I-U).$$

In the next lines we shall show that $R(I-U) = D_A$.

Indeed, let $v \in H$. *Thus, from the last lines above, there exists* $u \in D_A$ *such that*

$$(I-U)v = 2iu$$

so that $R(I-U) \subset D_A$.

Reciprocally, let $u \in D_A$ *and define* $v = (A+i)u$.

Therefore,

$$2iu = (I-U)v$$

so that $u \in R(I-U), \; \forall u \in D_A$. *Thus,*

$$D_A \subset R(I-U)$$

so that

$$R(I-U) = D_A.$$

From such last results, we may infer that

$$A = i(I+U)(I-U)^{-1},$$

in D_A.

33 Alguns resultados sobre operadores compactos e normais

Teorema 33.1. *Seja* H *um espaço de Hilbert complexo e seja* $A : H \to H$ *um operador linear, limitado e normal.*

Sejam $\lambda_1 \in \mathbb{C}$ *e* $\lambda \in \mathbb{C}$ *tais que* $\lambda_1 \neq \lambda_2$.

Sejam $u_1, u_2 \in H$ *tais que*

$$Au_1 = \lambda_1 u_1,$$

$$Au_2 = \lambda_2 u_2.$$

Sob tais hipóteses,

$$(u_1, u_2)_H = 0.$$

Proof. Sem perda de generalidade assuma $\lambda_1 \neq 0$.

Logo,

$$\begin{aligned}(\lambda_1 u_1, \lambda_2 u_2) &= (Au_1, Au_2)_H \\ &= (A^* A u_1, u_2)_H \\ &= (A^*(\lambda_1 u_1), u_2)_H \\ &= \lambda_1 (A^* u_1, u_2)_H \\ &= \lambda_1 (\overline{\lambda}_1 u_1, u_2)_H \\ &= \lambda_1 \overline{\lambda}_1 (u_1, u_2)_H. \end{aligned} \tag{191}$$

Resumindo, obtivemos

$$\lambda_1 \overline{\lambda}_2 (u_1, u_2)_H = \lambda_1 \overline{\lambda}_1 (u_1, u_2)_H,$$

e como $\lambda_1 \neq 0$, disto obtemos

$$(\overline{\lambda}_2 - \overline{\lambda}_1)(u_1, u_2)_H = 0,$$

Por outro lado, das hipóteses

$$\overline{\lambda}_2 - \overline{\lambda}_1 \neq 0,$$

de modo que

$$(u_1, u_2)_H = 0.$$

A prova está completa. ☐

Teorema 33.2. *Seja H um espaço de Hilbert complexo e seja $A : H \to H$ um operador linear e compacto.*

Sob tais hipóteses,

$$N(A - \lambda)$$

é finito-dimensional, $\forall \lambda \in \mathbb{C}$ tal que $\lambda \neq 0$.

Proof. Suponha, para obter contradição, que exista $\lambda \in \mathbb{C}$, $\lambda \neq 0$, tal que $N(A - \lambda)$ é infinito-dimensional.

Assim existe uma sequência $\{u_n\} \subset N(A - \lambda)$ de vetores linearmente independentes (por Gram-Schimidt podemos considerá-los ortonormais).

Sejam $m, n \in \mathbb{N}$ tais que $m \neq n$.

Logo

$$\begin{aligned}\|Au_n - Au_m\|_H &= |\lambda u_n - \lambda u_m| \\ &= |\lambda| \|u_n - u_m\|_H \\ &= 2|\lambda|. \end{aligned} \tag{192}$$

Assim $\{Au_n\}$ não possui qualquer subsequência de Cauchy, o que contradiz A ser compacto.

A prova está completa.

☐

Teorema 33.3. *Seja H um espaço de Hilbert complexo e seja $A : H \to H$ um operador linear e compacto.*

Sob tais hipóteses, se $\lambda \neq 0$ e $\lambda \in \pi(A)$, então

$$\lambda \in P\sigma(A).$$

Proof. Seja $\lambda \neq 0$ tal que $\lambda \in \pi(A)$.

Seja $\{u_n\} \subset H$ tal que

$$\|u_n\|_H = 1, \ \forall n \in \mathbb{N}$$

e

$$\|Au_n - \lambda u_n\|_H \to 0, \text{ quando } n \to \infty.$$

Sendo A compacto, existem uma subsequência $\{u_{n_k}\}$ de $\{u_n\}$ e $u \in H$ tais que

$$Au_{n_k} \to u, \text{ quando } k \to \infty.$$

Portanto

$$\begin{aligned} \|u - \lambda u_{n_k}\|_H &= \|u - Au_{n_k} + Au_{n_k} - \lambda u_{n_k}\|_H \\ &\leq \|u - Au_{n_k}\|_H + \|Au_{n_k} - \lambda u_{n_k}\|_H \\ &\to 0, \text{ quando } k \to \infty. \end{aligned} \tag{193}$$

Disto obtemos,

$$\begin{aligned} Au &= A\left(\lim_{k\to\infty} \lambda u_{n_k}\right) \\ &= \lambda \lim_{k\to\infty} Au_{n_k} \\ &= \lambda u. \end{aligned} \tag{194}$$

Além disso,

$$\|u\|_H = \lim_{k\to\infty} \|\lambda u_{n_k}\|_H = |\lambda| \neq 0,$$

de modo que

$$u \neq \mathbf{0}.$$

Disto concluímos que

$$\lambda \in P\sigma(A).$$

A prova está completa.

□

Teorema 33.4. *Seja H um espaço de Hilbert complexo e seja $A : H \to H$ um operador normal.*

Mostre que

$$r_\sigma(A) = \|A\|.$$

Mostre também que existe $\lambda \in \sigma(A)$ tal que

$$|\lambda| = \|A\|.$$

Proof. Exercício. ☐

Teorema 33.5. *Seja H um espaço de Hilbert complexo e seja $A : H \to H$ um operador normal.*

Mostre que

$$\pi(A) = \sigma(A).$$

Mostre também que

$$R\sigma(A) = \emptyset.$$

Proof. Exercício. ☐

Teorema 33.6. *Seja H um espaço de Hilbert complexo e seja $A : H \to H$ um operador compacto.*

Seja $\lambda \neq 0$.

Mostre que se $\lambda \in \pi(A)$, então $\lambda \in P\sigma(A)$.

Conclua que nesse caso

$$\pi(A) \setminus \{0\} = P\sigma(A) \setminus \{0\}.$$

Proof. Exercício. ☐

Teorema 33.7. *Seja H um espaço de Hilbert complexo e seja $A : H \to H$ um operador compacto e normal.*

Mostre que $P\sigma(A) \neq \emptyset$ e que existe $\lambda \in P\sigma(A)$ tal que

$$|\lambda| = \|A\|.$$

Proof. Exercício. ☐

Teorema 33.8. *Seja H um espaço de Hilbert complexo e seja $A \in L(H, H)$ um operador compacto e normal. Assuma que*

$$u \perp N(A - \lambda), \ \forall \lambda \in \mathbb{C}.$$

Sob tais hipóteses,

$$u = \mathbf{0}.$$

Proof. Denotemos

$$L = \cup_{\lambda \in \mathbb{C}} N(A - \lambda).$$

Assim

$$L^\perp = [\cup_{\lambda \in \mathbb{C}} N(A - \lambda)]^\perp = \cap_{\lambda \in \mathbb{C}} N(A - \lambda)^\perp.$$

Mostraremos que

$$L^\perp = \{0\}.$$

Seja $\lambda \in \mathbb{C}$. Observe que sendo A normal, A e A^* comutam com

$$A - \lambda.$$

Mostraremos agora que

$$A(N(A - \lambda)) \subset N(A - \lambda),$$

e

$$A^*(N(A-\lambda)) \subset N(A-\lambda).$$

De fato, seja $u \in A(N(A-\lambda))$.

Assim, existe $v \in N(A-\lambda)$ tal que

$$u = Av.$$

Observe que

$$Av = \lambda v$$

de modo que

$$u = \lambda v \in N(A-\lambda), \forall u \in A(N(A-\lambda)).$$

Resumindo,

$$A(N(A-\lambda)) \subset N(A-\lambda).$$

Por outro lado, seja $u \in A^*(N(A-\lambda)$.

Assim, existe $v \in N(A-\lambda)$ tal que

$$u = A^*v.$$

Observe que sendo A normal,

$$A^*v = \overline{\lambda}v$$

de modo que

$$u = \overline{\lambda}v \in N(A-\lambda), \forall u \in A^*(N(A-\lambda)).$$

Resumindo,

$$A^*(N(A-\lambda)) \subset N(A-\lambda).$$

Podemos então concluir que

$$A(L) \subset L$$

e

$$A^*(L) \subset L.$$

Agora, sejam $z \in L$ e $y \in L^\perp$.

Assim

$$A^*z \in L$$

de modo que

$$(z, Ay)_H = (A^*z, y)_H = 0.$$

Como $z \in L$ e $y \in L^\perp$ foram arbitrários, podemos concluir que

$$A(L^\perp) \subset L^\perp.$$

Similarmente, de

$$(z, A^*y)_H = (Az, y)_H = 0, \; \forall z \in L \text{ e } y \in L^\perp,$$

concluímos que

$$A^*(L^\perp) \subset L^\perp.$$

Suponha, para obter contradição que

$$L^\perp \neq \{0\}.$$

Defina $B = A|_{L^\perp}$ (A restrito a $L^\perp$). Assim $B^* = A^*|_{L^\perp}$ de modo que B é normal.

