

Cálculo Avançado - Sexta Lista de Exercícios

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1. Let $D \subset \mathbb{R}^3$ be an open non-empty set. Let $f : D \rightarrow \mathbb{R}$ be a continuous function.

Assume

$$\int_V f(\mathbf{x}) \, d\mathbf{x} = 0$$

for each compact set $V \subset D$ such that ∂V is of C^1 class.

Under such hypotheses, show that

$$f(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in D.$$

2. Let $D \subset \mathbb{R}^n$ be an open, bounded, non-empty set which the boundary ∂D is such that $m(\partial D) = 0$. Let $f : \overline{D} \rightarrow \mathbb{R}$ be a continuous function on D , such that

$$f(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in D.$$

Suppose

$$\int_{\overline{D}} f(\mathbf{x}) \, d\mathbf{x} = 0.$$

Under such hypotheses, show that

$$f(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in D.$$

3. Find the domain of the one variable vectorial functions \mathbf{r} below indicated.

(a)

$$\mathbf{r}(t) = \frac{1}{t^2 + 1} \mathbf{i} + \sqrt{(t-1)(t+3)} \mathbf{j},$$

(b)

$$\mathbf{r}(t) = \ln(t^2 - 16) \mathbf{i} + \sqrt{t^2 + 2t - 15} \mathbf{j} + \tan(t+1) \mathbf{k},$$

(c)

$$\sqrt{25 - t^2} \mathbf{i} + \sqrt{t^2 + 2t - 8} \mathbf{j}.$$

4. Through the formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

calculate the derivatives of the functions defined by the parametric equations indicated,

(a)

$$\mathbf{r}(t) = \frac{e^t}{1 + e^t} \mathbf{i} + t^2 \ln(t) \mathbf{j},$$

(b)

$$\mathbf{r}(t) = \frac{\cos(t)}{5 + \sin(t)} \mathbf{i} + \ln(\sqrt{t^4 + t^2}) \mathbf{j}.$$

5. Let $\mathbf{r} : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}^2$ be defined by

$$\mathbf{r}(t) = \frac{t}{t+1}\mathbf{i} + \ln(t^2 + 1)\mathbf{j}.$$

Find the equation of the tangent line to the graph of the curve defined by \mathbf{r} at the point corresponding to $t = 1$.

6. Let $\mathbf{r}, \mathbf{s} : \mathbb{R} \rightarrow \mathbb{R}^2$ be defined by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

and

$$\mathbf{s}(t) = (t^2 + t)\mathbf{i} + t^3\mathbf{j}.$$

Calculate the angle between $\mathbf{r}'(t)$ and $\mathbf{s}'(t)$ at the point corresponding to $t = 1$.

7. Let $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3$ be defined by

$$\mathbf{r}(t) = \frac{2t}{1+t^2}\mathbf{i} + \frac{1-t^2}{1+t^2}\mathbf{j} + \mathbf{k}.$$

Show that the angle between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ is constant.

8. Let $\mathbf{s} : [a, b] \rightarrow \mathbb{R}^3$ be a three times differentiable function.

Let $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ be defined by

$$\mathbf{r}(t) = \mathbf{s}(t) \times \mathbf{s}'(t).$$

Find

$$\mathbf{r}''(t)$$

on $[a, b]$.

9. Let $\mathbf{s} : [a, b] \rightarrow \mathbb{R}^3$ be a three times differentiable function.

Let $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ be defined by

$$\mathbf{r}(t) = \mathbf{s}(t) \cdot (\mathbf{s}'(t) \times \mathbf{s}''(t)).$$

Find

$$\mathbf{r}'(t)$$

on $[a, b]$.

10. A vectorial function \mathbf{r} satisfies the equation,

$$t\mathbf{r}'(t) = \mathbf{r}(t) + t\mathbf{A}, \quad \forall t > 0$$

where

$$\mathbf{A} \in \mathbb{R}^3.$$

Suppose that $\mathbf{r}(1) = 2\mathbf{A}$. Calculate $\mathbf{r}''(1)$ and $\mathbf{r}(3)$ as functions of \mathbf{A} .

11. Find a function $\mathbf{r} : (0, +\infty) \rightarrow \mathbb{R}^3$ such that

$$\mathbf{r}(x) = xe^x\mathbf{A} + \frac{1}{x} \int_1^x \mathbf{r}(t) dt.$$

where $\mathbf{A} \in \mathbb{R}^3$, $\mathbf{A} \neq \mathbf{0}$.

12. Calculate $I = \int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$ and where C is the curve defined by $\mathbf{r}(t) = a \cos(t)\mathbf{i} + b \sin(t)\mathbf{j}$, $0 \leq t \leq \pi/2$, and where $a, b \neq 0$.

13. Calculate $I = \int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = y^2\mathbf{i} + x\mathbf{j}$ and where C is the curve defined by $\mathbf{r}(t) = a \cos(t)\mathbf{i} + b \sin(t)\mathbf{j}$, and where $0 \leq t \leq \pi/2$.

