Notas de Aulas - Tópicos em Superfícies no $\mathbb{R}^n$

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Resumo

A primeira parte destas notas são um resumo de alguns tópicos do livro do Prof. Elon Lages Lima, Variedades Diferenciáveis, publicado pelo Impa em 2011. A segunda parte é baseado no capítulo 10 do livro Real Analysis and Applications, de F.S. Botelho.

1 Definições iniciais

Começamos com algumas definições iniciais.

Definição 1.1. *Sejam $U \subset \mathbb{R}^m$ e $V \subset \mathbb{R}^n$. Um homeomorfismo entre U e V é uma bijeção contínua $\mathbf{f} : U \to V$ cuja inversa também é contínua. Um difeomorfismo $\mathbf{h} : U \to V$ é uma bijeção diferenciável cuja inversa $\mathbf{h}^{-1} : V \to U$ também é diferenciável.*

Definição 1.2 (Imersão)**.** *Seja $U \subset \mathbb{R}^m$ um conjunto aberto. Dizemos que uma aplicação diferenciável $\mathbf{f} : U \to \mathbb{R}^n$ é uma imersão, quando para cada $\mathbf{u} \in U$, $\mathbf{f}'(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^n$ é uma transformação linear injetiva (observe que nesse caso necessariamente $m \leq n$).*

Definição 1.3 (Submersão)**.** *Seja $U \subset \mathbb{R}^m$ um conjunto aberto. Dizemos que uma aplicação diferenciável $\mathbf{f} : U \to \mathbb{R}^n$ é uma submersão, quando para cada $\mathbf{u} \in U$, $\mathbf{f}'(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^n$ é uma transformação linear sobrejetiva (observe que nesse caso necessariamente $m \geq n$).*

2 A forma local das submersões

Teorema 2.1. *Seja $D \subset \mathbb{R}^{n+m}$ um conjunto aberto e não-vazio e seja $\mathbf{f} : D \to \mathbb{R}^m$ uma função vetorial de classe C^1.*

Denotemos $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+m}$ onde $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ e $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$.

Suponha que $(\mathbf{x}_0, \mathbf{y}_0) \in D$ seja tal que

$$\det[\mathbf{f}_y(\mathbf{x}_0, \mathbf{y}_0)] \neq 0.$$

Sob tais hipóteses, existem conjuntos abertos V, Z, W *tais que*

$$(\mathbf{x}_0, \mathbf{y}_0) \in Z \subset \mathbb{R}^{n+m},\ \mathbf{x}_0 \in V \subset \mathbb{R}^n \ e\ \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) \in W \subset \mathbb{R}^m,$$

e também uma função $h : V \times W \to Z$ *de classe* C^1 *tal que*

$$\mathbf{f}(h(\mathbf{x}, \mathbf{w})) = \mathbf{w},\ \forall (\mathbf{x}, \mathbf{w}) \in V \times W.$$

Demonstração. Defina $\mathbf{c} = \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)$. Sem perda de generalidade assuma $\mathbf{c} = \mathbf{0}$.

Defina $\varphi : D \to \mathbb{R}^n \times \mathbb{R}^m$ por

$$\varphi(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{f}(\mathbf{x}, \mathbf{y})).$$

Observe que

$$\varphi'(\mathbf{x}_0, \mathbf{y}_0) = \begin{bmatrix} I_{n\times n} & 0_{n\times m} \\ \mathbf{f}_x(\mathbf{x}_0, \mathbf{y}_0) & \mathbf{f}_y(\mathbf{x}_0, \mathbf{y}_0) \end{bmatrix} \tag{1}$$

onde $I_{n\times n}$ denota a matriz identidade $n \times n$ e $0_{n\times m}$ denota a matriz com entradas todas zeros $n \times m$.

Observe também que

$$\det(\varphi'(\mathbf{x}_0, \mathbf{y}_0)) = \det \mathbf{f}_y(\mathbf{x}_0, \mathbf{y}_0) \neq 0.$$

Do Teorema da função inversa temos que $h = \varphi^{-1}$ existe e é de classe C^1 num aberto $V_1 \times W$ tal que

$$\varphi(\mathbf{x}_0, \mathbf{y}_0) = (\mathbf{x}_0, \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)) \in V_1 \times W.$$

Também do mesmo teorema existe um aberto $U_1 \times U_2$ tal que $\varphi(U_1 \times U_2) = V_1 \times W$ e onde $\mathbf{x}_0 \in U_1$ e $\mathbf{y}_0 \in U_2$.

Defina $V = U_1 \cap V_1$ e $Z = \varphi^{-1}(V \times W)$.

Observe que φ^{-1} é de classe C^1, também do teorema da função inversa e além disso, como φ deixa a primeira coordenada fixa, φ^{-1} também a deixará, de modo que existe uma função h_2 tal que

$$h(\mathbf{x}, \mathbf{y}) = \varphi^{-1}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, h_2(\mathbf{x}, \mathbf{y})),$$

Logo para $(\mathbf{x}, \mathbf{w}) \in V \times W$ temos que

$$\begin{aligned} (\mathbf{x}, \mathbf{w}) &= \varphi[\varphi^{-1}(\mathbf{x}, \mathbf{w})] \\ &= \varphi[h(\mathbf{x}, \mathbf{w})] \\ &= \varphi(\mathbf{x}, h_2(\mathbf{x}, \mathbf{w})) \\ &= (\mathbf{x}, \mathbf{f}(\mathbf{x}, h_2(\mathbf{x}, \mathbf{w}))) \\ &= (\mathbf{x}, \mathbf{f}(h(\mathbf{x}, \mathbf{w}))), \end{aligned} \tag{2}$$

ou seja

$$(\mathbf{x}, \mathbf{w}) = (\mathbf{x}, \mathbf{f}(h(\mathbf{x}, \mathbf{w}))),$$

isto é

$$\mathbf{w} = \mathbf{f}(h(\mathbf{x}, \mathbf{w})),\ \forall (\mathbf{x}, \mathbf{w}) \in V \times W.$$

A prova está completa. □

3 Forma local das imersões

Teorema 3.1. *Seja $D \subset \mathbb{R}^n$ um conjunto aberto e não-vazio. Seja $\mathbf{f} : D \to \mathbb{R}^{n+m}$ uma função de classe C^1 tal que*

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_{n+m}(\mathbf{x}) \end{bmatrix}. \tag{3}$$

Seja $\mathbf{x}_0 \in D$. Defina o conjunto A por

$$A = \left\{ \sum_{k=1}^{n} \langle \mathbf{f}'(\mathbf{x}_0)\mathbf{v}, e_k \rangle e_k \ : \ \mathbf{v} \in \mathbb{R}^n \right\},$$

onde $\{e_1, e_2, \ldots, e_{n+m}\}$ é a base canônica do $\mathbb{R}^{n+m}$.

Asuma que existem $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^{n+m}$ linearmente independentes tais que

$$\mathbb{R}^{n+m} = A \oplus F,$$

onde F é o subespaço gerado por $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$.

Sob tais hipóteses, existem conjuntos abertos V, Z, W tais que

$$(\mathbf{x}_0, \mathbf{0}) \in V \times W \subset \mathbb{R}^n \times \mathbb{R}^m,$$

$\mathbf{f}(\mathbf{x}_0) \in Z \subset \mathbb{R}^{n+m}$ e existe uma função $h : Z \to V \times W$ de classe C^1 tal que

$$h(\mathbf{f}(\mathbf{x})) = (\mathbf{x}, \mathbf{0}), \ \forall \mathbf{x} \in V.$$

Demonstração. Defina $\varphi : D \times \mathbb{R}^m \to \mathbb{R}^{n+m}$, para $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$ por

$$\varphi(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} f_1(\mathbf{x}) + \sum_{i=1}^m y_i \langle \mathbf{v}_i, e_1 \rangle \\ \vdots \\ f_{n+m}(\mathbf{x}) + \sum_{i=1}^m y_i \langle \mathbf{v}_i, e_{n+m} \rangle \end{bmatrix} \tag{4}$$

Assim,

$$\varphi'(\mathbf{x}_0, \mathbf{0})$$
$$= \begin{bmatrix} (f_1)_{x_1}(\mathbf{x}_0) & (f_1)_{x_2}(\mathbf{x}_0) & \cdots & (f_1)_{x_n}(\mathbf{x}_0) & \langle \mathbf{v}_1, e_1 \rangle & \cdots & \langle \mathbf{v}_m, e_1 \rangle \\ (f_2)_{x_1}(\mathbf{x}_0) & (f_2)_{x_2}(\mathbf{x}_0) & \cdots & (f_2)_{x_n}(\mathbf{x}_0) & \langle \mathbf{v}_1, e_2 \rangle & \cdots & \langle \mathbf{v}_m, e_2 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (f_{n+m})_{x_1}(\mathbf{x}_0) & (f_{n+m})_{x_2}(\mathbf{x}_0) & \cdots & (f_{n+m})_{x_n}(\mathbf{x}_0) & \langle \mathbf{v}_1, e_{n+m} \rangle & \cdots & \langle \mathbf{v}_m, e_{n+m} \rangle \end{bmatrix}$$

Da hipótese $A \oplus F = \mathbb{R}^{n+m}$, logo

$$\varphi'(\mathbf{x}_0, \mathbf{0})[\mathbb{R}^{n+m}] = \mathbb{R}^{n+m}.$$

Assim, $\det(\varphi'(\mathbf{x}_0, \mathbf{0})) \neq 0$ e portanto do teorema da função inversa existem abertos V, W, Z tais que $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ e $Z \subset \mathbb{R}^{n+m}$ e tais que

$$h \equiv \varphi^{-1} : Z \to V \times W$$

é de classe C^1 em Z, $\mathbf{f}(\mathbf{x}_0) \in Z$, $\mathbf{x}_0 \in V$ e $\mathbf{0} \in W$.

Além disso,

$$\varphi(V \times W) = Z.$$

Observe que $\varphi(\mathbf{x}, \mathbf{0}) = \mathbf{f}(\mathbf{x})$.