Mostraremos agora que B é compacto.

Seja $\{u_n\} \subset L^\perp$ tal que

$$\|u_n\|_H \leq M, \ \forall n \in \mathbb{N}$$

para algum $M \in \mathbb{R}^+$.

Sendo A compacto, existe uma subsequência $\{u_{n_k}\}$ de $\{u_n\}$ e $v \in L^\perp$ tais que

$$Bu_{n_k} = Au_{n_k} \to v, \text{ as } k \to \infty.$$

Portanto B é compacto.

Assim B é compacto e normal, de modo que B possui um auto-valor $\lambda \neq 0$ e correspondente auto-vetor o qual denotaremos por $u \neq \mathbf{0}$, ($u \in L^\perp$) de modo que

$$Au = Bu = \lambda u.$$

Assim

$$u \in L \cap L^\perp$$

o que contradiz

$$u \neq \mathbf{0}.$$

Resumindo, deve-se ter

$$L^\perp = \{0\}.$$

A prova está completa.

□

Teorema 33.9. *Seja H um espaço de Hilbert complexo. Seja $A \in L(H,H)$ um operador normal. Sob tais hipóteses*

1. *Se $Im(A)$ tem dimensão finita, então $P\sigma(A)$ é finito.*

2. *Se A é compacto e tem espectro pontual finito, então $Im(A)$ tem dimensão finita. Nesse caso, denotando*

$$P\sigma(A) = \{\lambda_1, \cdots, \lambda_k\},$$

temos que

   (a)

$$A = \sum_{j=1}^{k} \lambda_j E_j,$$

onde E_j é a projeção ortogonal sobre $N(A - \lambda_j)$.

(b) $E_j \perp E_l, \forall j \neq l.$

(c)

$$\sum_{j=1}^{k} E_j = I_d.$$

Proof. Para A linear, limitado e normal, assuma que $Im(A)$ tem dimensão finita.

Suponha, para obter contradição, que $\{\lambda_n\}_{n\in\mathbb{N}}$ seja uma sequência de elementos distintos de auto-valores de A com correpondentes auto-vetores $\{u_n\}_{n\in\mathbb{N}}$.

Observe que sendo A normal, os vetores u_n são ortogonais.

Observe também que

$$Au_n = \lambda u_n,$$

de modo que

$$A(\lambda_n^{-1} u_n) = u_n, \ \forall n \in \mathbb{N}.$$

Portanto $\{u_n\} \subset Im(A)$ o que contradiz $Im(A)$ ser de dimensão finita.

A prova deste primeiro item está completa.

Suponha agora que

$$P\sigma(A) = \{\lambda_1, \cdots, \lambda_k\},$$

onde A é normal e compacto.

Como A é normal, autovetores associados a distintos autovalores são ortogonais.

Logo $\{N(A-\lambda_j)\} \ : \ j \in \{1, \cdots, k\}\}$ é uma família de espaços ortogonais.

Denotemos $M = \left[\cup_{j=1}^{k} N(A-\lambda_j)\right]$.

Assim

$$M = N(A-\lambda_1) \oplus \cdots \oplus N(A-\lambda_k).$$

Mostraremos que $M = H$.

Observe que M é fechado e mostraremos agora que

$$M^{\perp} = \{0\}.$$

Seja $\lambda \notin P\sigma(A)$.

Logo,

$$N(A-\lambda) = \{0\}.$$

Assim

$$M = \left[\cup_{j=1}^{k} N(A-\lambda_j)\right] = \left[\cup_{\lambda\in\mathbb{C}} N(A-\lambda)\right].$$

Defina

$$L = \cup_{j=1}^{k} N(A-\lambda_j) = \cup_{\lambda\in\mathbb{C}} N(A-\lambda)$$

Sendo A normal e compacto, do Teorema 33.8, temos que

$$L^{\perp} = \{0\}.$$

Portanto $M = H$.

Seja $u \in H$.

Assim

$$u = \sum_{j=1}^{k} u_j$$

onde

$$u_j \in N(A - \lambda_j), \ \forall j \in \{1, \cdots, k\}.$$

Sendo A normal, para $i \neq j$, se $u_i \in N(A - \lambda_i)$, então

$$u_i \in N(A - \lambda_j)^{\perp}.$$

Logo

$$E_j u_i = \mathbf{0}, \quad \text{se } i \neq j.$$

Disto temos que

$$E_i \perp E_j, \text{ se } i \neq j.$$

Portanto,

$$E_j u = E_j \sum_{i=1}^{k} u_i = E_j u_j = u_j.$$

Disto podemos escrever,

$$u = \sum_{j=1}^{k} E_j u = \sum_{j=1}^{k} u_j = \left(\sum_{j=1}^{k} E_j \right) u, \ \forall u \in H$$

de modo que

$$\sum_{j=1}^{k} E_j = I_d.$$

Finalmente,

$$\begin{aligned} Au &= A \left(\sum_{j=1}^{k} u_j \right) \\ &= \sum_{j=1}^{k} \lambda_j u_j \\ &= \sum_{j=1}^{k} \lambda_j E_j u, \ \forall u \in H. \end{aligned} \tag{195}$$

Assim

$$A = \sum_{j=1}^{k} \lambda_j E_j.$$

Finalmente, disto temos que

$$dim(I_m(A)) = \sum_{j=1,\lambda_j\neq 0}^{k} dim(N-\lambda_j)$$

de modo que $dim(I_m(A))$ é finita.

A prova está completa.

□

Teorema 33.10 (Teorema espectral para operadores compactos e normais)**.** *Seja H um espaço de Hilbert complexo. Seja $A \in L(H,H)$ um operador compacto e normal.*

Sob tais hipóteses,

1.

$$\begin{aligned} H &= N(A) \oplus \sum_{\lambda\in P\sigma(A),\lambda\neq 0} \oplus N(A-\lambda) \\ &= \sum_{\lambda\in\mathbb{C}} \oplus N(A-\lambda) \\ &= \sum_{\lambda\in\mathbb{C}} \oplus N(A^*-\overline{\lambda}). \end{aligned} \tag{196}$$

2.

$$\overline{R(A)} = \overline{R(A^*)} = \sum_{\lambda\neq 0} \oplus N(A-\lambda).$$

3.

$$A = \sum_{\lambda\in P\sigma(A)} \lambda E_\lambda,$$

$$I = \sum_{\lambda\in\mathbb{C}} E_\lambda,$$

onde E_λ é a projeção ortogonal sobre $N(A-\lambda)$.

Proof. Como A é compacto, apenas para uma quantidade enumerável de λs, temos que

$$N(A-\lambda) \neq \{\mathbf{0}\}.$$

Sendo A normal, $\{N(A-\lambda) \,:\, \lambda \in \mathbb{C}\}$ é uma família de espaços fechados e ortogomais.

Como A é compacto e normal, do Teorema 33.8, se

$$u \perp N(A-\lambda),\ \forall \lambda \in \mathbb{C},$$

então

$$u = \mathbf{0}.$$

Logo

$$H = [\cup_{\lambda\in\mathbb{C}} N(A-\lambda)] = \sum_{\lambda\in P\sigma(A)} \oplus N(A-\lambda).$$

Sendo A normal temos que

$$N(A) = N(A^*),$$

e assim

$$\overline{R(A)} = N(A^*)^\perp = N(A)^\perp.$$

Observe que

$$\begin{aligned} H &= N(A) \oplus \overline{R(A)} \\ &= N(A) \oplus \sum_{\lambda\neq 0} \oplus N(A-\lambda), \end{aligned} \tag{197}$$

de modo que

$$\overline{R(A)} = \sum_{\lambda\neq 0} \oplus N(A-\lambda).$$

Portanto, sendo A^* compacto e normal, obtemos,

$$\overline{R(A^*)} = \sum_{\lambda\neq 0} \oplus N(A^* - \overline{\lambda})$$

Como $A-\lambda$ é normal, obtemos

$$N(A-\lambda) = N(A^* - \overline{\lambda}),$$

de modo que

$$\begin{aligned} H &= N(A) \oplus \sum_{\lambda\in P\sigma(A),\lambda\neq 0} \oplus N(A-\lambda) \\ &= \sum_{\lambda\in\mathbb{C}} \oplus N(A-\lambda) \\ &= \sum_{\lambda\in\mathbb{C}} \oplus N(A^* - \overline{\lambda}). \end{aligned} \tag{198}$$

e

$$\overline{R(A)} = \overline{R(A^*)} = \sum_{\lambda\neq 0} \oplus N(A-\lambda).$$

Portanto,

$$\sum_{\lambda\in\mathbb{C}} E_\lambda = I_d.$$

Além disso para $u \in H$ temos que

$$u = \sum_{\lambda\in P\sigma(A)} u_\lambda,$$

onde

$$u_\lambda = E_\lambda u \in N(A-\lambda).$$

Podemos então escrever,

$$\begin{aligned} Au &= A\left(\sum_{\lambda\in P\sigma(A)} u_\lambda\right) \\ &= \sum_{\lambda\in P\sigma(A)} Au_\lambda \\ &= \sum_{\lambda\in P\sigma(A)} \lambda u_\lambda \\ &= \sum_{\lambda\in P\sigma(A)} \lambda E_\lambda u,\ \forall u \in H. \end{aligned} \tag{199}$$

Finalmente, disto podemos denotar

$$A = \sum_{\lambda\in P\sigma(A)} \lambda E_\lambda.$$

A prova está completa. □

34 About the spectrum of a linear operator defined on a Banach space

Definição 34.1. *Let U be a Banach space and let $A \in \mathcal{L}(U)$. We recall that a complex number λ is said to be in the resolvent set $\rho(A)$ of A, if*

$$\lambda I - A$$

is a bijection with a bounded inverse. As previously indicated, we call

$$R_\lambda(A) = (\lambda I - A)^{-1}$$

the resolvent of A in λ.

