

# Cálculo Avançado - Segunda Lista de Exercícios

Prof. Fabio Silva Botelho

March 26, 2016

**Remark 0.1.** Para entregar dia 11/Abril: 4(b), 4(c), 5(e), 6(c), 6(d), 7(b), 9, 14, 15, 17.

1. Let  $A, B \subset \mathbb{R}^n$  be open sets.  
Prove that  $A \cup B$  and  $A \cap B$  are open.
2. Let  $A, B \subset \mathbb{R}^n$  be closed sets.  
Prove that  $A \cup B$  and  $A \cap B$  are closed.
3. Calculate the limits:

(a)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}},$$

(b)

$$\lim_{(x,y) \rightarrow (2,2)} \frac{x+y-4}{\sqrt{x+y}-2},$$

4. Prove formally that:

- (a)  $\lim_{(x,y) \rightarrow (1,3)} 3x - 5y + 7 = -5,$
- (b)  $\lim_{(x,y) \rightarrow (-2,1)} -x + 4y + 4 = 10,$
- (c)  $\lim_{(x,y) \rightarrow (1,3)} x^2 + y^2 - 2x + 1 = 9,$
- (d)  $\lim_{(x,y) \rightarrow (-1,2)} 3x^2 - 2y^2 - 2x + 3y + 5 = 8.$

5. For the functions below indicated, show that  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist, where

(a)

$$f(x, y) = \frac{x}{\sqrt{x^2 + y^2}},$$

(b)

$$f(x, y) = \frac{xy}{|xy|},$$

(c)

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2},$$

(d)

$$f(x, y) = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2 + y^2)^2},$$

(e)

$$f(x, y) = \frac{x^9 y}{(x^6 + y^2)^2}.$$

6. For the functions  $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$  below indicated, show that  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  exists and calculate its value, where,

(a)

$$f(x, y) = x \cos \left( \frac{1}{x^2 + y^2} \right),$$

(b)

$$f(x, y) = \frac{x^2 + 3xy}{\sqrt{x^2 + y^2}},$$

(c)

$$f(x, y) = \frac{x^2 y + xy^2}{x^2 + y^2},$$

(d)

$$f(x, y) = \cos \left( \frac{x^3 - y^3}{x^2 + y^2} \right).$$

7. For the functions  $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$  below indicated, calculate, if they exist, the limits  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  and discuss about the possibility or not of such functions to be continuously extended to  $(0, 0)$ , by appropriately defining  $f(0, 0)$ , where

(a)

$$f(x, y) = \ln \left( \frac{3x^4 - x^2 y^2 + 3y^4}{x^2 + y^2} + 2 \right),$$

(b)

$$f(x, y) = \ln \left( x^2 \cos^2 \left( \frac{1}{x^2 + y^2} \right) + 3 \right).$$

8. Let  $a, b, c \in \mathbb{R}$ , where  $a \neq 0$  or  $b \neq 0$ .

Prove formally that,

$$\lim_{(x,y) \rightarrow (x_0, y_0)} ax + by + c = ax_0 + by_0 + c.$$

9. Let  $a, b, c, d, e, f \in \mathbb{R}$  where  $a \neq 0$ ,  $b \neq 0$  or  $c \neq 0$ .

Prove formally that,

$$\lim_{(x,y) \rightarrow (x_0, y_0)} ax^2 + by^2 + cxy + dx + ey + f = ax_0^2 + by_0^2 + cx_0 y_0 + dx_0 + ey_0 + f.$$

10. Let  $D \subset \mathbb{R}^n$  be an open set, let  $f, g : D \rightarrow \mathbb{R}$  be real functions and let  $\mathbf{x}_0 \in D$ .

Suppose there exists  $K > 0$  and  $\delta > 0$  such that  $|g(\mathbf{x})| < K$ , if  $0 < |\mathbf{x} - \mathbf{x}_0| < \delta$ .

Assume

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = 0.$$

Under such hypotheses, prove that

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x})g(\mathbf{x}) = 0.$$

11. Use the item ?? to prove that

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0,$$

where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined by

$$f(x,y) = \begin{cases} (x^2 + y^2 + x - y) \sin\left(\frac{1}{x^2 + y^2}\right), & \text{if } (x,y) \neq (0,0) \\ 5, & \text{if } (x,y) = (0,0). \end{cases}$$

12. Let  $D \subset \mathbb{R}^n$  be an open set and let  $\mathbf{x}_0 \in D$ .

Assume  $f, g : D \rightarrow \mathbb{R}$  are such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = L \in \mathbb{R}$$

and

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} g(\mathbf{x}) = M \in \mathbb{R},$$

where  $L < M$ .

Prove that there exists  $\delta > 0$  such that if  $0 < |\mathbf{x} - \mathbf{x}_0| < \delta$ , then

$$f(\mathbf{x}) < \frac{L + M}{2} < g(\mathbf{x}).$$

Hint: Define  $\varepsilon = \frac{M-L}{2}$ .

13. Let  $D \subset \mathbb{R}^n$  be an open set and let  $f : D \rightarrow \mathbb{R}$  be a continuous function. Assume  $A \subset \mathbb{R}$  is open. Prove that  $f^{-1}(A)$  is open, where,

$$f^{-1}(A) = \{\mathbf{x} \in D : f(\mathbf{x}) \in A\}.$$

14. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function and let  $c \in \mathbb{R}$ .

Prove that the sets  $B$  and  $C$  are closed, where,

$$(a) \ B = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq c\}.$$

$$(b) \ C = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = c\}.$$

15. Let  $A, F \subset \mathbb{R}^n$  be such that  $A$  is open and  $F$  is closed. Prove that  $A \setminus F$  is open and  $F \setminus A$  is closed.

16. Let  $D \subset \mathbb{R}^n$  be an open set and let  $f : D \rightarrow \mathbb{R}$  be a continuous function.

Assume  $F \subset \mathbb{R}$  is closed. Prove that there exists a closed set  $F_1 \subset \mathbb{R}^n$  such that  $f^{-1}(F) = D \cap F_1$ , where

$$f^{-1}(F) = \{\mathbf{x} \in D : f(\mathbf{x}) \in F\}.$$

17. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be such that

$$f(x,y) = \begin{cases} \frac{\sin(x+y)}{x+y}, & \text{if } x+y \neq 0 \\ 1, & \text{if } x+y = 0. \end{cases}$$

Prove that  $f$  is continuous on  $\mathbb{R}^2$ .

18. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be such that

$$f(x,y) = \begin{cases} \frac{xy}{|x|+|y|}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that  $f$  is continuous on  $\mathbb{R}^2$ .