Logo

$$h(\mathbf{f}(\mathbf{x})) = \varphi^{-1}(\mathbf{f}(\mathbf{x})) = (\mathbf{x}, \mathbf{0}),$$

ou seja

$$h(\mathbf{f}(\mathbf{x})) = (\mathbf{x}, \mathbf{0}), \; \forall \mathbf{x} \in V.$$

A prova está completa. □

Observação 3.2. *Seja* $\pi : V \times W \to V$ *onde*

$$\pi(\mathbf{x}, \mathbf{w}) = \mathbf{x}.$$

Considerando o contexto do último teorema, seja também

$$\xi = \pi \circ h : Z \to V.$$

Assim,

$$(\xi \circ f)(\mathbf{x}) = \pi \circ h \circ f(\mathbf{x}) = \pi(\mathbf{x}, \mathbf{0}) = \mathbf{x}.$$

Logo, $\xi|_{f(V)} = (f|_V)^{-1}$. *Resumindo,* $\xi : Z \to V$ *de classe* C^k, *quando restrita à* $\mathbf{f}(V)$, *corresponde à inversa de*

$$\mathbf{f} : V \to \mathbf{f}(V).$$

4 Parametrizações e Superfícies no $\mathbb{R}^n$

Definição 4.1. *Seja* $U_0 \subset \mathbb{R}^m$ *um conjunto aberto. Uma imersão de classe* C^k $\mathbf{r} : U_0 \to \mathbb{R}^n$ *(a qual é também um homeomorfismo sobre* $\mathbf{r}(U_0)$*) é dita ser uma parametrização de classe* C^k *de*

$$U \equiv \mathbf{r}(U_0).$$

Observação 4.2. *Quanto á injetividade de* $\mathbf{r}'(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^n$, *temos que as seguintes condições são equivalentes.*

1. $\mathbf{r}'(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^n$ *é injetiva.*
2. $$\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1}, \ldots, \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_m}$$
   são vetores linearmente independentes.
3. *A matriz Jacobiana*
   $$\left\{ \frac{\partial r_i(\mathbf{u})}{\partial u_j} \right\}$$
   tem posto m, *isto é, algum dos seus sub-determinantes* $m \times m$ *é diferente de zero.*

Definição 4.3. *Seja* $1 \leq m \leq n$. *Dizemos que um conjunto não-vazio* $M \subset \mathbb{R}^n$ *é uma superfície* m*-dimensional de classe* C^k *quando para cada* $p \in M$, *existe um aberto* U *aberto em* M *tal que* $p \in U$ *e existe uma parametrização* $\mathbf{r} : U_0 \to U = \mathbf{r}(U_0)$, *para algum aberto* $U_0 \subset \mathbb{R}^m$.

O número $n - m$ *é chamado de co-dimensão de* M.

4.1 Mudança de coordenadas

Seja $M \subset \mathbb{R}^n$ uma superfície m-dimensional onde $(1 \leq m \leq n)$ de classe C^k e seja $\mathbf{r} : U_0 \to U$ uma parametrização do conjunto aberto em M, $U \subset M$. Seja $V_0 \subset \mathbb{R}^m$ um aberto e seja $\xi : V_0 \to U_0$ um difeomorfismo de classe C^k.

Assim $\mathbf{r} \circ \xi : V_0 \to U$ ainda é uma parametrização de U.

Observe que ξ representa uma mudança de coordenadas.

Observação 4.4. *Se* $\mathbf{r} : U_0 \to U$ *e* $\mathbf{s} : V_0 \to V$ *são parametrizações de* M *tais que* $U \cap V \neq \emptyset$,, *então*

$$\xi = \mathbf{s}^{-1} \circ \mathbf{r} : \mathbf{r}^{-1}(U \cap V) \to \mathbf{s}^{-1}(U \cap V)$$

é um homeomorfismo entre abertos do $\mathbb{R}^m$.

De fato, vejamos o próximo teorema.

Teorema 4.5. *Seja* V_0 *um subconjunto aberto do* $\mathbb{R}^m$ *e seja* $\mathbf{s} : V_0 \to \mathbb{R}^n$ *uma parametrização de classe* C^k *do conjunto* $V \subset \mathbb{R}^n$. *Dados* $U_0 \subset \mathbb{R}^r$ *aberto e* $\mathbf{f} : U_0 \to V \subset \mathbb{R}^n$ *de classe* C^k, *então*

1. *a composta* $\mathbf{s}^{-1} \circ \mathbf{f} : U_0 \to V_0 \subset \mathbb{R}^m$ *é de classe* C^k.

2. *Para* $\mathbf{x} \in U_0$ *e*

$$\mathbf{z} = (\mathbf{s}^{-1} \circ \mathbf{f})(\mathbf{x})$$

temos que

$$(\mathbf{s}^{-1} \circ \mathbf{f})'(\mathbf{x}) = [\mathbf{s}'(\mathbf{z})]^{-1} \circ \mathbf{f}'(\mathbf{x}).$$

Demonstração. 1. Como $\mathbf{s} : V_0 \to V$ é uma imersão injetora de classe C^k, para cada ponto $p \in V$, existe um aberto $Z \subset \mathbb{R}^n$ que o contém e uma função de classe C^k, $\mathbf{g} : Z \to \mathbb{R}^m$ (isto decorre da observação 3.2) tal que

$$\mathbf{g}|(V \cap Z) = \mathbf{s}^{-1}.$$

Seja $p \in \mathbf{f}(U_0) \subset V$. Assim

$$\mathbf{s}^{-1} \circ \mathbf{f} = \mathbf{g} \circ \mathbf{f} : \mathbf{f}^{-1}(V \cap Z) \subset \mathbb{R}^r \to \mathbb{R}^m$$

de modo que $\mathbf{s}^{-1} \circ \mathbf{f}$ é de classe C^k, pois $\mathbf{f}$ e $\mathbf{g}$ o são.

2. Escreva $h = \mathbf{s}^{-1} \circ \mathbf{f}$ e aplique a regra da cadeia a igualdade

$$\mathbf{s} \circ h = \mathbf{f}.$$

□

Corolário 4.6. *Sejam* U_0 *e* $V_0 \subset \mathbb{R}^m$ *e sejam* $\mathbf{r} : U_0 \to V$ *e* $\mathbf{s} : V_0 \to V$, *parametrizações de classe* C^k *do mesmo conjunto* $V \subset \mathbb{R}^n$ *Sob tais hipóteses, a mudança de coordenadas*

$$\xi = \mathbf{s}^{-1} \circ \mathbf{r}$$

é um difeomorfismo de classe C^k.

4.2 Funções diferenciáveis definidas em superfícies

Definição 4.7. *Seja $M \subset \mathbb{R}^n$ uma superfície de classe C^k. Dizemos que uma função $\mathbf{f} : M \to \mathbb{R}^s$ é diferenciável em um ponto $p \in M$ quando existe uma parametrização $\mathbf{r} : U_0 \to U$ de classe C^k com $p \in U$, tal que*

$$\mathbf{f} \circ \mathbf{r} : U_0 \to \mathbb{R}^s$$

é diferenciável em $\mathbf{u}_0 \in U_0$, onde $\mathbf{r}(\mathbf{u}_0) = p$.

Da última proposição e corolário, temos que

$$\mathbf{f} \circ \mathbf{s} = (\mathbf{f} \circ \mathbf{r}) \circ (\mathbf{r}^{-1} \circ \mathbf{s})$$

será também diferenciável em $\mathbf{u}_0 = \mathbf{r}^{-1}(p)$, seja qual for a parametrização $\mathbf{s}$ de classe C^k de uma vizinhança de p.

Concluímos então que a definição em questão não depende da parametrização escolhida.

Se tivermos superfícies $M \subset \mathbb{R}^r$ e $N \subset \mathbb{R}^s$ demensões m_1 e m_2 respectivamente, dizemos que $\mathbf{f} : M \to N$ é diferenciável no ponto $p \in M$ quando considerada como uma função de M em $\mathbb{R}^s$, $\mathbf{f}$ for diferenciável nesse ponto.

Similarmente, dizemos que $\mathbf{f} : M \to N$ é de classe C^k quando para cada $p \in M$ existe uma parametrização $\mathbf{r} : U_0 \to U \subset M$ de classe C^k, com $p \in U$ tal que

$$\mathbf{f} \circ \mathbf{r} : U_0 \to N \subset \mathbb{R}^s$$

é de classe C^k.

Observe, que nesse caso, da última proposição e seu corolário, $\mathbf{f} \circ \mathbf{r}$ é de classe C^k seja qual for a parametrização de classe C^k $\mathbf{r} : U_0 \to U$, tal que $p \in U$.

Vejamos então o próximo teorema.

Teorema 4.8. *No contexto das últimas observações acima, para que $\mathbf{f} : M \to N$ seja de classe C^k é necessário e suficiente que, para cada $p \in M$, existam parametrizações de classe C^k*

$$\mathbf{s} : V_0 \to V \subset N$$

e

$$\mathbf{r} : U_0 \to U \subset M$$

com $p \in U$, $\mathbf{f}(U) \subset V$ e tais que

$$\mathbf{s}^{-1} \circ \mathbf{f} \circ \mathbf{r} : U_0 \to V_0 \subset \mathbb{R}^{m_2}$$

seja de classe C^k

Demonstração. Seja $\mathbf{f} : M \to N$ de classe C^k. Seja $p \in M$. Assim existe uma parametrização $s : V_0 \to V \subset N$ de classe C^k com $\mathbf{f}(p) \in V$ e $V_0 \subset \mathbb{R}^{m_2}$.

Sendo $\mathbf{f}$ contínua, existe um aberto U_1 em M, tal que $p \in U_1$ e $\mathbf{f}(U_1) \subset V$. Podemos obter uma parametrização $\mathbf{r} : \hat{U}_0 \to \mathbf{r}(\hat{U}_0) \subset M$ com $p \in \mathbf{r}(\hat{U}_0)$, onde $\hat{U}_0 \subset \mathbb{R}^{m_1}$ é aberto. Defina

$$U_0 = \mathbf{r}^{-1}(U_1 \cap \mathbf{r}(\hat{U}_0))$$

e $U = U_1 \cap \mathbf{r}(\hat{U}_0)$.

Assim

$$\mathbf{f}(U) \subset \mathbf{f}(U_1) \subset V.$$

Pela definição, de $\mathbf{f}$ ser de classe C^k temos que

$$\mathbf{f} \circ \mathbf{r} : U_0 \to V \subset \mathbb{R}^s$$

é de classe C^k de modo que do último teorema e seu corolário

$$\mathbf{s}^{-1} \circ \mathbf{f} \circ \mathbf{r} : U_0 \to V_0$$

é de classe C^k.

A prova da recíproca é deixada como exercício. □

Corolário 4.9. *Sejam $M, N, P \subset \mathbb{R}^n$ superfícies de classe C^k de dimensões m_1, m_2 e m_3 respectivamente. Sejam $\mathbf{f} : M \to N$ e $\mathbf{g} : N \to P$ funções de classe C^k.*

Sob tais hipóteses $\mathbf{g} \circ \mathbf{f} : M \to P$ é também de classe C^k.

A prova deste corolário é deixada como exercício.

5 Superfícies orientáveis

Definição 5.1 (Atlas)**.** *Um atlas de classe C^k de uma superfície m-dimensional $M \subset \mathbb{R}^n$ é uma coleção $\mathcal{P}$ de parametrizações $\mathbf{r} : U_0 \to U \subset M$ de classe C^k tal que os conjuntos U formam uma cobertura de M.*

Duas parametrizações de classe C^k $\mathbf{r} : U_0 \to U$ e $\mathbf{s} : V_0 \to V$ são ditas coerentes se $U \cap V = \emptyset$ ou, se $U \cap V \neq \emptyset$, então $\xi = \mathbf{r}^{-1} \circ \mathbf{s}$ tem determinante Jacobiano positivo em todos os pontos de $\mathbf{s}^{-1}(U \cap V)$.

Um atlas é dito ser coerente quando todos os pares de parametrizações $\mathbf{r}, \mathbf{s}$ são coerentes.