If $\lambda \notin \rho(A)$, we write

$$\lambda \in \sigma(A) = \mathbb{C} - \rho(A),$$

where $\sigma(A)$ is said to be the spectrum of A.

Definição 34.2. *Let $A \in \mathcal{L}(U)$.*

1. *If $u \neq \theta$ and $Au = \lambda u$ for some $\lambda \in \mathbb{C}$ then u is said to be an eigenvector of A and λ the corresponding eigenvalue. If λ is an eigenvalue, then $(\lambda I - A)$ is not injective and therefore $\lambda \in \sigma(A)$.*

   The set of eigenvalues is said to be the point spectrum of A.

2. If λ is not an eigenvalue but

$$R(\lambda I - A)$$

is not dense in U and therefore $\lambda I - A$ is not a bijection, we have that $\lambda \in \sigma(A)$. In this case we say that λ is in the residual spectrum of A, or briefly $\lambda \in Res[\sigma(A)]$.

Teorema 34.3. *Let U be a Banach space and suppose that $A \in \mathcal{L}(U)$. Then $\rho(A)$ is an open subset of $\mathbb{C}$ and*

$$F(\lambda) = R_\lambda(A)$$

is an analytic function with values in $\mathcal{L}(U)$ on each connected component of $\rho(A)$. For $\lambda,\ \mu \in \sigma(A)$, $R_\lambda(A)$ and $R_\mu(A)$ commute and

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda)R_\mu(A)R_\lambda(A).$$

Proof. Let $\lambda_0 \in \rho(A)$. We will show that λ_0 is an interior point of $\rho(A)$.

Observe that symbolically we may write

$$\begin{aligned} \frac{1}{\lambda - A} &= \frac{1}{\lambda - \lambda_0 + (\lambda_0 - A)} \\ &= \frac{1}{\lambda_0 - A}\left[\frac{1}{1 - \left(\frac{\lambda_0 - \lambda}{\lambda_0 - A}\right)}\right] \\ &= \frac{1}{\lambda_0 - A}\left(1 + \sum_{n=1}^{\infty}\left(\frac{\lambda_0 - \lambda}{\lambda_0 - A}\right)^n\right). \end{aligned} \tag{200}$$

Define,

$$\hat{R}_\lambda(A) = R_{\lambda_0}(A)\{I + \sum_{n=1}^{\infty}(\lambda - \lambda_0)^n (R_{\lambda_0})^n\}. \tag{201}$$

Observe that

$$\|(R_{\lambda_0})^n\| \leq \|R_{\lambda_0}\|^n.$$

Thus, the series indicated in (201) will converge in norm if

$$|\lambda - \lambda_0| < \|R_{\lambda_0}\|^{-1}. \tag{202}$$

Hence, for λ satisfying (202), $\hat{R}(A)$ is well defined and we can easily check that

$$(\lambda I - A)\hat{R}_\lambda(A) = I = \hat{R}_\lambda(A)(\lambda I - A).$$

Therefore

$$\hat{R}_\lambda(A) = R_\lambda(A),\ \text{if } |\lambda - \lambda_0| < \|R_{\lambda_0}\|^{-1},$$

so that λ_0 is an interior point. Since $\lambda_0 \in \rho(A)$ is arbitrary, we have that $\rho(A)$ is open. Finally, observe that

$$\begin{aligned} R_\lambda(A) - R_\mu(A) &= R_\lambda(A)(\mu I - A)R_\mu(A) - R_\lambda(A)(\lambda I - A)R_\mu(A) \\ &= R_\lambda(A)(\mu I)R_\mu(A) - R_\lambda(A)(\lambda I)R_\mu(A) \\ &= (\mu - \lambda)R_\lambda(A)R_\mu(A) \qquad (203) \end{aligned}$$

Interchanging the roles of λ and μ we may conclude that R_λ and R_μ commute. □

Corolário 34.4. *Let U be a Banach space and $A \in \mathcal{L}(U)$. Then the spectrum of A is non-empty.*

Proof. Observe that if

$$\frac{\|A\|}{|\lambda|} < 1$$

we have

$$\begin{aligned} (\lambda I - A)^{-1} &= [\lambda(I - A/\lambda)]^{-1} \\ &= \lambda^{-1}(I - A/\lambda)^{-1} \\ &= \lambda^{-1}\left(I + \sum_{n=1}^{\infty}\left(\frac{A}{\lambda}\right)^n\right). \qquad (204) \end{aligned}$$

Therefore we may obtain

$$R_\lambda(A) = \lambda^{-1}\left(I + \sum_{n=1}^{\infty}\left(\frac{A}{\lambda}\right)^n\right).$$

In particular

$$\|R_\lambda(A)\| \to 0, \text{ as } |\lambda| \to \infty. \qquad (205)$$

Suppose, to obtain contradiction, that

$$\sigma(A) = \emptyset.$$

In such a case $R_\lambda(A)$ would be a entire bounded analytic function. From Liouville's theorem, $R_\lambda(A)$ would be constant, so that from (205) we would have

$$R_\lambda(A) = \theta, \forall \lambda \in \mathbb{C},$$

which is a contradiction.

Proposição 34.5. *Let H be a Hilbert space and $A \in \mathcal{L}(H)$.*

1. *If $\lambda \in Res[\sigma(A)]$ then $\overline{\lambda} \in P\sigma(A^*)$.*
2. *If $\lambda \in P\sigma(A)$ then $\overline{\lambda} \in P\sigma(A^*) \cup Res[\sigma(A^*)]$.*

Proof. 1. If $\lambda \in Res[\sigma(A)]$ then
$$R(A - \lambda I) \neq H.$$
Therefore there exists $v \in (R(A - \lambda I))^{\perp}$, $v \neq \theta$ such that
$$(v, (A - \lambda I)u)_H = 0, \forall u \in H$$
that is
$$((A^* - \overline{\lambda} I)v, u)_H = 0, \forall u \in H$$
so that
$$(A^* - \overline{\lambda} I)v = \theta,$$
which means that $\overline{\lambda} \in P\sigma(A^*)$.

2. Suppose there exists $v \neq \theta$ such that
$$(A - \lambda I)v = \theta,$$
and
$$\overline{\lambda} \notin P\sigma(A^*).$$
Thus
$$(u, (A - \lambda I)v))_H = 0, \forall u \in H,$$
so that
$$((A^* - \overline{\lambda} I)u, v)_H, \forall u \in H.$$
Since
$$(A^* - \overline{\lambda} I)u \neq \theta, \forall u \in H, u \neq \theta,$$
we get $v \in (R(A^* - \overline{\lambda} I))^{\perp}$, so that $R(A^* - \overline{\lambda} I) \neq H$.
Hence $\overline{\lambda} \in Res[\sigma(A^*)]$.

□

Teorema 34.6. *Let $A \in \mathcal{L}(H)$ be a self-adjoint operator. then*

1. $\sigma(A) \subset \mathbb{R}$.
2. *Eigenvectors corresponding to distinct eigenvalues of A are orthogonal.*

Proof. Let $\mu, \lambda \in \mathbb{R}$. Thus, given $u \in H$ we have
$$\|(A - (\lambda + \mu i))u\|^2 = \|(A - \lambda)u\|^2 + \mu^2 \|u\|^2,$$
so that
$$\|(A - (\lambda + \mu i))u\|^2 \geq \mu^2 \|u\|^2.$$

Therefore if $\mu \neq 0$, $A-(\lambda+\mu i)$ has a bounded inverse on its range, which is closed. If $R(A-(\lambda+\mu i)) \neq H$ then by the last result $(\lambda - \mu i)$ would be in the point spectrum of A, which contradicts the last inequality. Hence, if $\mu \neq 0$ then $\lambda + \mu i \in \rho(A)$. To complete the proof, suppose

$$Au_1 = \lambda_1 u_1,$$

and

$$Au_2 = \lambda_2 u_2,$$

where

$$\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2 \text{ and } u_1, u_2 \neq \theta.$$

Thus

$$\begin{aligned}(\lambda_1 - \lambda_2)(u_1, u_2)_H &= \lambda_1(u_1, u_2)_H - \lambda_2(u_1, u_2)_H \\ &= (\lambda_1 u_1, u_2)_H - (u_1, \lambda_2 u_2)_H \\ &= (Au_1, u_2)_H - (u_1, Au_2)_H \\ &= (u_1, Au_2)_H - (u_1, Au_2)_H \\ &= 0. \end{aligned} \tag{206}$$

Since $\lambda_1 - \lambda_2 \neq 0$ we get

$$(u_1, u_2)_H = 0.$$

□

We finish this section with an exercise and its solution.