14. Through the Green Theorem, calculate the areas of the regions D , where,

(a)

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } y \geq 1/2\}.$$

(b)

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } -1/2 \leq y \leq \sqrt{3}/2\}.$$

(c)

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } 0 \leq x \leq 1/2\}.$$

15. Calculate the area of surface S , where

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } \frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2} \right\}.$$

16. Calculate the area of surface S , where

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } \frac{-\sqrt{3}}{2} \leq z \leq \frac{1}{2} \right\}.$$

17. Calculate the area of surface S , where

$$S = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2 \text{ and } x^2 + y^2 \leq 2ax\},$$

where $a \in \mathbb{R}$.

18. Calculate $I = \int \int_S x \, dS$, where

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = R^2 \text{ and } |z| \leq 1\}.$$

19. Through the Divergence Theorem, calculate $I = \int \int_S (y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n} \, dS$, where

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : x = \sqrt{R^2 - y^2 - z^2} \text{ and } x \geq \frac{\sqrt{3}R}{2} \right\},$$

where $R > 0$.

20. Through the Divergence Theorem, calculate $I = \int \int_S \mathbf{F} \cdot \mathbf{n} \, dS$ where

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2R_0x \text{ and } z \geq 0\}$$

and where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ e $R_0 > 0$.

21. Let $u : V \rightarrow \mathbb{R}$ be a scalar field and let $\mathbf{F} : V \rightarrow \mathbb{R}^3$ be a vectorial one, where $V \subset \mathbb{R}^3$ is open u, \mathbf{F} are of C^1 class. Show that

$$\text{div}(u\mathbf{F}) = (\nabla u) \cdot \mathbf{F} + u (\text{div}\mathbf{F}).$$

22. Let $u, v : V \rightarrow \mathbb{R}$ be C^2 class scalar fields, where $V \subset \mathbb{R}^3$ is open and its closure is simple. Defining

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

show that $\text{div}(\nabla u) = \nabla^2 u$ and prove the Green identities,

(a)

$$\int \int \int_V (v \nabla^2 u + \nabla v \cdot \nabla u) dV = \int \int_S v (\nabla u \cdot \mathbf{n}) dS$$

where $S = \partial V$ (that is, S is the boundary of V .)

(b)

$$\int \int \int_V (v \nabla^2 u - u \nabla^2 v) dV = \int \int_S \left(v \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial v}{\partial \mathbf{n}} \right) dS,$$

where $S = \partial V$ and $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$.

23. Let $u : V \rightarrow \mathbb{R}$, $\mathbf{F} : V \rightarrow \mathbb{R}^3$ be C^2 class fields on the open set $V \subset \mathbb{R}^3$.

Prove that $\text{curl}(\nabla u) = \mathbf{0}$ and $\text{div}(\text{curl}(\mathbf{F})) = 0$, on V .

24. Let $M \subset \mathbb{R}^n$ be a 3-dimensional C^1 class manifold, where $n \geq 4$,

$$M = \{\mathbf{r}(\mathbf{u}) = X_i(\mathbf{u})\mathbf{e}_i : \mathbf{u} \in D\},$$

$D \subset \mathbb{R}^3$ and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis for \mathbb{R}^n ,

Let $\omega = dX_1 \wedge dX_4 \wedge dX_3$ be a 3-form on M , where,

$$dX_1(\mathbf{u}) = \frac{\partial X_1(\mathbf{u})}{\partial u_1} du_1 + \frac{\partial X_1(\mathbf{u})}{\partial u_2} du_2 + \frac{\partial X_1(\mathbf{u})}{\partial u_3} du_3,$$

$$dX_4(\mathbf{u}) = \frac{\partial X_4(\mathbf{u})}{\partial u_1} du_1 + \frac{\partial X_4(\mathbf{u})}{\partial u_2} du_2 + \frac{\partial X_4(\mathbf{u})}{\partial u_3} du_3,$$

and

$$dX_3(\mathbf{u}) = \frac{\partial X_3(\mathbf{u})}{\partial u_1} du_1 + \frac{\partial X_3(\mathbf{u})}{\partial u_2} du_2 + \frac{\partial X_3(\mathbf{u})}{\partial u_3} du_3.$$

Compute

$$(dX_1(\mathbf{u}) \wedge dX_4(\mathbf{u}) \wedge dX_3(\mathbf{u}))(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3),$$

where

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1} \Delta u_1,$$

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_2} \Delta u_2$$

and

$$\mathbf{s}_3 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_3} \Delta u_3.$$

25. Consider the vectorial field $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ where $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$.

Through the Stokes Theorem, calculate

$$I = \int \int_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} dS$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = 8 - x^2 - 2y^2 \text{ and } z \geq 2\}.$$

26. Consider the vectorial field $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + (z - x^2)\mathbf{k}$.

Through the Stokes Theorem, calculate

$$I = \int \int_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} dS$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = 8 - x^2 - 2y^2 \text{ and } 2 \leq z \leq 4\}.$$