Se M admite um atlas $\mathcal{P}$ coerente, é dita ser orientável e também é dita ser positivamente orientada por $\mathcal{P}$.

Teorema 5.2. *Seja $M \subset \mathbb{R}^n$ uma superfície de classe C^k.*

Se existem $n-m$ campos contínuos de vetores $\mathbf{n}_1, \ldots, \mathbf{n}_{n-m} : M \to \mathbb{R}^n$ tais que $\mathbf{n}_1(p), \ldots, \mathbf{n}_{n-m}(p) \in (T_pM)^\perp$ são linearmente independentes, $\forall p \in M$, então M é orientável.

Demonstração. Seja $\mathcal{P}$ o conjunto das parametrizações de classe C^k, $\mathbf{r} : U_0 \to U \subset M$ tais que

1. U_0 é conexo.
2. Para cada $\mathbf{u} \in U_0$, $A(\mathbf{r}(\mathbf{u}))$ a matriz $n \times n$ cujas colunas são

$$\mathbf{r}'(\mathbf{u})e_1, \ldots, \mathbf{r}'(\mathbf{u})e_m, \mathbf{n}_1(\mathbf{r}(\mathbf{u})), \ldots, \mathbf{n}_{n-m}(\mathbf{r}(\mathbf{u})),$$

   tem determinante positivo. Aqui $\{e_1, \ldots, e_m\}$ é a base canônica do $\mathbb{R}^m$.

Vamos mostrar que $\mathcal{P}$ é um atlas coerente em M.

Seja $p \in M$. Considere uma parametrização de classe C^k $\mathbf{r} : U_0 \to U \subset M$ com $p \in U$ e U_0 conexo.

Então por continuidade ou $\det A(\mathbf{r}(\mathbf{u})) > 0,\ \forall \mathbf{u} \in U_0$ e nesse caso $\mathbf{r} \in \mathcal{P}$, ou então $\det A(\mathbf{r}(\mathbf{u})) < 0,\ \forall \mathbf{u} \in U_0$ e nesse caso basta substituir $\mathbf{r}$ por $\mathbf{r}_1$, onde $\mathbf{r}_1(u_1, u_2, \ldots, u_m) = \mathbf{r}(-u_1, u_2, \ldots u_m)$ que obteremos $\mathbf{r}_1 \in \mathcal{P}$.

Como $p \in M$ é arbitrário, mostramos assim que as imagens das parametrizações de $\mathcal{P}$ cobrem M.

Sejam $\mathbf{r} : U_0 \to U$ e $\mathbf{s} : V_0 \to V$ elementos de $\mathcal{P}$ tais que $U \cap V \neq \emptyset$.

Temos que mostar que

$$\mathbf{r}^{-1} \circ \mathbf{s} : \mathbf{s}^{-1}(U \cap V) \to \mathbf{r}^{-1}(U \cap V)$$

tem determinante Jacobiano positivo em cada ponto

$$\mathbf{u} \in \mathbf{s}^{-1}(U \cap V).$$

Seja

$$p = \mathbf{r}(\mathbf{u}_1) = \mathbf{s}(\mathbf{u}_2) \in U \cap V \subset M.$$

Observe que

$$\mathbf{s}'(\mathbf{u}_2)e_j = \sum_{i=1}^{m} \alpha_j^i \mathbf{r}'(\mathbf{u}_1)e_i,\ \forall j \in \{1, \ldots, m\}.$$

Disto obtemos,

$$\det A(\mathbf{s}(\mathbf{u}_2)) = \det\{\alpha_j^i\} \det A(\mathbf{r}(\mathbf{u}_1)),$$

de modo que

$$\det\{\alpha_j^i\} > 0.$$

Observe que a matriz Jacobiana de $\mathbf{r}^{-1} \circ \mathbf{s}$ em $\mathbf{u}_2$ é exatamente $\{\alpha_j^i\}$.

A prova está completa.

□

Teorema 5.3. *Seja $M \subset \mathbb{R}^n$ uma superfície m-dimensional de classe C^k, onde $1 \leq m \leq n$. Seja $p \in M$.*

Os elementos de T_pM são os vetores velocidade em p dos caminhos diferenciáveis contidos em M e que passam por p.

Mais precisamente,

$$\begin{aligned} T_pM \quad = \quad & \{\mathbf{v} = \lambda'(0) \ : \ \lambda : (-\varepsilon, \varepsilon) \to M \subset \mathbb{R}^n \\ & \textit{é diferenciável e } \lambda(0) = p\}. \end{aligned} \qquad (5)$$

Demonstração. Seja $\mathbf{v} \in T_pM$. Assim existe $\mathbf{w} \in \mathbb{R}^m$ tal que

$$\mathbf{v} = \mathbf{r}'(\mathbf{u})\mathbf{w},$$

onde $p = \mathbf{r}(\mathbf{u})$, para uma paremetrização apropriada

$$\mathbf{r} : U_0 \to U.$$

Portanto

$$\mathbf{v} = \lim_{t \to 0} \frac{\mathbf{r}(\mathbf{u} + t\mathbf{w}) - \mathbf{r}(\mathbf{u})}{t}.$$

Para $\varepsilon > 0$ suficientemente pequeno, defina $\lambda : (-\varepsilon, \varepsilon) \to M$ por

$$\lambda(t) = \mathbf{r}(\mathbf{u} + t\mathbf{w}).$$

Logo,

$$\mathbf{v} = \lambda'(0)$$

e

$$\lambda(0) = \mathbf{r}(\mathbf{u}) = p.$$

Reciprocamente, suponha que $\lambda : (-\varepsilon, \varepsilon) \to M$ seja um caminho diferenciável com $\lambda(0) = p$. Seja

$$\mathbf{v} = \lambda'(0).$$

Observe que existe uma parametrização $\mathbf{r} : U_0 \to U$, onde $p \in \mathbf{r}(\mathbf{u}) \in U$, para $\mathbf{u} \in U_0$, de modo que para $0 < \varepsilon_1 < \varepsilon$ suficientemente pequeno,

$$\lambda(-\varepsilon_1, \varepsilon_1) \subset U.$$

Observe que

$$\mathbf{r}^{-1} \circ \lambda : (-\varepsilon_1, \varepsilon_1) \to U_0$$

é diferenciável.

Defina

$$\mathbf{w} = (\mathbf{r}^{-1} \circ \lambda)'(0),$$

assim,

$$\mathbf{w} = [\mathbf{r}'(\mathbf{u})]^{-1} \lambda'(0),$$

de modo que

$$\mathbf{v} = \lambda'(0) = \mathbf{r}'(\mathbf{u})\mathbf{w} \in T_p M.$$

A prova está completa. □

Proposição 5.4. *Seja $U \subset \mathbb{R}^{m+n}$ um conjunto aberto e seja $\mathbf{f} : U \to \mathbb{R}^n$ uma aplicação de classe C^k.*

Seja $\mathbf{c} \in \mathbb{R}^n$.

Defina

$$M = \{p \in U : \mathbf{f}(p) = \mathbf{c} \text{ e } \mathbf{f}'(p) : \mathbb{R}^{m+n} \to \mathbb{R}^n \text{ é sobrejetora } \}.$$

Sob tais hipóteses, se $M \neq \emptyset$, então M é uma superfície de classe C^k.

Além disso,

$$T_p(M) = Ker \mathbf{f}'(p), \ \forall p \in M.$$

Demonstração. Denotemos $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n}$ e seja

$$p = (\mathbf{x}_0, \mathbf{y}_0) \in M.$$

Logo

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{c}.$$

Renomeando as variáveis, se necessário, obtemos $\det[\mathbf{f}_y(\mathbf{x_0},\mathbf{y}_0)] \neq 0$, e assim, do Teorema da função implícita existem $\delta_1 > 0$ e $\delta_2 > 0$ tais que para cada $\mathbf{x} \in B_{\delta_1}(\mathbf{x}_0)$ existe um único $\mathbf{y} \in B_{\delta_2}(\mathbf{y}_0)$ tal que

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{c},$$

onde denotamos

$$\mathbf{y} = \xi(\mathbf{x}),$$

onde tal função é de classe C^k, de modo que

$$\mathbf{f}(\mathbf{x},\xi(\mathbf{x})) = \mathbf{c},\ \forall \mathbf{x} \in B_{\delta_1}(\mathbf{x}_0).$$

Logo a vizinhaça $Z = B_{\delta_1}(\mathbf{x}_0) \times B_{\delta_2}(\mathbf{y}_0)$ de p é tal que a parametrização $\mathbf{s} : V = B_{\delta_1}(\mathbf{x}_0) \to Z \cap f^{-1}(c)$, onde $\mathbf{s}(\mathbf{x}) = (\mathbf{x},\xi(\mathbf{x}))$, é bijetiva.

Resumindo $p \in Z \cap f^{-1}(c) \subset M$.

Sendo tal p arbitrário, segue-se que M é uma superfície.

Seja agora $\mathbf{v} \in T_pM$. Seja $\lambda : (-\varepsilon,\varepsilon) \to M$ tal que $\lambda(0) = p$ e $\lambda'(0) = \mathbf{v}$.

Assim,

$$\begin{aligned} \mathbf{0} &= (\mathbf{f} \circ \lambda)'(0) \\ &= \mathbf{f}'(\lambda(0))\lambda'(0) \\ &= \mathbf{f}'(p)\mathbf{v} \end{aligned} \qquad (6)$$

Logo $\mathbf{v} \in Ker\mathbf{f}'(p)$.

Como T_pM e $Ker\mathbf{f}'(p)$ são dois subespaços de dimensão m de $\mathbb{R}^{m+n}$ e do exposto acima $T_pM \subset Ker\mathbf{f}'(p)$, obtemos que

$$T_pM = Ker\mathbf{f}'(p).$$

A prova está completa.

□

6 Superfícies no $\mathbb{R}^n$ com bordo

Definição 6.1 (Semi-espaço)**.** *Considere a função linear* $\alpha : \mathbb{R}^m \to \mathbb{R}$, *onde*

$$\alpha(\mathbf{u}) = \sum_{k=1}^{m} a_k u_k,\ \forall \mathbf{u} \in \mathbb{R}^m,$$

e onde $a_k \in \mathbb{R}, \forall k \in \{1,\ldots,m\}$.

Definimos o semi-espaço $H \subset \mathbb{R}^m$ *por*

$$H = \{\mathbf{u} \in \mathbb{R}^m \ :\ \alpha(\mathbf{u}) \leq 0\}.$$

Observe que ne nesse caso, a fronteira de H, *denotada por* ∂H, *será*

$$\partial H = \{\mathbf{u} \in \mathbb{R}^m \ :\ \alpha(\mathbf{u}) = 0\}.$$

Observação 6.2. *Seja $A \subset H \subset \mathbb{R}^m$ um conjunto aberto em H. Seja $\mathbf{f} : A \to \mathbb{R}^n$ uma função diferenciável.*

Mostraremos que $f'(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^n$ está bem definida, $\forall \mathbf{u} \in A$.

Seja $F : U \to \mathbb{R}^n$, uma extensão diferenciável de F onde $U \supset A$ é aberto.