Exercício 34.7. *Let H be a complex Hilbert space e let $A \in L(H)$ be a self-adjoint operator. Prove that $\lambda \in \sigma(A)$ if, and only if, there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\|_H = 1$, $\forall n \in \mathbb{N}$ and*

$$\|Au_n - \lambda u_n\|_H \to 0, \text{ as } n \to \infty.$$

Solution: Suppose $\lambda \in \mathbb{C}$ is such that there exists $\{u_n\} \subset H$ such that $\|u_n\| = 1$, $\forall n \in \mathbb{N}$ and

$$\|Au_n - \lambda u_n\|_H \to 0, \text{ as } n \to \infty. \tag{207}$$

Suppose, to obtain contradiction, that $\lambda \in \rho(A)$. Thus, $(A - \lambda I)^{-1}$ exists and it is bounded, so that there exists $K > 0$ such that

$$\|Au - \lambda I u\|_H \geq K\|u\|_H, \forall u \in H.$$

From this we obtain

$$\|Au_n - \lambda u_n\|_H \geq K,\ \forall n \in \mathbb{N}$$

which contradicts (207).

Thus $\lambda \notin \rho(A)$ so that $\lambda \in \sigma(A)$.

Reciprocally, suppose $\lambda \in \sigma(A)$. Suppose, to obtain contradiction, that there exists $K > 0$ such that

$$\|Au - \lambda u\|_H \geq K\|u\|_H, \ \forall u \in H. \tag{208}$$

Thus $(A - \lambda I)^{-1}$ exists and it is bounded. Since $\lambda \in \sigma(A)$, we must have that $R(A - \lambda I)$ is not dense, so that $\lambda \in Res[\sigma(A)]$.

From Proposition 34.5, we have $\overline{\lambda} \in P\sigma(A^*)$.

Since $A = A^*$, from this we obtain $\lambda = \overline{\lambda} \in P(\sigma(A))$ which contradicts $(A - I\lambda)^{-1}$ to exist.

Thus, we may infer that it does not exist $K > 0$ such that (208) holds.

From this, for each $n \in \mathbb{N}$ there exists $u_n \in H$ such that

$$\|u_n\|_H = 1$$

and

$$\|Au_n - \lambda u_n\|_H < 1/n$$

so that

$$\|Au_n - \lambda u_n\|_H \to 0.$$

The solution is complete.

35 The spectral theorem for bounded self-adjoint operators

Let H be a complex Hilbert space. Consider $A : H \to H$ a linear bounded operator, that is $A \in \mathcal{L}(H)$, and suppose also that such an operator is self-adjoint. Define

$$m = \inf_{u \in H}\{(Au, u)_H \mid \|u\|_H = 1\},$$

and

$$M = \sup_{u \in H}\{(Au, u)_H \mid \|u\|_H = 1\}.$$

Observação 35.1. *It is possible to prove that for a linear self-adjoint operator $A : H \to H$ we have*

$$\|A\| = \sup\{|(Au, u)_H| \mid u \in H, \ \|u\|_H = 1\}.$$

This propriety, which prove in the next lines, is crucial for the subsequent results, since for example for A, B linear and self adjoint and $\varepsilon > 0$ we have

$$-\varepsilon I \leq A - B \leq \varepsilon I,$$

we also would have

$$\|A - B\| < \varepsilon.$$

So, we present the following basic result.

Teorema 35.2. *Let H be a real Hilbert space and let $A : H \to H$ be a bounded linear self adjoint operator. Define*

$$\alpha = \max\{|m|, |M|\},$$

where

$$m = \inf_{u \in H}\{(Au, u)_H \mid \|u\|_H = 1\},$$

and

$$M = \sup_{u \in H}\{(Au, u)_H \mid \|u\|_H = 1\}.$$

Then

$$\|A\| = \alpha.$$

Proof. Observe that

$$(A(u+v), u+v)_H = (Au, u)_H + (Av, v)_H + 2(Au, v)_H,$$

and

$$(A(u-v), u-v)_H = (Au, u)_H + (Av, v)_H - 2(Au, v)_H.$$

Thus,

$$4(Au, v) = (A(u+v), u+v)_H - (A(u-v), u-v)_H \leq M\|u+v\|_U^2 - m\|u-v\|_U^2,$$

so that

$$4(Au, v)_H \leq \alpha(\|u+v\|_U^2 + \|u-v\|_U^2).$$

Hence, replacing v by $-v$ we obtain

$$-4(Au, v)_H \leq \alpha(\|u+v\|_U^2 + \|u-v\|_U^2),$$

and therefore

$$4|(Au, v)_H| \leq \alpha(\|u+v\|_U^2 + \|u-v\|_U^2).$$

Replacing v by βv, we get

$$4|(A(u), v)_H| \leq 2\alpha(\|u\|_U^2/\beta + \beta\|v\|_U^2).$$

Minimizing the last expression in $\beta > 0$, for the optimal

$$\beta = \|u\|_U/\|v\|_U,$$

we obtain

$$|(Au, v)_H| \leq \alpha\|u\|_U\|v\|_U, \forall u, v \in U.$$

Thus

$$\|A\| \leq \alpha.$$

On the other hand,

$$|(Au, u)_H| \leq \|A\| \|u\|_U^2,$$

so that

$$|M| \leq \|A\|$$

and

$$|m| \leq \|A\|,$$

so that

$$\alpha \leq \|A\|.$$

The proof is complete. $\square$

Observação 35.3. *A similar result is valid as H is a complex Hilbert space.*

At this point we start to develop the spectral theory. Define by P the set of all real polynomials defined in $\mathbb{R}$. Define

$$\Phi_1 : P \to \mathcal{L}(H),$$

by

$$\Phi_1(p(\lambda)) = p(A), \forall p \in P.$$

Thus we have

1. $\Phi_1(p_1 + p_2) = p_1(A) + p_2(A)$,
2. $\Phi_1(p_1 \cdot p_2) = p_1(A)p_2(A)$,
3. $\Phi_1(\alpha p) = \alpha p(A), \forall \alpha \in \mathbb{R},\ p \in P$
4. if $p(\lambda) \geq 0,$ on $[m, M],$ then $p(A) \geq \theta,$

We will prove (4):

Consider $p \in P$. Denote the real roots of $p(\lambda)$ less or equal to m by $\alpha_1, \alpha_2, ..., \alpha_n$ and denote those that are greater or equal to M by $\beta_1, \beta_2, ..., \beta_l$. Finally denote all the remaining roots, real or complex by

$$v_1 + i\mu_1, ..., v_k + i\mu_k.$$

Observe that if $\mu_i = 0$ then $v_i \in (m, M)$. The assumption that $p(\lambda) \geq 0$ on $[m, M]$ implies that any real root in (m, M) must be of even multiplicity.

Since complex roots must occur in conjugate pairs, we have the following representation for $p(\lambda)$:

$$p(\lambda) = a \prod_{i=1}^{n} (\lambda - \alpha_i) \prod_{i=1}^{l} (\beta_i - \lambda) \prod_{i=1}^{k} ((\lambda - v_i)^2 + \mu_i^2),$$

where $a \geq 0$. Observe that

$$A - \alpha_i I \geq \theta,$$

since,

$$(Au, u)_H \geq m(u, u)_H \geq \alpha_i (u, u)_H, \forall u \in H,$$

and by analogy

$$\beta_i I - A \geq \theta.$$

On the other hand, since $A - v_k I$ is self-adjoint, its square is positive and hence since the sum of positive operators is positive, we obtain

$$(A - v_k I)^2 + \mu_k^2 I \geq \theta.$$

Therefore

$$p(A) \geq \theta.$$

The idea is now to extend de domain of Φ_1 to the set of upper semi-continuous functions, and such set we will denote by C^{up}.

Observe that if $f \in C^{up}$, there exists a sequence of continuous functions $\{g_n\}$ such that

$$g_n \downarrow f, \text{ pointwise },$$

that is

$$g_n(\lambda) \downarrow f(\lambda), \forall \lambda \in \mathbb{R}.$$

Considering the Weierstrass Theorem, since $g_n \in C([m, M])$ we may obtain a sequence of polynomials $\{p_n\}$ such that

$$\left\| \left(g_n + \frac{1}{2^n} \right) - p_n \right\|_\infty < \frac{1}{2^n},$$

where the norm $\| \cdot \|_\infty$ refers to $[m, M]$. Thus

$$p_n(\lambda) \downarrow f(\lambda), \text{ on } [m, M].$$

Therefore

$$p_1(A) \geq p_2(A) \geq p_3(A) \geq ... \geq p_n(A) \geq ...$$

Since $p_n(A)$ is self-adjoint for all $n \in \mathbb{N}$, we have

$$p_j(A) p_k(A) = p_k(A) p_j(A), \forall j, k \in \mathbb{N}.$$

Then the $\lim_{n\to\infty} p_n(A)$ (in norm) exists, and we denote

$$\lim_{n\to\infty} p_n(A) = f(A).$$

Now recall the Dini's Theorem:

Teorema 35.4 (Dini)**.** *Let $\{g_n\}$ be a sequence of continuous functions defined on a compact set $K \subset \mathbb{R}$. Suppose $g_n \to g$ point-wise and monotonically on K. Under such assumptions the convergence in question is also uniform.*

