Cálculo Avançado - Terceira Lista de Exercícios

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Remark 0.1. *Para entregar dia 3/5/2016, os exercícios 2,3c,7,8,9c,12,16,20,21,23.*

1. Through the definition of partial derivative, for $(x,y) \in \mathbb{R}^2$ such that $3x+2y>0$, calculate
$$\frac{\partial f(x,y)}{\partial x} \text{ and } \frac{\partial f(x,y)}{\partial y},$$
where
$$f(x,y) = \frac{1}{\sqrt{3x+2y}}.$$

2. Through the definition of partial derivative, for $(x,y) \in \mathbb{R}^2$ such that $x^2-y \neq 0$, calculate
$$\frac{\partial f(x,y)}{\partial y},$$
where
$$f(x,y) = \frac{x+2y}{x^2-y}.$$

3. Through the definition of differentiability, prove that the functions below indicated are differentiable on the respective domains,
   (a) $f(x,y) = 3x^2 - 2xy + 5y^2$,
   (b) $f(x,y) = 2xy^2 - 3xy$,
   (c) $f(x,y) = \frac{x^2}{y}$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} \frac{(x^3+y^3)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
Calculate $f_x(0,0)$ e $f_y(0,0)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x,y) = \begin{cases} \frac{3x^2y^2}{x^4+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
Prove that $f_x(0,0)$ and $f_y(0,0)$ exist however f is not differentiable at $(0,0)$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
Prove that $f_x(0,0)$ and $f_y(0,0)$ exist and f is differentiable at $(0,0)$.

7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,y,z) = \begin{cases} \frac{xyz^2}{x^2+y^2+z^2}, & \text{if } (x,y,z) \neq (0,0,0) \\ 0, & \text{if } (x,y,z) = (0,0,0). \end{cases}$$

   Prove that f is differentiable at $(0,0,0)$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Obtain $\Delta f(0,0,\Delta x,\Delta y)$.
   (b) Calculate $f_x(0,0)$ e $f_y(0,0)$.
   (c) Through the definition of differentiability, show that f is differentiable at $(0,0)$.

9. For the functions below indicated, obtain the respective domains and prove that they are differentiable (on the domains in question):
   (a) $f(x,y) = \frac{x+y}{x^2+5y}$
   (b) $f(x,y) = y\ln x - x/y$,
   (c) $f(x,y) = \arctan(x^2-y) + \frac{1}{\sqrt{x^2-y}}$,

10. Let $D \subset \mathbb{R}^n$ be an open connected set. Suppose all partial derivatives of f are zero on D.

    Prove that f é constant on D.

11. Let $D \subset \mathbb{R}^2$ be an open rectangle and let $f : D \to \mathbb{R}$ be a function. Assume f has partial derivatives well defined on D. Let (x,y) and $(x+u,y+v) \in D$.

    Prove that there exists $\lambda \in (0,1)$ such that

$$f(x+u,y+v) - f(x,y) = f_x(x+\lambda u, y+v)u + f_y(x, y+\lambda v)v.$$

12. Let $D \subset \mathbb{R}^n$ be an open convex set and let $f : D \to \mathbb{R}$ be a function. Suppose there exists $K > 0$ such that

$$\left|\frac{\partial f(\mathbf{x})}{\partial x_j}\right| \leq K, \ \forall \mathbf{x} \in D, \ \ j \in \{1,..,n\}.$$

    Prove that

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq Kn|\mathbf{x}-\mathbf{y}|, \ \forall \mathbf{x},\mathbf{y} \in D.$$

13. Let $D \subset \mathbb{R}^n$ be an open set and let $f : D \to \mathbb{R}$ be a differentiable function at $\mathbf{x_0} \in D$. Prove that there exist $\delta > 0$ and $K > 0$ such that if $|\mathbf{h}| < \delta$, then $\mathbf{x_0} + \mathbf{h} \in D$ and

$$|f(\mathbf{x_0}+\mathbf{h}) - f(\mathbf{x_0})| < K|\mathbf{h}|.$$

14. Let $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be defined by $f(\mathbf{x}) = |\mathbf{x}|^c$, where $c \in \mathbb{R}$. Let $\mathbf{x} = (x_1,...,x_n)$ and $\mathbf{v} = (v_1,...,v_n) \in \mathbb{R}^n$.

    Calculate

$$\nabla f(\mathbf{x}) \cdot \mathbf{v}.$$

15. let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, t) = \frac{t^2 + y}{e^t + x^2 + t^2}.$$

Suppose the functions $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ be defined by

$$x(t) = \cos^2(t^3),$$

and

$$y(t) = e^{t^2}.$$

Through the chain rule, calculate $g'(t)$ where $g(t) = f(x(t), y(t), t), \ \ \forall t \in \mathbb{R}$.

Finally, obtain the equation of the tangent line to the graph of g at the points $t = 0$ and $t = \pi$.

16. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$g(x, y, z) = \frac{x^2 + y^2 + xy}{z^2 + e^x + cos^2(y)}.$$

Let $z(x, y) = cos^2(x^2 + y^2)$ and define $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x, y) = g(x, y, z(x, y)).$$

Through the chain rule, calculate $h_x(x, y)$ and $h_y(x, y)$.

Find the equation of the normal line and the equation of the tangent plane, to the graph of h at the point $(1, 0)$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $u(x, y) = bx - ay$. Show that $z(x, y) = f(u(x, y))$ satisfies the equation,

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 0.$$

18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function.

Denoting $u(r, \theta) = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$ show that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r},$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}.$$

19. Consider the ellipsoid of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where $a, b, c > 0$.

Find the closest points on such surface to the origin $(0, 0, 0)$.

20. Let A be a symmetric matrix $m \times n$. Let $\mathbf{y}_0 \in \mathbb{R}^m$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(\mathbf{x}) = \langle (A\mathbf{x}), \mathbf{y}_0 \rangle,$$

where $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ denotes the usual inner product in $\mathbb{R}^m$. Through the method of Lagrange multipliers, find the points of minimum and maximum of $f(\mathbf{x})$ subject to $|\mathbf{x}| = 1$.

21. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if

$$f(\lambda\mathbf{x} + (1 - \lambda\mathbf{y})) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}), \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \ \lambda \in [0, 1].$$

(a) Suppose that$f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Show that f is convex if, and only if,

$$f(\mathbf{y}) - f(\mathbf{x}) \geq \nabla f(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}), \ \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

(b) Prove that if f is convex, differentiable and $\nabla f(\mathbf{x}) = \mathbf{0}$ then $\mathbf{x} \in \mathbb{R}^n$ is a point of global minimum for f.

22. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function such that

$$H(\mathbf{x}) = \left\{ \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \right\}$$

is a positive definite matrix. $\forall \mathbf{x} \in \mathbb{R}^n$.

Show that f is convex on $\mathbb{R}^n$.

23. Let $F, G : \mathbb{R}^4 \to \mathbb{R}$ be defined by $F(x, y, u, v) = x^2 + y^3 - u + v^2$ and $G(x, y, u, v) = e^{2x} + e^{3y} + 2uv + 3v^2$. Assuming the hypotheses of the vectorial case of implicit function theorem, consider the functions $u(x, y)$ and $v(x, y)$ implicitly defined on a neighborhood of a point $(x, y, u, v) \in \mathbb{R}^4$ such that

$$F(x, y, u, v) = 0 \quad \text{and} \quad G(x, y, u, v) = 0.$$

Find u_x, u_y, v_x and v_y on such neighborhood.