Seja $\mathbf{u} \in A$. Se $\mathbf{u} \in H^\circ$, obviamente,

$$F'(\mathbf{u}) = \mathbf{f}'(\mathbf{u}).$$

Assuma então que $\mathbf{u} \in A \cap \partial H$. Seja $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset H$ uma base do $\mathbb{R}^m$.

De fato, tal base existe, pois dada uma base qualquer do $\mathbb{R}^m$, trocando o sinal de cada elemento se necessário, podemos obter que cada elemento pertence a H, (mesmo trocando alguns sinais ainda teremos uma base).

Seja $t \geq 0$. Assim

$$\mathbf{u} + t\mathbf{v}_k \in H, \forall k \in \{1, \ldots, m\},$$

pois

$$\alpha(\mathbf{u} + t\mathbf{v}_k) = \alpha(\mathbf{u}) + t\alpha(\mathbf{v}_k) \leq 0, \ \forall k \in \{1, \ldots, m\}.$$

Como A é aberto em H e $\mathbf{u} \in A \cap \partial H$, para todo $t \geq 0$ suficientemente pequeno, temos que

$$\mathbf{u} + t\mathbf{v}_k \in A.$$

Seja, $k \in \{1, \ldots, m\}$.

Assim

$$\begin{aligned} F'(\mathbf{u})\mathbf{v}_k &= \lim_{t \to 0} \frac{F(\mathbf{u} + t\mathbf{v}_k) - F(\mathbf{u})}{t} \\ &= \lim_{t \to 0^+} \frac{F(\mathbf{u} + t\mathbf{v}_k) - F(\mathbf{u})}{t} \\ &= \lim_{t \to 0^+} \frac{\mathbf{f}(\mathbf{u} + t\mathbf{v}_k) - \mathbf{f}(\mathbf{u})}{t} \\ &= \mathbf{f}'(\mathbf{u})\mathbf{v}_k. \end{aligned} \tag{7}$$

Logo

$$F'(\mathbf{u})\mathbf{v}_k = \mathbf{f}'(\mathbf{u})\mathbf{v}_k, \ \forall k \in \{1, \ldots, m\}.$$

Disto podemos concluir que $F'(\mathbf{u})$ não depende da extensão escolhida.

Vale também a regra da cadeia, isto é, se $\mathbf{f} : A \to \mathbb{R}^n$ e $\mathbf{g} : B \to \mathbb{R}^p$ diferenciáveis, onde $A \subset H$ é aberto e H e $\mathbf{f}(A) \subset B \subset H_1$, onde H é semi-espaço do $\mathbb{R}^m$ e H_1 é semi-espaço do $\mathbb{R}^n$, então $(\mathbf{g} \circ \mathbf{f}) : A \to \mathbb{R}^p$ é diferenciável e

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{u}) = \mathbf{g}'(\mathbf{f}(\mathbf{u}))\mathbf{f}'(\mathbf{u}), \ \forall \mathbf{u} \in A.$$

Observação 6.3. *Considere novamente um semi-espaço $H \subset \mathbb{R}^m$ e seja $A \subset H$ um conjunto aberto em H. Definiremos o bordo de A em H, denotado por ∂A, como*

$$\partial A = A \cap \partial H.$$

Observe que o bordo ∂A é uma hiperfície em $\mathbb{R}^m$.

De fato, sendo A aberto em H, temos que $A = U \cap H$, para algum aberto $U \subset \mathbb{R}^m$.

Portanto,

$$
\begin{aligned}
U \cap \partial H &= U \cap (H \cap \partial H) \\
&= (U \cap H) \cap \partial H \\
&= A \cap \partial H \\
&= \partial A, \qquad (8)
\end{aligned}
$$

de modo que ∂A é um subconjunto aberto na hiperfície

$$\partial H = \alpha^{-1}(0).$$

Teorema 6.4. *Seja $A \subset H$ um conjunto aberto em H e seja $B \subset H_1$ um conjunto aberto em H_1, onde H, H_1 são semi-espaços em $\mathbb{R}^m$*

Seja $\mathbf{f} : A \to B$ um difeomorfismo de calsse C^1.

Sob tais hipóteses

$$\mathbf{f}(\partial A) = \partial B.$$

Em particular a restrição $\mathbf{f}|_{\partial A}$ é um difeomorfismo entre ∂A e ∂B.

Demonstração. Seja $\mathbf{u} \in A \cap H^0 \equiv U$. Assim $U \subset \mathbb{R}^m$ é um aberto tal que $\mathbf{u} \in U \subset A \subset H$.

Restrito a U, $\mathbf{f}$ é um difeomorfismo de classe C^1 sobre $\mathbf{f}(U)$. Pelo Teorema da Função Inversa $\mathbf{f}(U)$ é aberto no $\mathbb{R}^m$. Como $\mathbf{f}(U) \subset B \subset H_1$ concluímos que $\mathbf{f}(U) \subset B \cap H_1^\circ$. Assim

$$\mathbf{f}(A \cap H^\circ) \subset B \cap H_1^\circ.$$

Portanto

$$\mathbf{f}^{-1}(\partial B) \subset \partial A.$$

Similarmente, invertendo os papéis de $\mathbf{f}$, $\mathbf{f}^{-1}$ e, A e B, podemos obter

$$\mathbf{f}(\partial A) \subset \partial B,$$

de modo que

$$\mathbf{f}(\partial A) = \partial B.$$

□

6.1 Parametrizações para superfícies no $\mathbb{R}^n$ com bordo

Definição 6.5. *Uma parametrização (de classe C^k e dimensão m) de um conjunto $U \subset \mathbb{R}^n$ é um homeomorfisno $\mathbf{r} : U_0 \to U$ de classe C^k definido no aberto $U_0 \subset H$ em H, onde H é um semi-espaço, tal que $\mathbf{r}'(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^n$ é uma transformação linrear injetiva para cada $\mathbf{u} \in U_0$.*

Definição 6.6. *No contexto da última definição, um conjunto $M \subset \mathbb{R}^n$ é dito ser uma superfície de dimensão m e classe C^k, com bordo, quando para cada $p \in M$, existe uma parametrização $\mathbf{r} : U_0 \to U \subset M$, com $p = \mathbf{r}(\mathbf{u})$, para algum $\mathbf{u} \in U_0 \subset H$, onde U_0 é aberto em H e $H \subset \mathbb{R}^m$ é um semi-espaço.*

Teorema 6.7. *Seja $M \subset \mathbb{R}^n$ uma superfície m-dimensional de classe C^k com bordo. Sejam $\mathbf{r} : U_0 \to U$ e $\mathbf{s} : V_0 \to V$ parametrizações de classe C^k de abertos (em M) $U, V \subset M$, com $U \cap V \neq \emptyset$. Sob tais hipóteses,*

$$\mathbf{s}^{-1} \circ \mathbf{r} : \mathbf{r}^{-1}(U \cap V) \to \mathbf{s}^{-1}(U \cap V)$$

é um difeomorfismo de classe C_k

Demonstração. Seja $\mathbf{u} \in \mathbf{r}^{-1}(U \cap V)$.

Seja $p = \mathbf{r}(\mathbf{u})$ e seja $\mathbf{v} = \mathbf{s}^{-1}(\mathbf{r}(\mathbf{u}))$.

Observe que do exposto anteriormente, $\mathbf{s}$ poder ser estendido a uma aplicação $\mathbf{s}_1 : W \to \mathbb{R}^n$ de classe C^k num aberto $W \subset \mathbb{R}^m$ tal que $\mathbf{v} \in W$. Como $\mathbf{s}_1'(\mathbf{v}) : \mathbb{R}^m \to \mathbb{R}^n$ é injetiva, temos que, da forma local das imersões, restringindo-se W se necessário, $\mathbf{s}_1$ é um homeomorfismo de W sobre sua imagem $\mathbf{s}_1(W)$, de modo que o homeomorfismo inverso $\mathbf{s}_1^{-1} : \mathbf{s}_1(W) \to W$ é a restrição de uma aplicação $\mathbf{g}$ de classe C^k definida num aberto do $\mathbb{R}^n$.

Definindo $A = \mathbf{r}^{-1}(\mathbf{s}_1(W))$ temos que $\mathbf{u} \in A \cap H$.

Observe que

$$(\mathbf{s}^{-1} \circ \mathbf{r})|_{A \cap H} = (\mathbf{g} \circ \mathbf{r})|_{A \cap H}$$

onde $\mathbf{g} \circ \mathbf{r}$ é de classe C^k.

Em particular $(\mathbf{s}^{-1} \circ \mathbf{r})$ é de classe C^k em $\mathbf{u}$, $\forall \mathbf{u} \in \mathbf{r}^{-1}(U \cap V)$.

A prova está completa. □

Observação 6.8. *Para um a superfície $M \subset \mathbb{R}^n$ com bordo, o bordo de M é o conjunto dos pontos $p \in M$ tais que toda parametrização $\mathbf{r} : U_0 \to U$ de classe C^1 de um aberto em M, $U \subset M$ com $p = \mathbf{r}(\mathbf{u})$, necessariamente temos que $\mathbf{u} \in \partial U_0$.*

Pelo Teorema 6.4 juntamente com o fato de que cada mudança de parametrização é um difeomorfismo, para $p \in M$, basta que exista uma parametrização $\mathbf{r} : U_0 \to U$ de classe C^1 de aberto U em M, com $p = \mathbf{r}(\mathbf{u})$ e $\mathbf{u} \in \partial U_0$, para que se tenha $p \in \partial M$.

Observação 6.9. *Se $M \subset \mathbb{R}^n$ é uma superfície com bordo de classe C^k e dimensão $m+1$, seu bordo ∂M é uma superfície (sem bordo) de classe C^k e dimensão m (isto pois $\partial H \subset \mathbb{R}^{m+1}$ é um subespaço de dimensão m).*

As parametrizações que definem ∂M como superfície são as restrições ao bordo $\partial U_0 = U_0 \cap \partial H$, das parametrizações $\mathbf{r} : U_0 \to U$ de classe C^k, que tem como imagem, um aberto U em M, tal que $U \cap \partial M \neq \emptyset$. Observe que a restrição $\mathbf{r}|_{\partial U_0} : \partial U_0 \to \partial U$ tem $\partial U = U \cap \partial M$ como imagem. Podemos parametrizar ∂U da seguinte forma.

Escreva $\mathbf{u} = (u_0, u_1, \ldots, u_m) \in H$, e defina

$$H_0 = \{\mathbf{u} = (u_0, u_1, \ldots, u_m) \in \mathbb{R}^{m+1} \ : \ u_0 \leq 0\}.$$

Assim identificamos ∂H_0 com $\mathbb{R}^m$, onde

$$\partial H_0 = \{(0, u_1, \ldots, u_m) \ : \ (u_1, \ldots, u_m) \in \mathbb{R}^m\}.$$

Observe que para todo semi-espaço $H \subset \mathbb{R}^{m+1}$ existe um isomorfismo linear $T : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ tal que $T(H_0) = H$.