Now suppose that $\{p_n\}$ and $\{q_n\}$ are sequences of polynomial such that

$$p_n \downarrow f, \text{ and } q_n \downarrow f,$$

we will show that

$$\lim_{n\to\infty} p_n(A) = \lim_{n\to\infty} q_n(A).$$

First observe that being $\{p_n\}$ and $\{q_n\}$ sequences of continuous functions we have that

$$\hat{h}_{nk}(\lambda) = \max\{p_n(\lambda), q_k(\lambda)\}, \forall \lambda \in [m, M]$$

is also continuous, $\forall n, k \in \mathbb{N}$. Now fix $n \in \mathbb{N}$ and define

$$h_k(\lambda) = \max\{p_k(\lambda), q_n(\lambda)\}.$$

observe that

$$h_k(\lambda) \downarrow q_n(\lambda), \forall \lambda \in \mathbb{R},$$

so that by Dini's theorem

$$h_k \to q_n, \text{ uniformly on } [m, M].$$

It follows that for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that if $k > k_n$ then

$$h_k(\lambda) - q_n(\lambda) \leq \frac{1}{n}, \forall \lambda \in [m, M].$$

Since

$$p_k(\lambda) \leq h_k(\lambda), \forall \lambda \in [m, M],$$

we obtain

$$p_k(\lambda) - q_n(\lambda) \leq \frac{1}{n}, \forall \lambda \in [m, M].$$

By analogy, we may show that for each $n \in \mathbb{N}$ there exists $\hat{k}_n \in \mathbb{N}$ such that if $k > \hat{k}_n$ then

$$q_k(\lambda) - p_n(\lambda) \leq \frac{1}{n}.$$

From above we obtain

$$\lim_{k\to\infty} p_k(A) \leq q_n(A) + \frac{1}{n}.$$

Since the self adjoint $q_n(A) + 1/n$ commutes with the

$$\lim_{k\to\infty} p_k(A)$$

we obtain

$$\begin{aligned} \lim_{k\to\infty} p_k(A) &\leq \lim_{n\to\infty} \left(q_n(A) + \frac{1}{n}\right) \\ &\leq \lim_{n\to\infty} q_n(A). \end{aligned} \tag{209}$$

Similarly we may obtain

$$\lim_{k\to\infty} q_k(A) \le \lim_{n\to\infty} p_n(A),$$

so that

$$\lim_{n\to\infty} q_n(A) = \lim_{n\to\infty} p_n(A) = f(A).$$

Hence, we may extend $\Phi_1 : P \to \mathcal{L}(H)$ to $\Phi_2 : C^{up} \to \mathcal{L}(H)$ where C^{up} as earlier indicated, denotes the set of upper semi-continuous functions, where

$$\Phi_2(f) = f(A).$$

Observe that Φ_2 has the following properties

1. $\Phi_2(f_1 + f_2) = \Phi_2(f_1) + \Phi_2(f_2)$,
2. $\Phi_2(f_1 \cdot f_2) = f_1(A)f_2(A)$,
3. $\Phi_2(\alpha f) = \alpha\Phi_2(f), \forall \alpha \in \mathbb{R},\ \alpha \ge 0.$
4. if $f_1(\lambda) \ge f_2(\lambda), \forall \lambda \in [m, M]$, then
$$f_1(A) \ge f_2(A).$$

The next step is to extend Φ_2 to $\Phi_3 : C^{up}_{-} \to \mathcal{L}(H)$, where

$$C^{up}_{-} = \{f - g \mid f, g \in C^{up}\}.$$

For $h = f - g \in C^{up}_{-}$ we define

$$\Phi_3(h) = f(A) - g(A).$$

Now we will show that Φ_3 is well defined. Suppose that $h \in C^{up}_{-}$ and

$$h = f_1 - g_1 \text{ and } h = f_2 - g_2.$$

Thus

$$f_1 - g_1 = f_2 - g_2,$$

that is

$$f_1 + g_2 = f_2 + g_1,$$

so that from the definition of Φ_2 we obtain

$$f_1(A) + g_2(A) = f_2(A) + g_1(A),$$

that is

$$f_1(A) - g_1(A) = f_2(A) - g_2(A).$$

Therefore Φ_3 is well defined. Finally observe that for $\alpha < 0$

$$\alpha(f - g) = -\alpha g - (-\alpha)f,$$

where $-\alpha g \in C^{up}$ and $-\alpha f \in C^{up}$. Thus

$$\Phi_3(\alpha f) = \alpha f(A) = \alpha\Phi_3(f), \forall \alpha \in \mathbb{R}.$$

35.1 The spectral theorem

Consider the upper semi-continuous function

$$h_\mu(\lambda) = \begin{cases} 1, & \text{if } \lambda \leq \mu, \\ 0, & \text{if } \lambda > \mu. \end{cases} \tag{210}$$

Denote

$$E(\mu) = \Phi_3(h_\mu) = h_\mu(A).$$

Observe that

$$h_\mu(\lambda)h_\mu(\lambda) = h_\mu(\lambda), \forall \lambda \in \mathbb{R},$$

so that

$$[E(\mu)]^2 = E(\mu), \forall \mu \in \mathbb{R}.$$

Therefore

$$\{E(\mu) \mid \mu \in \mathbb{R}\}$$

is a family of orthogonal projections. Also observe that if $\nu \geq \mu$ we have

$$h_\nu(\lambda)h_\mu(\lambda) = h_\mu(\lambda)h_\nu(\lambda) = h_\mu(\lambda),$$

so that

$$E(\nu)E(\mu) = E(\mu)E(\nu) = E(\mu), \forall \nu \geq \mu.$$

If $\mu < m$, then $h_\mu(\lambda) = 0, \text{ on } [m, M]$, so that

$$E(\mu) = 0, \text{ if } \mu < m.$$

Similarly, if $\mu \geq M$ them $h_\mu(\lambda) = 1, \text{ on } [m, M]$, so that

$$E(\mu) = I, \text{ if } \mu \geq M.$$

Next we show that the family $\{E(\mu)\}$ is strongly continuous from the right. First we will establish a sequence of polynomials $\{p_n\}$ such that

$$p_n \downarrow h_\mu,$$

and

$$p_n(\lambda) \geq h_{\mu+\frac{1}{n}}(\lambda), \text{ on } [m, M].$$

Observe that for any fixed n there exists a sequence of polynomials $\{p_j^n\}$ such that

$$p_j^n \downarrow h_{\mu+1/n}, \text{ point-wise }.$$

Consider the monotone sequence

$$g_n(\lambda) = \min\{p_s^r(\lambda) \mid r, s \in \{1, ..., n\}\}.$$

Thus

$$g_n(\lambda) \geq h_{\mu+\frac{1}{n}}(\lambda), \forall \lambda \in \mathbb{R},$$

and we obtain

$$\lim_{n\to\infty} g_n(\lambda) \geq \lim_{n\to\infty} h_{\mu+\frac{1}{n}}(\lambda) = h_\mu(\lambda).$$

On the other hand

$$g_n(\lambda) \leq p_n^r(\lambda), \forall \lambda \in \mathbb{R}, \forall r \in \{1, ..., n\},$$

so that

$$\lim_{n\to\infty} g_n(\lambda) \leq \lim_{n\to\infty} p_n^r(\lambda).$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} g_n(\lambda) &\leq \lim_{r\to\infty} \lim_{n\to\infty} p_n^r(\lambda) \\ &= h_\mu(\lambda). \end{aligned} \tag{211}$$

Thus

$$\lim_{n\to\infty} g_n(\lambda) = h_\mu(\lambda).$$

Observe that g_n are not necessarily polynomials. To set a sequence of polynomials, observe that we may obtain a sequence $\{p_n\}$ of polynomials such that

$$|g_n(\lambda) + 1/n - p_n(\lambda)| < \frac{1}{2^n}, \forall \lambda \in [m, M],\ n \in \mathbb{N}.$$

so that

$$p_n(\lambda) \geq g_n(\lambda) + 1/n - 1/2^n \geq g_n(\lambda) \geq h_{\mu+1/n}(\lambda).$$

Thus

$$p_n(A) \to E(\mu),$$

and

$$p_n(A) \geq h_{\mu+\frac{1}{n}}(A) = E(\mu + 1/n) \geq E(\mu).$$

Therefore we may write

$$E(\mu) = \lim_{n\to\infty} p_n(A) \geq \lim_{n\to\infty} E(\mu + 1/n) \geq E(\mu).$$

Thus

$$\lim_{n\to\infty} E(\mu + 1/n) = E(\mu).$$

From this we may easily obtain the strong continuity from the right.

