Cálculo Avançado - Primeira Lista de Exercícios

Prof. Fabio Silva Botelho

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1. Assume $A \subset B \subset \mathbb{R}^n$. Show that $A' \subset B'$.
2. Let $A, B \subset \mathbb{R}^n$. Show that
$$A' \cup B' = (A \cup B)'.$$
3. Let $E \subset \mathbb{R}^n$. Show that E' is closed.
4. Let $B_1, B_2, ...$ be subsets of the metric space $\mathbb{R}^n$.
   (a) Show that if
   $$B_n = \cup_{i=1}^n B_i \text{ then } \overline{B}_n = \cup_{i=1}^n \overline{B}_i.$$
   (b) Show that if
   $$B = \cup_{i=1}^\infty B_i \text{ then } \overline{B} \supset \cup_{i=1}^\infty \overline{B}_i.$$
5. Let $E \subset \mathbb{R}^n$. Recall that the interior of E, denoted by E°, is defined as the set of all interior points of E.
   (a) Show that E° is open.
   (b) Show that E is open, if and only if, $E = E^\circ$.
   (c) Show that if $G \subset E$ and G is open, then $G \subset E^\circ$.
   (d) Prove que $(E^\circ)^c = \overline{E^c}$.
   (e) Do E and $\overline{E}$ have always the same interior? If not, present a counter example.
   (f) Do E and E^0 have always the same closure? If not, present a counter example.
6. Prove that $\mathbb{Q}$, the rational set, has empty interior.
7. Prove that $\mathbb{I}$, the set of irrationals, has empty interior.
8. Prove that given $x, y \in \mathbb{R}$ such that $x < y$, there exists $\alpha \in \mathbb{I}$, such that
$$x < \alpha < y.$$
9. Prove that $\mathbb{I}$ is dense in $\mathbb{R}$.
   Hint: Prove that
   $$x \in \mathbb{I}', \ \forall x \in \mathbb{R},$$
   where $\mathbb{I}'$ denotes the set of limit of points of $\mathbb{I}$.
10. Let $B \subset \mathbb{R}$ be an open set. Show that for all $x \in \mathbb{R}$ the set
$$x + B = \{x + y \mid y \in B\}$$
is open.

11. Let $A, B \subset \mathbb{R}$ be open sets. Show that the set

$$A + B = \{x + y \ : \ x \in A \text{ and } y \in B\},$$

is open.

12. Let $B \subset \mathbb{R}$ be an open set. Show that for all $x \in \mathbb{R}$ such that $x \neq 0$ the set

$$x \cdot B = \{x \cdot y \mid y \in B\}$$

is open.

13. Let $A, B \subset \mathbb{R}^n$, show that

   (a)
   $$(A \cap B)^\circ = A^\circ \cap B^\circ,$$

   (b)
   $$(A \cup B)^\circ \supset A^\circ \cup B^\circ,$$

   and give an example in which the inclusion is proper.

14. Let $A \subset \mathbb{R}^n$ be an open set and $a \in A$. Prove that $A \setminus \{a\}$ is open.

15. Let $A, B \subset \mathbb{R}^n$. Prove that:

   (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

   (b) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, and give an example for which the last inclusion is proper.

16. Show that a set A is dense in $\mathbb{R}$ if, and only if, X^c has empty interior.

17. Let $F \subset \mathbb{R}$ be a closed set and let $x \in F$. Show that x is an isolated point of F if, and only if, $F \setminus \{x\}$ is closed.

18. Show that if $A \subset \mathbb{R}$ is uncountable, then so is A'.

19. Show that if $A \subset \mathbb{R}$ then $\overline{A} \setminus A'$ is countable.

20. Let $A \subset \mathbb{R}^n$ be an open set. Assume $a_1, .., a_n \in A$.
    Prove that $A \setminus \{a_1, ..., a_n\}$ is open.

21. Let $A \subset \mathbb{R}^n$ be an open set and let $F \subset \mathbb{R}^n$ be a closed one.
    Show that $A \setminus F$ is open and $F \setminus A$ is closed.

22. Let $A \subset \mathbb{R}$ be an uncountable set. Prove that $A \cap A' \neq \emptyset$.