Então para cada parametrização $\mathbf{s} : V_0 \to U$ de calsse C^k no aberto V_0 em H, definimos, $U_0 = T^{-1}(V_0)$ e obtemos $\mathbf{r} = \mathbf{s} \circ T : U_0 \to U$, uma parametrização padronizada definida num aberto em H_0, de classe C^k e com a mesma imagem de $\mathbf{s}$.

Se $\mathbf{r} : U_0 \to U$ *é padronizada e* $U \cap \partial M \neq \emptyset$*, a restrição* $\mathbf{r}|_{\partial U_0} : \partial U_0 \to \partial U$ *na superfície* ∂M *está definida em um subconjunto aberto* $\partial U_0 \subset \mathbb{R}^m$*, considerando que identificamos* ∂H_0 *com o* $\mathbb{R}^m$

Definição 6.10 (Espaço tangente)**.** *Seja* $M \subset \mathbb{R}^m$ *uma superfície com bordo de classe* C^1 *e dimensão* $m+1$*. Para cada ponto* $p \in M$ *podemos definir o espaço tangente de dimensão* $m+1$*, denotado por* $T_pM \subset \mathbb{R}^n$*, por*

$$T_pM = \mathbf{r}'(\mathbf{u})[\mathbb{R}^{m+1}],$$

onde $\mathbf{r}(\mathbf{u}) = p$ *e* $\mathbf{r} : U_0 \to U$ *é qualquer parametrização de classe* C^1 *de um aberto em M,* $U \subset M$ *e* $U_0 \subset \mathbb{R}^{m+1}$ *é um conjunto aberto em semi-espaço* $H \subset \mathbb{R}^{m+1}$*.*

Observação 6.11. *Seja* $p \in \partial M$*.*

Assim, para uma paremetrização $\mathbf{r} : U_0 \to U \subset M$*, com* $p = \mathbf{r}(\mathbf{u})$*, temos que* $\mathbf{u} \in \partial U_0$*, de modo que* $\mathbf{r}'(\mathbf{u})[\partial U_0] = T_p(\partial M)$ *é o subespaço tangente ao bordo de M em* $p \in \partial M$*.*

Observe que

$$T_p(\partial M) \subset T_pM,$$

onde $T_p(\partial M)$ *é um subespaço de* $T_p(M)$ *de dimensão m (co-dimensão 1).*

Proposição 6.12. *Seja* $M \subset \mathbb{R}^n$ *uma superficie m-dimensional com bordo. A definição de* T_pM *não depende da parametrização uilizada.*

Demonstração. De fato sejam $\mathbf{r} : U_0 \to U$ e $\mathbf{r} : V_0 \to V$ parametrizações em M onde $p = \mathbf{r}(\mathbf{u}) = \mathbf{s}(\mathbf{v}) \in U \cap V$.

Observe que

$$\xi = (\mathbf{s}^{-1} \circ \mathbf{r}) : \mathbf{r}^{-1}(U \cap V) \to \mathbf{s}^{-1}(U \cap V)$$

é um difeoformismo, onde

$$\mathbf{s} \circ \xi = \mathbf{r}.$$

assim

$$\mathbf{s}'(\mathbf{v})\xi'(\mathbf{u}) = \mathbf{r}'(\mathbf{u}).$$

Como $\xi' : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ é um isomorfismo, temos que

$$\mathbf{r}'(\mathbf{u})[\mathbb{R}^{m+1}] = \mathbf{s}'(\mathbf{v})[\mathbb{R}^{m+1}].$$

A prova está completa. □

Observação 6.13. *Seja* $p \in \partial M$*. Temos então o espaço tangente* T_pM *e o seu subespaço tangente* $T_p(\partial M)$*.*

Além disso, em T_pM *temos o semi-espaço composto pelos vetores de* T_pM *que apontam para fora de M e pelos vetores de* $T_p(\partial M)$*.*

Vejamos com mais rigor essa ideia intuitiva.

Definição 6.14. *Dizemos que um vetor* $\mathbf{w} \in \mathbb{R}^m$ *aponta para fora do semi-espaço H, quando* $\mathbf{w} \notin H$*.*

Nesse caso, se

$$H = \{\mathbf{u} \in \mathbb{R}^m \ : \ \alpha(\mathbf{u}) \leq 0\},$$

teremos $\alpha(\mathbf{w}) > 0$*.*

Vejamos também o próximo teorema.

Teorema 6.15. *Seja* $\mathbf{f} : A \to B$ *um difeomorfismo entre* $A \subset H$ *e* $B \subset H_1$*, onde* A *é aberto em* H *e* B *é aberto em* H_1 *e,* $H, H_1 \subset \mathbb{R}^m$ *são semi-espaços.*

Suponha que $\mathbf{w}$ *aponta para fora de* H*. Sob tais hipóteses, para cada* $\mathbf{u} \in \partial A$ *temos que* $\mathbf{f}'(\mathbf{u})\mathbf{w}$ *aponta para fora de* H_1*.*

Demonstração. Observe que

$$H = \{\mathbf{u} \in \mathbb{R}^m \ : \ \alpha(\mathbf{u}) \leq 0\},$$

e

$$H_1 = \{\mathbf{u} \in \mathbb{R}^m \ : \ \beta(\mathbf{u}) \leq 0\},$$

onde $\alpha, \beta : \mathbb{R}^m \to \mathbb{R}$ são funções lineares apropriadas.

Seja $\mathbf{u} \in \partial A = A \cap \partial H$. Pelo Teorema 6.4, $\mathbf{f}$ é um difeomorfismo entre ∂A e ∂B, de modo que $\mathbf{f}'(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^m$ transforma isomorficamente ∂H em ∂H_1.

Assim, dado $\mathbf{v} \in \mathbb{R}^m$, temos que

$$\beta[\mathbf{f}'(\mathbf{u})\mathbf{v}] = 0,$$

se e somente se,

$$\alpha(\mathbf{v}) = 0.$$

Como $\alpha(\mathbf{w}) > 0$, basta então mostar que

$$\beta[\mathbf{f}'(\mathbf{u})\mathbf{w}] \geq 0.$$

Observe que para $t < 0$ temos que $\mathbf{u} + t\mathbf{w} \in H$, pois $\alpha(\mathbf{u} + t\mathbf{w}) = 0 + t\alpha(\mathbf{w}) < 0$.

Logo, para $t < 0$ suficientemente pequeno em módulo, temos que

$$\mathbf{u} + t\mathbf{w} \in A \setminus \partial A,$$

de modo que

$$\mathbf{f}(\mathbf{u} + t\mathbf{w}) \in B \setminus \partial B$$

e portanto

$$\beta(\mathbf{f}(\mathbf{u} + t\mathbf{w})) < 0.$$

Assim para tais valores de t, obtemos

$$\frac{\beta(\mathbf{f}(\mathbf{u} + t\mathbf{w})) - \beta(\mathbf{f}(\mathbf{u}))}{t} = \frac{\beta(\mathbf{f}(\mathbf{u} + t\mathbf{w}))}{t} > 0.$$

Fazendo $t \to 0^-$, obtemos

$$\beta(\mathbf{f}'(\mathbf{u})\mathbf{w}) \geq 0.$$

A prova está completa. □

Definição 6.16. *Seja* $M \subset \mathbb{R}^n$ *uma superfície* $m+1$*-dimensional de classe* C^1 *com bordo e seja* $p \in \partial M$*.*

Dizemos que um vetor $\mathbf{w} \in T_pM$ *aponta para fora da superfície* M*, quando existe uma parametrização* $\mathbf{r} : U_0 \to U$ *de classe* C^1 *de um aberto* U_0 *em* H*, onde* $H \subset \mathbb{R}^{m+1}$ *é um semi-espaço,* $U \subset M$ *é aberto em* M*, e onde* $p = \mathbf{r}(\mathbf{u}) \in U$*,* $\mathbf{u} \in \partial U_0$ *e* $\mathbf{w} = \mathbf{r}'(\mathbf{u})\mathbf{w}_0$*, para algum* $\mathbf{w}_0 \in \mathbb{R}^{n+1}$ *o qual aponta para fora de* H*.*

Observe que nesse caso, do último teorema, para qualquer outra parametrização $\mathbf{s} : V_0 \to V$, com $p = \mathbf{s}(\mathbf{v}) \in V$, $\mathbf{v} \in \partial V_0$, temos que

$$\mathbf{w} = \mathbf{s}'(\mathbf{v})\mathbf{w}_1,$$

para algum $\mathbf{w}_1 \in \mathbb{R}^{m+1}$, o qual aponta para fora do semi-espaço H_1, tal que $\partial V_0 = V_0 \cap \partial H_1$ e no qual V_0 é aberto.

De fato, definindo

$$\xi = \mathbf{s}^{-1}\mathbf{r},$$

e

$$\mathbf{w}_1 = \xi'(\mathbf{u})\mathbf{w}_0$$

temos que

$$\begin{aligned} \mathbf{s}'(\mathbf{v})\mathbf{w}_1 &= \mathbf{s}'(\mathbf{v})\xi'(\mathbf{u})\mathbf{w}_0 \\ &= \mathbf{r}'(\mathbf{u})\mathbf{w}_0 \\ &= \mathbf{w}. \end{aligned} \tag{9}$$

Pelo último teorema, $\mathbf{w}_1$ aponta para fora de H_1, pois $\mathbf{w}_0$ aponta para fora de H.

Observação 6.17. *Em cada ponto $p \in \partial M$, os vetores tangentes a ∂M e os vetores que apontam para fora de M, formam um semi-espaço de T_pM. Como $dim(T_pM) = m+1$, temos que $dim(T_p(\partial M)) = m$, de modo que existe um único vetor $\mathbf{n}(p)$ o qual é ortogonal a $T_p(\partial M)$, está em $T_p(M)$ e aponta para fora de M. O vetor $\mathbf{n}(p)$ é dito ser o vetor normal exterior a ∂M em $p \in \partial M$. (normal exterior ao bordo de M em p).*

Seja $p \in \partial M$, onde $p = \mathbf{r}(\mathbf{u})$ e $\mathbf{u} = (0, u_1, \ldots, u_m) \in \partial U_0$ (para uma parametrização apropriada padronizada).

Observe que $\mathbf{n}(p) \in T_pM$ tem a forma

$$\mathbf{n}(p) = a_0\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_0} + a_1\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1} + \ldots + a_m\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_m},$$

onde a base de $T_p(\partial M)$ é

$$\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1}, \ldots, \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_m}.$$

Deve-se ter então,

$$\mathbf{n}(p) \cdot \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_k} = 0,\ \forall k \in \{1, \ldots, m\}. \tag{10}$$

Temos então m equações lineares e $m+1$ incógnitas, $a_0, a_1, \ldots, a_{m+1}$.

Obtemos então mediante (10) $a_1(a_0), \ldots, a_m(a_0)$, onde tais relações são lineares.

Mediante a condição $\mathbf{n}(p) \cdot \mathbf{n}(p) = 1$, obtemos a_0^2, ou seja $|a_0|$.

Finalmente, o sinal de a_0 deve ser obtido de modo que $\mathbf{n}(p)$ aponte para fora de M.