For $\mu \leq \nu$ we have

$$\begin{aligned} \mu(h_\nu(\lambda) - h_\mu(\lambda)) &\leq \lambda(h_\nu(\lambda) - h_\mu(\lambda)) \\ &\leq \nu(h_\nu(\lambda) - h_\mu(\lambda)). \end{aligned} \tag{212}$$

To verify this observe that if $\lambda < \mu$ or $\lambda > \nu$ then all terms involved in the above inequalities are zero. On the other hand if

$$\mu \leq \lambda \leq \nu$$

then

$$h_\nu(\lambda) - h_\mu(\lambda) = 1,$$

so that in any case (212) holds. From the monotonicity property we have

$$\begin{aligned} \mu(E(\nu) - E(\mu)) &\leq A(E(\nu) - E(\mu)) \\ &\leq \nu(E(\nu) - E(\mu)). \end{aligned} \tag{213}$$

Now choose $a, b \in \mathbb{R}$ such that

$$a < m \text{ and } b \geq M.$$

Suppose given $\varepsilon > 0$. Choose a partition P_0 of $[a, b]$, that is

$$P_0 = \{a = \lambda_0, \lambda_1, ..., \lambda_n = b\},$$

such that

$$\max_{k \in \{1,...,n\}} \{|\lambda_k - \lambda_{k-1}|\} < \varepsilon.$$

Hence

$$\begin{aligned} \lambda_{k-1}(E(\lambda_k) - E(\lambda_{k-1})) &\leq A(E(\lambda_k) - E(\lambda_{k-1})) \\ &\leq \lambda_k(E(\lambda_k) - E(\lambda_{k-1})). \end{aligned} \tag{214}$$

Summing up on k and recalling that

$$\sum_{k=1}^{n} E(\lambda_k) - E(\lambda_{k-1}) = I,$$

we obtain

$$\begin{aligned} \sum_{k=1}^{n} \lambda_{k-1}(E(\lambda_k) - E(\lambda_{k-1})) &\leq A \\ &\leq \sum_{k=1}^{n} \lambda_k(E(\lambda_k) - E(\lambda_{k-1})). \end{aligned} \tag{215}$$

Let $\lambda_k^0 \in [\lambda_{k-1}, \lambda_k]$. Since $(\lambda_k - \lambda_k^0) \leq (\lambda_k - \lambda_{k-1})$ from (214) we obtain

$$\begin{aligned} A - \sum_{k=1}^{n} \lambda_k^0(E(\lambda_k) - E(\lambda_{k-1})) &\leq \varepsilon \sum_{k=1}^{n}(E(\lambda_k) - E(\lambda_{k-1})) \\ &= \varepsilon I. \end{aligned} \tag{216}$$

By analogy

$$-\varepsilon I \leq A - \sum_{k=1}^{n} \lambda_k^0 (E(\lambda_k) - E(\lambda_{k-1})). \tag{217}$$

Since

$$A - \sum_{k=1}^{n} \lambda_k^0 (E(\lambda_k) - E(\lambda_{k-1}))$$

is self-adjoint we obtain

$$\|A - \sum_{k=1}^{n} \lambda_k^0 (E(\lambda_k) - E(\lambda_{k-1}))\| < \varepsilon.$$

Being $\varepsilon > 0$ arbitrary, we may write

$$A = \int_a^b \lambda dE(\lambda),$$

that is

$$A = \int_{m^-}^{M} \lambda dE(\lambda).$$

Observação 35.5. *Consider again the function $h_\mu : \mathbb{R} \to \mathbb{R}$ where*

$$h_\mu(\lambda) = \begin{cases} 1, & \text{if } \lambda \leq \mu \\ 0, & \text{if } \lambda > \mu. \end{cases} \tag{218}$$

Let H be a complex Hilbert space and let $A \in L(H)$, where A is a self-adjoint opertaor.

Suppose $f \in C([m, M])$ where

$$m = \inf_{u \in H} \{(Au, u)_H \ : \ \|u\|_H = 1\},$$

and

$$M = \sup_{u \in H} \{(Au, u)_H \ : \ \|u\|_H = 1\}.$$

Let $\varepsilon > 0$. Since f is uniformly continuous on the compact set $[m, M]$, there exists $\delta > 0$ such that if $x, y \in [m, M]$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| < \varepsilon. \tag{219}$$

Let $P = \{\lambda_0 = m, \lambda_1, \ldots, \lambda_n = M\}$ be a partition of $[m, M]$, such that $\|P\| = \max\{\lambda_k - \lambda_{k-1} \ : \ k \in \{1, \ldots, n\}\} < \delta$.

Choose

$$\lambda_k^0 \in (\lambda_{k-1}, \lambda_k), \forall k \in \{1, \ldots, n\}$$

and observe that

$$h_{\lambda_k}(\lambda) - h_{\lambda_{k-1}}(\lambda) = \begin{cases} 1, & \text{if } \lambda_{k-1} < \lambda \leq \lambda_k \\ 0, & \text{otherwise.} \end{cases} \tag{220}$$

From this and (219), we may obtain

$$\left|f(\lambda) - \sum_{k=1}^{n} f(\lambda_k^0)[h_{\lambda_k}(\lambda) - h_{\lambda_{k-1}}(\lambda)]\right| < \varepsilon, \ \forall \lambda \in [m, M].$$

Therefore, for the corresponding operators, we have got

$$\left\|f(A) - \sum_{k=1}^{n} f(\lambda_k^0)[E(\lambda_k) - E(\lambda_{k-1})]\right\| < \varepsilon.$$

Since $\varepsilon > 0$*, the partition* P *and* $\{\lambda_k^0\}$ *have been arbitrary, we may denote*

$$f(A) = \int_{m^-}^{M} f(\lambda) dE(\lambda).$$

36 The spectral decomposition of unitary transformations

Definição 36.1. *Let* H *be a Hilbert space. A transformation* $U : H \to H$ *is said to be unitary if*

$$(Uu, Uu)_H = (u, u)_H, \forall u, u \in H.$$

Observe that in this case

$$U^* U = U U^* = I,$$

so that

$$U^{-1} = U^*.$$

Teorema 36.2. *Every Unitary transformation* U *has a spectral decomposition*

$$U = \int_{0^-}^{2\pi} e^{i\phi} dE(\phi),$$

where $\{E(\phi)\}$ *is a spectral family on* $[0, 2\pi]$*. Furthermore* $E(\phi)$ *is continuous at* 0 *and it is the limit of polynomials in* U *and* U^{-1}*.*

We present just a sketch of the proof. For the trigonometric polynomials

$$p(e^{i\phi}) = \sum_{k=-n}^{n} c_k e^{ik\phi},$$

consider the transformation

$$p(U) = \sum_{k=-n}^{n} c_k U^k,$$

where $c_k \in \mathbb{C}, \forall k \in \{-n, ..., 0, ..., n\}$.

Observe that

$$\overline{p(e^{i\phi})} = \sum_{k=-n}^{n} \overline{c}_k e^{-ik\phi},$$

so that the corresponding operator is

$$p(U)^* = \sum_{k=-n}^{n} \overline{c}_k U^{-k} = \sum_{k=-n}^{n} \overline{c}_k (U^*)^k.$$

Also if

$$p(e^{i\phi}) \geq 0$$

there exists a polynomial q such that

$$p(e^{i\phi}) = |q(e^{i\phi})|^2 = \overline{q(e^{i\phi})} q(e^{i\phi}),$$

so that

$$p(U) = [q(U)]^* q(U).$$

Therefore

$$(p(U)v, v)_H = (q(U)^* q(U)v, v)_H = (q(U)v, q(U)v)_H \geq 0, \forall v \in H,$$

which means

$$p(U) \geq 0.$$

Define the function $h_\mu(\phi)$ by

$$h_\mu(\phi) = \begin{cases} 1, \text{ if } 2k\pi < \phi \leq 2k\pi + \mu, \\ 0, \text{ if } 2k\pi + \mu < \phi \leq 2(k+1)\pi, \end{cases} \tag{221}$$

for each $k \in \{0, \pm 1, \pm 2, \pm 3, ...\}$. Define $E(\mu) = h_\mu(U)$. Observe that the family $\{E(\mu)\}$ are projections and in particular

$$E(0) = 0,$$

$$E(2\pi) = I$$

and if $\mu \leq \nu$, since

$$h_\mu(\phi) \leq h_\nu(\phi),$$

we have

$$E(\mu) \leq E(\nu).$$

Suppose given $\varepsilon > 0$. Let P_0 be a partition of $[0, 2\pi]$ that is,

$$P_0 = \{0 = \phi_0, \phi_1, ..., \phi_n = 2\pi\}$$

such that

$$\max_{j \in \{1,...,n\}} \{|\phi_j - \phi_{j-1}|\} < \varepsilon.$$

For fixed $\phi \in [0, 2\pi]$, let $j \in \{1, ..., n\}$ be such that

$$\phi \in [\phi_{j-1}, \phi_j].$$

$$\begin{aligned} |e^{i\phi} - \sum_{k=1}^{n} e^{i\phi_k}(h_{\phi_k}(\phi) - h_{\phi_{k-1}}(\phi))| &= |e^{i\phi} - e^{i\phi_j}| \\ &\leq |\phi - \phi_j| < \varepsilon. \end{aligned} \tag{222}$$

Thus,

$$0 \leq |e^{i\phi} - \sum_{k=1}^{n} e^{i\phi_k}(h_{\phi_k}(\phi) - h_{\phi_{k-1}}(\phi))|^2 \leq \varepsilon^2$$

so that, for the corresponding operators

$$\begin{aligned} 0 &\leq [U - \sum_{k=1}^{n} e^{i\phi_k}(E(\phi_k) - E(\phi_{k-1})]^*[U - \sum_{k=1}^{n} e^{i\phi_k}(E(\phi_k) - E(\phi_{k-1})] \\ &\leq \varepsilon^2 I \end{aligned} \tag{223}$$

and hence

$$\|U - \sum_{k=1}^{n} e^{i\phi_k}(E(\phi_k) - E(\phi_{k-1})\| < \varepsilon.$$

Being $\varepsilon > 0$ arbitrary, we may infer that

$$U = \int_0^{2\pi} e^{i\phi} dE(\phi).$$

37 Unbounded operators

37.1 Introduction

Let H be a Hilbert space. Let $A : D(A) \to H$ be an operator, where unless indicated $D(A)$ is a dense subset of H. We consider in this section the special case where A is unbounded.

Definição 37.1. *Given $A : D \to H$ we define the graph of A, denoted by $\Gamma(A)$ by,*

$$\Gamma(A) = \{(u, Au) \mid u \in D\}.$$

Definição 37.2. *An operator $A : D \to H$ is said to be closed if $\Gamma(A)$ is closed.*

Definição 37.3. *Let $A_1 : D_1 \to H$ and $A_2 : D_2 \to H$ operators. We write $A_2 \supset A_1$ if $D_2 \supset D_1$ and*

$$A_2 u = A_1 u, \forall u \in D_1.$$

In this case we say that A_2 is an extension of A_1.

Definição 37.4. *A linear operator $A : D \to H$ is said to be closable if it has a linear closed extension. The smallest closed extension of A is denote by $\overline{A}$ and is called the closure of A.*

Proposição 37.5. *Let $A : D \to H$ be a linear operator. If A is closable then*

$$\Gamma(\overline{A}) = \overline{\Gamma(A)}.$$

Proof. Suppose B is a closed extension of A. Then

$$\overline{\Gamma(A)} \subset \overline{\Gamma(B)} = \Gamma(B),$$

so that if $(\theta, \phi) \in \overline{\Gamma(A)}$ then $(\theta, \phi) \in \Gamma(B)$, and hence $\phi = \theta$. Define the operator C by

$$D(C) = \{\psi \mid (\psi, \phi) \in \overline{\Gamma(A)} \text{ for some } \phi\},$$

and $C(\psi) = \phi$, where ϕ is the unique point such that $(\psi, \phi) \in \overline{\Gamma(A)}$. Hence

$$\Gamma(C) = \overline{\Gamma(A)} \subset \Gamma(B),$$

so that

$$A \subset C.$$

However $C \subset B$ and since B is an arbitrary closed extension of A we have

$$C = \overline{A}$$

so that

$$\Gamma(C) = \Gamma(\overline{A}) = \overline{\Gamma(A)}.$$

□

Definição 37.6. *Let $A : D \to H$ be a linear operator where D is dense in H. Define $D(A^*)$ by*

$$D(A^*) = \{\phi \in H \mid (A\psi, \phi)_H = (\psi, \eta)_H, \ \forall \psi \in D \text{ for some } \eta \in H\}.$$

In this case we denote

$$A^*\phi = \eta.$$

A^* defined in this way is called the adjoint operator related to A.

Observe that by the Riesz lemma, $\phi \in D(A^*)$ if and only if there exists $K > 0$ such that

$$|(A\psi, \phi)_H| \leq K\|\psi\|_H, \forall \psi \in D.$$

Also note that if

$$A \subset B \text{ then } B^* \subset A^*.$$

Finally, as D is dense in H then

$$\eta = A^*(\phi)$$

is uniquely defined. However the domain of A^* may not be dense, and in some situations we may have $D(A^*) = \{\theta\}$.

If $D(A^*)$ is dense we define

$$A^{**} = (A^*)^*.$$

Teorema 37.7. *Let $A : D \to H$ a linear operator, being D dense in H. Then*

1. *A^* is closed,*

2. *A is closable if and only if $D(A^*)$ is dense and in this case*

$$\overline{A} = A^{**}.$$

3. *If A is closable then $(\overline{A})^* = A^*$.*

Proof. 1. We define the operator $V : H \times H \to H \times H$ by

$$V(\phi, \psi) = (-\psi, \phi).$$

Let $E \subset H \times H$ be a subspace. Thus if $(\phi_1, \psi_1) \in V(E^\perp)$ then there exists $(\phi, \psi) \in E^\perp$ such that

$$V(\phi, \psi) = (-\psi, \phi) = (\phi_1, \psi_1).$$

Hence

$$\psi = -\phi_1 \text{ and } \phi = \psi_1,$$

so that for $(\psi_1, -\phi_1) \in E^\perp$ and $(w_1, w_2) \in E$ we have

$$((\psi_1, -\phi_1), (w_1, w_2))_{H\times H} = 0 = (\psi_1, w_1)_H + (-\phi_1, w_2)_H.$$

Thus

$$(\phi_1, -w_2)_H + (\psi_1, w_1)_H = 0,$$

and therefore

$$((\phi_1, \psi_1), (-w_2, w_1))_{H\times H} = 0,$$

that is

$$((\phi_1, \psi_1), V(w_1, w_2))_{H\times H} = 0, \forall (w_1, w_2) \in E.$$

This means that

$$(\phi_1, \psi_1) \in (V(E))^\perp,$$

so that

$$V(E^\perp) \subset (V(E))^\perp.$$

It is easily verified that the implications from which the last inclusion results are in fact equivalences, so that

$$V(E^\perp) = (V(E))^\perp.$$

Suppose $(\phi, \eta) \in H \times H$. Thus $(\phi, \eta) \in V(\Gamma(A))^\perp$ if and only if

$$((\phi, \eta), (-A\psi, \psi))_{H\times H} = 0, \forall \psi \in D,$$

which holds if and only if

$$(\phi, A\psi)_H = (\eta, \psi)_H, \forall \psi \in D,$$

that is, if and only if

$$(\phi, \eta) \in \Gamma(A^*).$$

Thus

$$\Gamma(A^*) = V(\Gamma(A))^\perp.$$

Since $(V(\Gamma(A))^\perp$ is closed, A^* is closed.

2. Observe that $\Gamma(A)$ is a linear subset of $H \times H$ so that

$$\begin{aligned} \overline{\Gamma(A)} &= [\Gamma(A)^\perp]^\perp \\ &= V^2[\Gamma(A)^\perp]^\perp \\ &= [V[V(\Gamma(A))^\perp]]^\perp \\ &= [V(\Gamma(A^*)]^\perp \end{aligned} \tag{224}$$

so that from the proof of item 1, if A^* is densely defined we get

$$\overline{\Gamma(A)} = \Gamma[(A^*)^*].$$

Conversely, suppose $D(A^*)$ is not dense. Thus there exists $\psi \in [D(A^*)]^\perp$ such that $\psi \neq \theta$. Let $(\phi, A^*\phi) \in \Gamma(A^*)$. Hence

$$((\psi, \theta), (\phi, A^*\phi))_{H \times H} = (\psi, \phi)_H = 0,$$

so that

$$(\psi, \theta) \in [\Gamma(A^*)]^\perp.$$

Therefore $V[\Gamma(A^*)]^\perp$ is not the graph of a linear operator. Since $\overline{\Gamma(A)} = V[\Gamma(A^*)]^\perp$ A is not closable.

3. Observe that if A is closable then

$$A^* = \overline{(A^*)} = A^{***} = (\overline{A})^*.$$

□

38 Symmetric and self-adjoint operators

Definição 38.1. *Let $A : D \to H$ be a linear operator, where D is dense in H. A is said to be symmetric if $A \subset A^*$, that is if $D \subset D(A^*)$ and*

$$A^*\phi = A\phi, \forall \phi \in D.$$

Equivalently, A is symmetric if and only if

$$(A\phi, \psi)_H = (\phi, A\psi)_H, \forall \phi, \psi \in D.$$

Definição 38.2. *Let $A : D \to H$ be a linear operator. We say that A is self-adjoint if $A = A^*$, that is if A is symmetric and $D = D(A^*)$.*

Definição 38.3. *Let $A : D \to H$ be a symmetric operator. We say that A is essentially self-adjoint if its closure $\overline{A}$ is self-adjoint. If A is closed, a subset $E \subset D$ is said to be a core for A if $\overline{A|_E} = A$.*

Teorema 38.4. *Let $A : D \to H$ be a symmetric operator. Then the following statements are equivalent*

1. *A is self-adjoint.*
2. *A is closed and $N(A^* \pm iI) = \{\theta\}$.*
3. *$R(A \pm iI) = H$.*

Proof.

- 1 implies 2:

  Suppose A is self-adjoint let $\phi \in D = D(A^*)$ be such that

$$A\phi = i\phi$$

  so that

$$A^*\phi = i\phi.$$

  Observe that

$$\begin{aligned} -i(\phi,\phi)_H &= (i\phi,\phi)_H \\ &= (A\phi,\phi)_H \\ &= (\phi, A\phi)_H \\ &= (\phi, i\phi)_H \\ &= i(\phi,\phi)_H, \end{aligned} \tag{225}$$

  so that $(\phi,\phi)_H = 0$, that is $\phi = \theta$. Thus

$$N(A - iI) = \{\theta\}.$$

  Similarly we prove that $N(A + iI) = \{\theta\}$. Finally, since $\overline{A^*} = A^* = A$, we get that $A = A^*$ is closed.