Mostraremos agora, de uma maneira mais formal, que se M é orientada, então seu bordo também o é.

Seja $M \subset \mathbb{R}^n$ uma superfície $m+1$-dimensional de classe C^1, orientada e com bordo.

Denotemos por $\mathcal{P}$ o conjunto das parametrizações $\mathbf{r} : U_0 \to U \subset M$, de classe C^1, com as seguintes propriedades.

1. U_0 é conexo.

2. U_0 é aberto no semi-espaço

$$H = \{(u_0, u_1, \ldots, u_m) \in \mathbb{R}^{m+1} \ : \ u_0 \leq 0\}.$$

3. $\mathbf{r}$ é positiva em relação à orientação de M.

Já vimos que o conjunto das parametrizações padronizadas $\mathbf{r} : U_0 \to U \subset M$ que satisfazem 1 e 2 (padronizadas) formam um atlas de M.

Se incluirmos a condição 3 também ainda teremos um atlas.

De fato, seja $\mathbf{s} : V_0 \to V$ a qual satisfaz 1 e 2. Se $\mathbf{s}$ é negativa podemos compô-la com a tranformação linear

$$T(u_0, u_1, \ldots, u_m) = (u_0, u_1, \ldots, -u_m)$$

e assim definindo $U_0 = T^{-1}(V_0)$, teremos que

$$\mathbf{r} = \mathbf{s} \circ T : U_0 \to V$$

satisfaz às condições 1, 2 e 3, pois $\det T < 0$.

Portanto $\mathcal{P}$ é um atlas.

Vamos agora identificar $\mathbb{R}^m$ com

$$\partial H_0 = \{\mathbf{u} = (u_0, u_1, \ldots, u_m) \ : \ u_0 = 0\}.$$

Seja $\mathcal{P}_0$ o conjunto das restrições $\mathbf{r}_0 = \mathbf{r}|_{\partial U_0}$ das paramerizações $\mathbf{r}$ de $\mathcal{P}$ tais que $\partial U_0 = U_0 \cap \partial H_0 \neq \emptyset$.

Assim $\mathcal{P}_0$ é um atlas de classe C^1 de ∂M.

Mostraremos agora que $\mathcal{P}_0$ é coerente.

De fato, sejam $\mathbf{r}_0 : \partial U_0 \to \partial U \in \mathcal{P}_0$ e $\mathbf{s}_0 : \partial V_0 \to \partial V \in \mathcal{P}_0$ tais que

$$\partial U \cap \partial V \neq \emptyset.$$

Assim, a mudança de parametrização $\xi_0 = \mathbf{s}_0^{-1} \circ \mathbf{r}_0$ é a restrição do difeomorfismo $\xi = \mathbf{s}^{-1} \circ \mathbf{r}$ ao bordo do domínio em questão.

Defina $A = \xi'(\mathbf{u}) : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ a derivada de ξ em $\mathbf{u}$ tal que

$$\mathbf{r}(\mathbf{u}) \in \partial U \cap \partial V$$

(domínio de ξ_0).

Como $\mathcal{P}$ é coerente, temos que $\det A > 0$.

Por outro lado, ξ é um difeomorfismo do aberto

$$\mathbf{r}^{-1}(U \cap V) \subset H_0$$

no aberto

$$\mathbf{s}^{-1}(U \cap V) \subset H_0.$$

Do Teorema 6.4, temos que

$$A(\partial H_0) = \partial H_0$$

ou seja

$$Ae_i = (0, a_{1i}, \dots, a_{mi}),$$

$\forall i \in \{1, \dots, m\}$.

Finalmente, como $e_0 = (1, 0, \dots, 0)$ *aponta para fora de* H_0*, temos do Teorema 6.15, que* $Ae_0 = (a_{00}, a_{10}, \dots a_{m_0})$ *também apontará.*

Portanto deve-se ter $a_{00} > 0$*, onde a matriz* A *tem a seguinte forma*

$$A = \begin{bmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{mm} \end{bmatrix}_{(m+1)\times(m+1)} . \tag{11}$$

Observe que $\det A = a_{00} \det A_0$ *onde*

$$A_0 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}_{m\times m} . \tag{12}$$

e onde A_0 *é a derivada de* ξ_0 *em* $\mathbf{u}$.

Como $\det A > 0$ *e* $a_{00} > 0$*, temos que* $\det A_0 > 0$.

Disto concluímos que o atlas $\mathcal{P}_0$ *é coerente.*

A orientação de ∂M *é dita ser induzida pela orientação de* M.

7 The tangent space

Let $M \subset \mathbb{R}^n$ be a m-dimensional surface, where $1 \leq m \leq n$, of C^1 class, with a boundary ∂M.

Observe that for each $p \in M$, there exists a parametrization $\mathbf{r}^p : U_0^p \to U^p \subset M$, where $U_0^p \subset H_0$ is open in the semi-space H_0 (for standard parametrization), for a local system of coordinates. Here

$$H_0 = \{(u_1, \dots, u_m) \in \mathbb{R}^m \ : \ u_1 \leq 0\}.$$

We denote $D = \cup_{p \in M} U_0^p$ and and generically also denote $M = \mathbf{r}(D) = \cup_{p\in M} \mathbf{r}^p(U_0^p)$. where, to simplify the notation, we have written

$$\mathbf{r}^p(\mathbf{u}) \equiv \mathbf{r}(\mathbf{u}),$$

emphasizing we are in fact referring to a particular system of coordinates relating U_0^p and for a specific parametrization defined on U_0^p.

Definição 7.1 (The tangent space to M)**.** *In the context of the last lines above, let* $p = \mathbf{r}(\mathbf{u}) \in M$*. We define the tangent space to* M *at* p*, denoted by* $T_p(M)$*, by*

$$T_p(M) = \left\{ \sum_{i=1}^m \alpha_i \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i} \ : \ \alpha_1, \cdots, \alpha_m \in \mathbb{R} \right\}.$$

We recall that for a C^k class surface,

$$\left\{\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^{m}$$

is linearly independent $\forall p \in M$.

Observação 7.2. *We recall that the dual space of a real vector space V, is formally defined as the set of all linear continuous functionals (in this context, in fact real functions) defined on V. From the Riesz representation theorem, considering the specific case where $V \subset \mathbb{R}^n$, given a linear continuous functional $F : V \to \mathbb{R}$ there exists $\alpha \in V$ such that*

$$F(\mathbf{u}) = \alpha \cdot \mathbf{u}, \ \forall \mathbf{u} \in V.$$

In such a case, as for any Hilbert space, we say that $V^* \approx V$, that is the dual space V^* is indeed identified with V.

In the general case, from elementary linear algebra, given a basis $(\mathbf{e}_1, \cdots, \mathbf{e}_m)$ for V, we may obtain a corresponding dual basis for V^*, given by $F_1, \cdots, F_m \in V^*$, such that

$$F_i(\mathbf{e}_j) = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

In the next lines we present the formal details on these last statements.

Teorema 7.3. *Let V be a real m-dimensional vector space and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ be a basis for V*

Under such hypotheses, there exists a unique basis $\{F_1, \cdots, F_m\}$ for V^* such that

$$F_i(v_j) = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \forall i, j \in \{1, \cdots, m\} \end{cases} \tag{14}$$

Demonstração. For each $i \in \{1, \cdots, m\}$, define $F_i : V \to \mathbb{R}$ by

$$F_i(\mathbf{v}) = a_i,$$

where,

$$\mathbf{v} = \sum_{j=1}^{m} a_j \mathbf{v}_j.$$

Therefore,

$$F_i(\mathbf{v}_j) = \delta_{ij}, \ \forall i, j \in \{1, \cdots, m\}.$$

Now we are going to show that such $\{F_i\}$ are unique.

Let $\tilde{F}_1, \cdots, \tilde{F}_m \in V^*$ be such that $\tilde{F}_i(\mathbf{v}_j) = \delta_{ij}, \ \forall i, j \in \{1, \cdots, m\}$.

Let $\mathbf{v} = \sum_{j=1}^{m} a_j \mathbf{v}_j \in V$.

Thus, fixing $i \in \{1, \cdots, m\}$, we have,

$$\begin{aligned} \tilde{F}_i(\mathbf{v}) &= \tilde{F}_i(\sum_{j=1}^m a_j \mathbf{v}_j) \\ &= \sum_{j=1}^m a_j \tilde{F}_i(\mathbf{v}_j) \\ &= \sum_{j=1}^m a_j \delta_{ij} \\ &= a_i \\ &= F_i(\mathbf{v}). \end{aligned} \tag{15}$$

Therefore,

$$\tilde{F}_i(\mathbf{v}) = F_i(\mathbf{v}),\ \forall \mathbf{v} \in V,$$

so that,

$$\tilde{F}_i = F_i,\ \forall i \in \{1, \cdots, m\}.$$

We may conclude that the $F_1, \cdots, F_m \in V^*$ in question are unique.

At this point, we are going to show that $F_1, \cdots, F_m \in V^*$ are linearly independent.

Suppose that $b_1, \cdots, b_m \in \mathbb{R}$ are such that

$$F = \sum_{i=1}^m b_i F_i = \mathbf{0}.$$

Thus,

$$0 = F(\mathbf{v}_j) = \sum_{i=1}^m b_i F_i(\mathbf{v}_j) = \sum_{i=1}^m b_i \delta_{ij} = b_i,\ \forall i \in \{1, \cdots, m\}.$$

Hence, $\{F_1, \cdots, F_m\}$ is a linearly independent set.

To finish the proof, we are going to show that $\{F_1, \cdots, F_m\}$ spans V^*.

Let $S \in V^*$.

Denote $b_i = S(\mathbf{v}_i), \forall i \in \{1, \cdots, m\}$.

Define

$$F = \sum_{i=1}^m b_i F_i \in V^*.$$

Thus

$$F(\mathbf{v}_j) = \sum_{i=1}^m b_i F_i(\mathbf{v}_j) = \sum_{i=1}^m b_i \delta_{ij} = b_j.$$

Let $\mathbf{v} = \sum_{j=1}^m a_j \mathbf{v}_j$.

Thus,

$$S(\mathbf{v}) = S(\sum_{j=1}^m a_j \mathbf{v}_j) = \sum_{j=1}^m a_j S(\mathbf{v}_j) = \sum_{j=1}^m a_j b_j.$$

On the other hand,

$$F(\mathbf{v}) = F\left(\sum_{j=1}^{m} a_j \mathbf{v}_j\right) = \sum_{j=1}^{m} a_j F(\mathbf{v}_j) = \sum_{j=1}^{m} a_j b_j.$$

Hence,

$$F(\mathbf{v}) = S(\mathbf{v}), \forall \mathbf{v} \in V,$$

so that

$$S = F = \sum_{i=1}^{m} b_i F_i.$$

Thus, $\{F_1, \cdots, F_m\}$ spans V^*, so that it is a basis for V^*.
This completes the proof.