- 2 implies 3:

  Suppose 2 holds. Thus the equation

$$A^*\phi = -i\phi$$

  has no non trivial solution. We will prove that $R(A - iI)$ is dense in H. If $\psi \in R(A - iI)^\perp$ then

$$((A - iI)\phi, \psi)_H = 0, \forall \phi \in D,$$

  so that $\psi \in D(A^*)$ and

$$(A - iI)^*\psi = (A^* + iI)\psi = \theta,$$

and hence by above $\psi = \theta$. Now we will prove that $R(A - iI)$ is closed and conclude that

$$R(A - iI) = H.$$

Given $\phi \in D$ we have

$$\|(A - iI)\phi\|_H^2 = \|A\phi\|_H^2 + \|\phi\|_H^2. \tag{226}$$

Let $\psi_0 \in H$ be a limit point of $R(A - iI)$. Thus we may find $\{\phi_n\} \subset D$ such that

$$(A - iI)\phi_n \to \psi_0.$$

From (226)

$$\|\phi_n - \phi_m\|_H \leq \|(A - iI)(\phi_n - \phi_m)\|_H, \forall m, n \in \mathbb{N}$$

so that $\{\phi_n\}$ is a Cauchy sequence, therefore converging to some $\phi_0 \in H$. Also from (226)

$$\|A\phi_n - A\phi_m\|_H \leq \|(A - iI)(\phi_n - \phi_m)\|_H, \forall m, n \in \mathbb{N}$$

so that $\{A\phi_n\}$ is a Cauchy sequence, hence also a converging one. Since A is closed, we get $\phi_0 \in D$ and

$$(A - iI)\phi_0 = \psi_0.$$

Therefore $R(A - iI)$ is closed, so that

$$R(A - iI) = H.$$

Similarly

$$R(A + iI) = H.$$

- 3 implies 1: Let $\phi \in D(A^*)$. Since $R(A - iI) = H$, there is an $\eta \in D$ such that

$$(A - iI)\eta = (A^* - iI)\phi,$$

and since $D \subset D(A^*)$ we obtain $\phi - \eta \in D(A^*)$, and

$$(A^* - iI)(\phi - \eta) = \theta.$$

Since $R(A + iI) = H$ we have $N(A^* - iI) = \{\theta\}$. Therefore $\phi = \eta$, so that $D(A^*) = D$. The proof is complete.

□

38.1 The spectral theorem using Cayley transform

In this section H is a complex Hilbert space. We suppose A is defined on a dense subspace of H, being A self-adjoint but possibly unbounded. We have shown that $(A + i)$ and $(A - i)$ are onto H and it is possible to prove that

$$U = (A - i)(A + i)^{-1},$$

exists on all H and it is unitary. Furthermore on the domain of A,

$$A = i(I + U)(I - U)^{-1}.$$

The operator U is called the Cayley transform of A. We have already proven that

$$U = \int_0^{2\pi} e^{i\phi} dF(\phi),$$

where $\{F(\phi)\}$ is a monotone family of orthogonal projections, strongly continuous from the right and we may consider it such that

$$F(\phi) = \begin{cases} 0, & \text{if } \phi \leq 0, \\ I, & \text{if } \phi \geq 2\pi. \end{cases} \tag{227}$$

Since $F(\phi) = 0$, for all $\phi \leq 0$ and

$$F(0) = F(0^+)$$

we obtain

$$F(0^+) = 0 = F(0^-),$$

that is, $F(\phi)$ is continuous at $\phi = 0$. We claim that F is continuous at $\phi = 2\pi$. Observe that $F(2\pi) = F(2\pi^+)$ so that we need only to show that

$$F(2\pi^-) = F(2\pi).$$

Suppose

$$F(2\pi) - F(2\pi^-) \neq \theta.$$

Thus there exists some $u, v \in H$ such that

$$(F(2\pi) - F(2(\pi^-)))u = v \neq \theta.$$

Therefore

$$F(\phi)v = F(\phi)[(F(2\pi) - F(2\pi^-))u],$$

so that

$$F(\phi)v = \begin{cases} 0, & \text{if } \phi < 2\pi, \\ v, & \text{if } \phi \geq 2\pi. \end{cases} \tag{228}$$

Observe that

$$U - I = \int_0^{2\pi} (e^{i\phi} - 1) dF(\phi),$$

and

$$U^* - I = \int_0^{2\pi} (e^{-i\phi} - 1) dF(\phi).$$

Let $\{\phi_n\}$ be a partition of $[0, 2\pi]$. From the monotonicity of $[0, 2\pi]$ and pairwise orthogonality of

$$\{F(\phi_n) - F(\phi_{n-1})\}$$

we can show that (this is not proved in details here)

$$(U^* - I)(U - I) = \int_0^{2\pi} (e^{-i\phi} - 1)(e^{i\phi} - 1)dF(\phi),$$

so that, given $z \in H$ we have

$$((U^* - I)(U - I)z, z)_H = \int_0^{2\pi} |e^{i\phi} - 1|^2 d\|F(\phi)z\|^2,$$

thus, for v defined above

$$\begin{aligned} \|(U - I)v\|^2 &= ((U - I)v, (U - I)v)_H \\ &= ((U - I)^*(U - I)v, v)_H \\ &= \int_0^{2\pi} |e^{i\phi} - 1|^2 d\|F(\phi)v\| \\ &= \int_0^{2\pi^-} |e^{i\phi} - 1|^2 d\|F(\phi)v\| \\ &= 0 \end{aligned} \tag{229}$$

The last two equalities results from $e^{2\pi i} - 1 = 0$ and $d\|F(\phi)v\| = \theta$ on $[0, 2\pi)$. Since $v \neq \theta$ the last equation implies that $1 \in P\sigma(U)$, which contradicts the existence of

$$(I - U)^{-1}.$$

Thus, F is continuous at $\phi = 2\pi$.

Now choose a sequence of real numbers $\{\phi_n\}$ such that $\phi_n \in (0, 2\pi)$, $n = 0, \pm 1, \pm 2, \pm 3, ...$ such that

$$-cot\left(\frac{\phi_n}{2}\right) = n.$$

Now define $T_n = F(\phi_n) - F(\phi_{n-1})$. Since U commutes with $F(\phi)$, U commutes with T_n. since

$$A = i(I + U)(I - U)^{-1},$$

this implies that the range of T_n is invariant under U and A. Observe that

$$\begin{aligned} \sum_n T_n &= \sum_n (F(\phi_n) - F(\phi_{n-1})) \\ &= \lim_{\phi \to 2\pi} F(\phi) - \lim_{\phi \to 0} F(\phi) \\ &= I - \theta = I. \end{aligned} \tag{230}$$

Hence

$$\sum_n R(T_n) = H.$$

Also, for $u \in H$ we have that

$$F(\phi)T_n u = \begin{cases} 0, \text{ if } \phi < \phi_{n-1}, \\ (F(\phi) - F(\phi_{n-1}))u, \text{ if } \phi_{n-1} \leq \phi \leq \phi_n, \\ F(\phi_n) - F(\phi_{n-1}))u, \text{ if } \phi > \phi_n, \end{cases} \tag{231}$$

so that

$$\begin{aligned} (I-U)T_n u &= \int_0^{2\pi} (1-e^{i\phi})dF(\phi)T_n u \\ &= \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})dF(\phi)u. \end{aligned} \tag{232}$$

Therefore

$$\begin{aligned} &\int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})^{-1}dF(\phi)(I-U)T_n u \\ &= \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})^{-1}dF(\phi) \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})dF(\phi)u \\ &= \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})^{-1}(1-e^{i\phi})dF(\phi)u \\ &= \int_{\phi_{n-1}}^{\phi_n} dF(\phi)u \\ &= \int_0^{2\pi} dF(\phi)T_n u = T_n u. \end{aligned} \tag{233}$$

Hence

$$\left[(I-U)|_{R(T_n)}\right]^{-1} = \int_{\phi_{n-1}}^{\phi_n} (1-e^{i\phi})^{-1}dF(\phi).$$

From this, from above and as

$$A = i(I+U)(I-U)^{-1}$$

we obtain

$$AT_n u = \int_{\phi_{n-1}}^{\phi_n} i(1+e^{i\phi})(1-e^{i\phi})^{-1}dF(\phi)u.$$

Therefore defining

$$\lambda = -cot\left(\frac{\phi}{2}\right),$$

and

$$E(\lambda) = F(-2cot^{-1}\lambda),$$

we get

$$i(1+e^{i\phi})(1-e^{i\phi})^{-1} = -cot\left(\frac{\phi}{2}\right) = \lambda.$$

Hence,

$$AT_n u = \int_{n-1}^{n} \lambda dE(\lambda)u.$$

Finally, from

$$u = \sum_{n=-\infty}^{\infty} T_n u,$$

we can obtain

$$\begin{aligned} Au &= A\left(\sum_{n=-\infty}^{\infty} T_n u\right) \\ &= \sum_{n=-\infty}^{\infty} AT_n u \\ &= \sum_{n=-\infty}^{\infty} \int_{n-1}^{n} \lambda dE(\lambda)u. \end{aligned} \tag{234}$$

Being the convergence in question in norm, we may write

$$Au = \int_{-\infty}^{\infty} \lambda dE(\lambda)u.$$

Since $u \in H$ is arbitrary, we may denote

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda). \tag{235}$$

□

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