□

Teorema 7.4. *Let $M \subset \mathbb{R}^n$ be a m dimensional surface and let $f : M \to \mathbb{R}$ be C^1 class function.*
Let $p = \mathbf{r}(\mathbf{u}) \in M$. Under such hypotheses, we may associate to such a function f, a functional

$$F : T_p(M) \to \mathbb{R}$$

where, for each

$$\mathbf{v} = v_i \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i},$$

we have

$$\begin{aligned} F(\mathbf{v}) &= F\left(v_i \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right) \\ &= \lim_{\varepsilon \to 0} \frac{(f \circ \mathbf{r})(\{u_i + \varepsilon v_i\}) - (f \circ \mathbf{r})(\{u_i\})}{\varepsilon} \\ &= df(\mathbf{v}). \end{aligned} \tag{16}$$

Conversely, if $F : T_p(M) \to \mathbb{R}$ is a continuous linear functional, that is, if there exists $\alpha \in \mathbb{R}^m$ such that

$$F(\mathbf{v}) = \alpha \cdot \left(v_i \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right), \ \forall \mathbf{v} \in T_p(M),$$

then there exists $f : M \to \mathbb{R}$ of C^1 class such that,

$$F(\mathbf{v}) \equiv df(\mathbf{v}), \ \forall \mathbf{v} \in T_p(M).$$

Demonstração. For $f \in C^1(M)$ and $p \in M$ in question, define $F : T_p(M) \to \mathbb{R}$ by

$$F(\mathbf{v}) = df(\mathbf{v}),$$

for all $\mathbf{v} = v_i \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i} \in T_p(M)$.

Hence,

$$\begin{aligned} F(\mathbf{v}) &= df(\mathbf{v}) \\ &= \lim_{\varepsilon\to 0} \frac{(f\circ\mathbf{r})(\{u_i+\varepsilon v_i\})-(f\circ\mathbf{r})(\{u_i\})}{\varepsilon} \\ &= \sum_{j=1}^{n} \frac{\partial (f\circ\mathbf{r})(u)}{\partial X_j}\frac{\partial X_j(\mathbf{u})}{\partial u_i} v_i \\ &= \alpha\cdot\left(v_i\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right), \end{aligned} \tag{17}$$

where

$$\alpha_j = \frac{\partial (f\circ\mathbf{r})(\mathbf{u})}{\partial X_j},\ \forall j\in\{1,\cdots,n\}.$$

We may conclude that F is continuous and linear on $T_p(M)$, that is,

$$F \in T_p(M)^*.$$

Conversely, assume $F: T_p(M)\to\mathbb{R}$ is linear and continuous, so that by the Riesz representation theorem, the exists, $\alpha\in\mathbb{R}^n$ such that

$$F(v) = \alpha\cdot\left(v_i\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right),\ \forall\mathbf{v}\in T_p(M).$$

Define $f: M\to\mathbb{R}$ by

$$f(\mathbf{w}) = \alpha\cdot\mathbf{w}, \forall\mathbf{w}\in M.$$

Thus $f(\mathbf{r}(u)) = \alpha\cdot\mathbf{r}(\mathbf{u}) = \alpha_j X_j(\mathbf{u})$.
From this,

$$\begin{aligned} df(\mathbf{v}) &= \lim_{\varepsilon\to 0}\frac{\alpha_j X_j(\{u_i+\varepsilon v_i\})-\alpha_j X_j(\{u_i\})}{\varepsilon} \\ &= \alpha_j\frac{\partial X_j(\mathbf{u})}{\partial u_i}v_i \\ &= \alpha\cdot\left(v_i\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right) \\ &= F(\mathbf{v}). \end{aligned} \tag{18}$$

This completes the proof.

□

Corolário 7.5. *In an appropriate sense, the corresponding dual basis to $T_p(M)^*$, to the primal basis for $T_p(M)$*

$$\left\{\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^{m},$$

is given by $\{dw_i(r(\mathbf{u}))\}$ where $\{w_i(\mathbf{r}(u)\} = \mathbf{r}^{-1}(\mathbf{r}(\mathbf{u})) = \{u_i\}$, so that we could denote such basis for $T_p(M)^*$ by

$$\{du_1,\cdots,du_m\}.$$

Demonstração. Let us denote

$$\left\{\tilde{\mathbf{e}}_i = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^{m},$$

and

$$\tilde{\mathbf{v}}_j = (0, 0, \cdots, 0, 1, 0, \cdots, 0),$$

that is, value 1 at the $j - th$ entry and value 0 at the remaining ones.

Hence

$$\begin{aligned} du_i(\tilde{\mathbf{e}}_j) &= dw_i(\mathbf{r}(\mathbf{u}))(\tilde{\mathbf{e}}_j) \\ &= \lim_{\varepsilon \to 0} \frac{w_i(\mathbf{r}(\mathbf{u} + \varepsilon \tilde{\mathbf{v}}_j)) - w_i(\mathbf{r}(\mathbf{u}))}{\varepsilon} \\ &= \frac{\partial w_i(r(\mathbf{u}))}{\partial u_j} \\ &= \frac{\partial u_i}{\partial u_j} = \delta_{ij}. \end{aligned} \tag{19}$$

Summarizing, we have obtained

$$du_i(\tilde{\mathbf{e}}_j) = \delta_{ij},$$

so that, in the context of last theorem, $\{du_i, \cdots, du_m\}$ is a basis for $T_p(M)^*$.

The proof is complete. □

8 Vector Fields

Definição 8.1. *Let $M \subset \mathbb{R}^n$ be an m-dimensional surface that is, $1 \leq m < n$.*

Generically denoting

$$M = \{\mathbf{r}(\mathbf{u}) \ : \ \mathbf{u} \in D\},$$

let $p = \mathbf{r}(\mathbf{u}) \in M$ and consider the set

$$\left\{\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^{m},$$

be a basis for $T_p(M)$.

We define a vector field

$$X : M \to \{T_p(M),\ p \in M\},$$

by

$$X = \{X_p,\ p \in M\},$$

where point-wise

$$X_p(\mathbf{u}) = \sum_{i=1}^{m} X_{p_i}(\mathbf{u}) \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}.$$

Moreover, for each $p \in M$ we define the operator

$$\tilde{X}_p : C^1(M) \to C^1(D)$$

where

$$X_p = \sum_{i=1}^{m} X_{p_i} \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i},$$

by

$$\begin{aligned} \tilde{X}_p(f) &= df(X_p) \\ &= \lim_{\varepsilon 0} \frac{(f \circ \mathbf{r})(\{u_i + \varepsilon X_{p_i}(\mathbf{u})\}) - (f \circ \mathbf{r})(\{u_i\})}{\varepsilon} \\ &= \frac{\partial (f \circ \mathbf{r})(\mathbf{u})}{\partial X_j} \frac{\partial X_j(\mathbf{u})}{\partial u_i} X_{p_i}(\mathbf{u}) \\ &= \nabla f(\mathbf{r}(u)) \cdot \left(X_{p_i}(\mathbf{u}) \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i} \right). \end{aligned} \tag{20}$$

Moreover, we denote by $\mathcal{X}(M)$ the set of C^1 class vector fields on M.

Definição 8.2 (The exterior product)**.** *Let V be a real vector space and V^* its dual one. Let $(F_1, \cdots, F_k) \in V^* \times V^* \times \cdots \times V^*$ (k times) and $(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k) \in V \times V \times \cdots \times V$ (k times). We define point-wise the exterior product $(F_1 \wedge F_2 \wedge \cdots \wedge F_k)(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k)$, by*

$$(F_1 \wedge F_2 \wedge \cdots \wedge F_k)(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k) = det\{F_i(\mathbf{v}_j)\}.$$

Observação 8.3. *Concerning a C^1 class m-dimensional surface $M \subset \mathbb{R}^n$, where $1 \leq m < n$ and*

$$M = \{\mathbf{r}(\mathbf{u}) \ : \ \mathbf{u} \in D\},$$

for a vector field $X \in \mathcal{X}(M)$ where

$$X = X_{p_i} \frac{\partial \mathbf{r}(u)}{\partial u_i}$$

and $f : M \to \mathbb{R}$ of C^1 class, we shall denote,

$$X \cdot f = df(X),$$

where point-wisely,

$$\begin{aligned} (X \cdot f)(\mathbf{r}(\mathbf{u})) &= df(X)[\mathbf{r}(\mathbf{u})] \\ &= \lim_{\varepsilon \to 0} \frac{(f \circ \mathbf{r})(\{u_i + \varepsilon X_{p_i}\}) - (f \circ \mathbf{r})(\mathbf{u})}{\varepsilon}, \end{aligned} \tag{21}$$

where $p = \mathbf{r}(\mathbf{u})$.

Definição 8.4 (Lie bracket)**.** *Let $M \subset \mathbb{R}^n$ be a C^1 class m-dimensional surface where $1 \leq m < n$.*

Let $X, Y \in \tilde{\mathcal{X}}(M)$, where $\tilde{\mathcal{X}}(M)$ denotes the set of the $C^\infty(M) = \cap_{k \in \mathbb{N}} C^k(M)$ class vector fields. We define the Lie bracket of X and Y, denoted by $[X, Y]$ by,

$$[X, Y] = (X \cdot Y_i - X \cdot Y_i) \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i},$$

where

$$X = X_i \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}$$

and

$$Y = Y_i \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}.$$

Teorema 8.5. *Let $M \subset \mathbb{R}^n$ be a C^1 class m-dimensional surface where $1 \leq m < n$. Let $X, Y \in \tilde{\mathcal{X}}(M)$ and let $f \in C^2(M)$.*

Under such hypotheses,

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f).$$

Demonstração. Observe that

$$\begin{aligned} X \cdot (Y \cdot f) &= X \cdot (df(Y)) \\ &= X \cdot \left(\frac{\partial (f \circ r)(\mathbf{u})}{\partial u_i} Y_i \right) \\ &= \left[d \left(\frac{\partial (f \circ r)(\mathbf{u})}{\partial u_i} Y_i \right) \right] (X) \\ &= \frac{\partial^2 (f \circ \mathbf{r})(\mathbf{u})}{\partial u_j \partial u_i} Y_i X_j + \frac{\partial (f \circ r)(\mathbf{u})}{\partial u_i} \frac{\partial Y_i}{\partial u_j} X_j. \end{aligned} \quad (22)$$

Similarly,

$$Y \cdot (X \cdot f) = \frac{\partial^2 (f \circ \mathbf{r})(\mathbf{u})}{\partial u_j \partial u_i} X_i Y_j + \frac{\partial (f \circ r)(\mathbf{u})}{\partial u_i} \frac{\partial X_i}{\partial u_j} Y_j,$$

so that

$$\begin{aligned} & X \cdot (Y \cdot f) - Y \cdot (X \cdot f) \\ = & \frac{\partial (f \circ r)(\mathbf{u})}{\partial u_i} \frac{\partial Y_i}{\partial u_j} X_j - \frac{\partial (f \circ r)(\mathbf{u})}{\partial u_i} \frac{\partial X_i}{\partial u_j} Y_j. \end{aligned} \quad (23)$$

On the other hand,

$$[X, Y] \cdot f = \left[(X \cdot Y_i - Y \cdot X_i) \frac{\partial \mathbf{r}(u)}{\partial u_i} \right] \cdot f. \quad (24)$$

Observe that

$$\begin{aligned} \left[(X \cdot Y_i) \frac{\partial \mathbf{r}(u)}{\partial u_i} \right] \cdot f &= df \left[(X \cdot Y_i) \frac{\partial \mathbf{r}(u)}{\partial u_i} \right] \\ &= \frac{\partial (f \circ \mathbf{r})(\mathbf{u})}{\partial u_i} (X \cdot Y_i) \\ &= \frac{\partial (f \circ \mathbf{r})(\mathbf{u})}{\partial u_i} dY_i(X) \\ &= \frac{\partial (f \circ \mathbf{r})(\mathbf{u})}{\partial u_i} \frac{\partial Y_i}{\partial u_j} X_j. \end{aligned} \quad (25)$$

Similarly,

$$\begin{aligned} & \left[(X \cdot Y_i)\frac{\partial \mathbf{r}(u)}{\partial u_i}\right] \cdot f \\ = & \frac{\partial (f \circ \mathbf{r})(\mathbf{u})}{\partial u_i}\frac{\partial X_i}{\partial u_j}Y_j. \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned} [X, Y] \cdot f &= \left[(X \cdot Y_i - Y \cdot X_i)\frac{\partial \mathbf{r}(u)}{\partial u_i}\right] \cdot f \\ &= \frac{\partial (f \circ \mathbf{r})(\mathbf{u})}{\partial u_i}\frac{\partial Y_i}{\partial u_j}X_j - \frac{\partial (f \circ \mathbf{r})(\mathbf{u})}{\partial u_i}\frac{\partial X_i}{\partial u_j}Y_j \\ &= X \cdot (Y \cdot f) - Y \cdot (X \cdot f). \end{aligned} \tag{27}$$

The proof is complete. □

Exercícios 8.6.

1. *Let $M \subset \mathbb{R}^n$ be a C^1 class m-dimensional surface where $1 \leq m < n$ and let $X, Y, Z \in \tilde{\mathcal{X}}(M)$. Show that*
   (a) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z],\ \forall \alpha, \beta \in \mathbb{R},$
   (b) $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z],\ \forall \alpha, \beta \in \mathbb{R}.$
   (c) *Antisymmetry:*
   $$[X, Y] = -[Y, X],$$
   (d) *Jacob Identity:*
   $$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$
   (e) *Leibnitz Rule:*
   $$[fX, gY] = fg[X, Y] + f(X \cdot g)Y - g(Y \cdot f)X,\ \forall f, g \in C^2(M).$$

At this point we introduce the definition of Lie Algebra.

Definição 8.7 (Lie Algebra)**.** *A Lie Algebra is a vector space V for which is defined an anti-symmetric bilinear form $[\cdot, \cdot] : V \times V \to V$, which satisfies the Jacob identity.*

Observação 8.8. *Observe that, from the exposed above, $\tilde{\mathcal{X}}(M)$ is a Lie Algebra with the Lie bracket*

$$[X, Y] = (X \cdot Y_i - Y \cdot X_i)\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}.$$

Definição 8.9 (Integral curve)**.** *Let $M \subset \mathbb{R}^n$ be a C^1 class m-dimensional surface where $1 \leq m < n$. Let $X \in \mathcal{X}(M)$.*

Let A curve $(\mathbf{r} \circ \mathbf{u}) : I = (-\varepsilon, \varepsilon) \to M$ is said to be an integral curve for X on I around $p = \mathbf{r}(u(0))$, if

$$\frac{d\mathbf{r}(\mathbf{u}(t))}{dt} = X(\mathbf{u}(t)),\ \forall t \in I.$$

Observe that in such a case:

$$\begin{aligned}\frac{d\mathbf{r}(\mathbf{u}(t))}{dt} &= \frac{\partial \mathbf{r}(\mathbf{u}(t))}{\partial u_i}\frac{du_i(t)}{dt} = X(\mathbf{u}(t)) \\ &= X_i(\mathbf{u}(t))\frac{\partial \mathbf{r}(\mathbf{u}(t))}{\partial u_i}, \end{aligned} \tag{28}$$

so that

$$\frac{du_i(t)}{dt} = X_i(\mathbf{u}(t)). \tag{29}$$

9 The generalized derivative

At this section we introduce the definition of Lie derivative. We start with a preliminary definition, namely, the generalized derivative.

Definição 9.1 (Generalized derivative)**.** *Let $M \subset \mathbb{R}^n$ be a C^1 class m-dimensional surface where $1 \leq m < n$. Let $X \in \mathcal{X}(M)$. Let $\mathbf{r}(\mathbf{u}(t))$ be the point-wise representation of the integral curve of X around $p = \mathbf{r}(\mathbf{u}(0))$.*

Let $Y \in \tilde{\mathcal{X}}(M)$ and $f \in C^2(M)$. We define the generalized derivative of f relating Y, along the direction X, at the point $p = \mathbf{r}(\mathbf{u}(0))$, denoted by $D_X Y(f)(p)$, by

$$D_X Y(f) = \frac{d[Y(\mathbf{u}(t)) \cdot f(\mathbf{r}(\mathbf{u}(t))]}{dt}|_{t=0}.$$

Teorema 9.2. *Let $M \subset \mathbb{R}^n$ be a C^1 class m-dimensional surface where $1 \leq m < n$. Let $X \in \mathcal{X}(M)$. Let $\mathbf{r}(\mathbf{u}(t))$ be the point-wise representation of the integral curve of X around $p = \mathbf{r}(\mathbf{u}(0))$.*

Let $Y \in \mathcal{X}(M)$ and $f \in C^2(M)$.

Under such assumptions,

$$D_X Y(f)(p) = X \cdot (Y \cdot f)(p).$$

Demonstração. Observe that,

$$\begin{aligned} D_X Y(f)(p) &= \frac{d[df(Y)]}{dt}|_{t=0} \\ &= \frac{d}{dt}\left[\frac{\partial (f\circ \mathbf{r})(\mathbf{u}(t))}{\partial u_i} Y_i(\mathbf{u}(t))\right]_{t=0} \\ &= \left[\frac{\partial^2 (f\circ \mathbf{r})(\mathbf{u}(t))}{\partial u_j \partial u_i}\frac{du_j(t)}{dt} Y_i(\mathbf{u}(t))\right]_{t=0} \\ &+ \left[\frac{\partial (f\circ \mathbf{r})(\mathbf{u}(t))}{\partial u_i}\frac{\partial Y_i(\mathbf{u}(t))}{\partial u_j}\frac{du_j(t)}{dt}\right]_{t=0} \\ &= \left[\frac{\partial^2 (f\circ \mathbf{r})(\mathbf{u}(t))}{\partial u_j \partial u_i} X_j(\mathbf{u}(t)) Y_i(\mathbf{u}(t))\right]_{t=0} \\ &+ \left[\frac{\partial (f\circ \mathbf{r})(\mathbf{u}(t))}{\partial u_i}\frac{\partial Y_i(\mathbf{u}(t))}{\partial u_j} X_j(\mathbf{u}(t))\right]_{t=0} \\ &= X \cdot (Y \cdot f)(p). \end{aligned} \tag{30}$$

□

Definição 9.3 (Derivative of $Y \in \mathcal{X}(M)$ on the direction of $X \in \mathcal{X}(M)$)**.** *Let $M \subset \mathbb{R}^m$ be a m-dimensional surface where $1 \leq m < n$. Let $X, Y \in \mathcal{X}(M)$. Let $p = \mathbf{r}(\mathbf{u}(0))$ where $\mathbf{r}(\mathbf{u}(t))$ is the integral curve of X about $p = \mathbf{r}(\mathbf{u}_0)$, where $\mathbf{u}_0 = \mathbf{u}(0)$.*

We define the derivative of Y on the direction X at the point $p = \mathbf{r}(\mathbf{u}_0)$, denoted by

$$(D_X Y)(p),$$

by

$$(D_X Y)(p) = \lim_{t \to 0} \frac{Y(\mathbf{u}(t)) - Y(\mathbf{u}(0))}{t} = \frac{dY(\mathbf{u}(t))}{dt}|_{t=0}.$$

Observação 9.4. *Observe that*

$$Y(\mathbf{u}(t)) = Y_i(\mathbf{u}(t)) \frac{\partial \mathbf{r}(\mathbf{u}(t))}{\partial u_i},$$

so that

$$\begin{aligned} (D_X Y)(p) &= \frac{\partial Y_i(\mathbf{u}(0))}{\partial u_j} \frac{du_j(0)}{dt} \frac{\partial \mathbf{r}(\mathbf{u}(0))}{\partial u_i} + Y_i(\mathbf{u}(0) \frac{\partial^2 \mathbf{r}(\mathbf{u}(0))}{\partial u_i \partial u_j} \frac{du_j(0)}{dt} \\ &= \frac{\partial Y_i(\mathbf{u}(0))}{\partial u_j} X_j(\mathbf{u}(0)) + Y_i(\mathbf{u}(0)) \frac{\partial^2 \mathbf{r}(\mathbf{u}(0))}{\partial u_i \partial u_j} X_j(\mathbf{u}(0)) \\ &= dY_i(X)(\mathbf{u}_0) \frac{\partial \mathbf{r}(\mathbf{u}_0)}{\partial u_i} + Y_i(\mathbf{u}_0) X_j(\mathbf{u}_0) \frac{\partial^2 \mathbf{r}(\mathbf{u}_0)}{\partial u_i \partial u_j}. \end{aligned} \tag{31}$$

Summarizing,

$$(D_X Y)(p) = dY_i(X)(\mathbf{u}_0) \frac{\partial \mathbf{r}(\mathbf{u}_0)}{\partial u_i} + Y_i(\mathbf{u}_0) X_j(\mathbf{u}_0) \frac{\partial^2 \mathbf{r}(\mathbf{u}_0)}{\partial u_i \partial u_j}.$$

Definição 9.5 (Lie derivative of a vector field)**.** *Let $M \subset \mathbb{R}^n$ be a C^1 class m-dimensional surface where $1 \leq m < n$. Let $X \in \tilde{\mathcal{X}}(M)$. Let $\mathbf{r}(\mathbf{u}(t))$ be the point-wise representation of the integral curve of X around $p = \mathbf{r}(\mathbf{u}(0))$.*

Let $Y \in \tilde{\mathcal{X}}(M)$ and $f \in C^2(M)$. We define the Lie derivative of Y, along the direction X, at $f \in C^2(M)$ at the point $p = \mathbf{r}(\mathbf{u}(0))$, denoted by $L_X Y(f)(p)$, by

$$L_X Y(f)(p) = X \cdot (Y \cdot f)(p) - Y \cdot (X \cdot f)(p).$$

Observe that in such a case

$$L_X Y(f)(p) = ([X, Y] \cdot f)(p), \ \forall f \in C^2(M),$$

so that we denote simply,

$$L_X Y = [X, Y].$$

Relating the statements of this last definition, consider the following exercises.

Exercícios 9.6.

1. *Prove that*

$$L_X[Y, Z] = [L_X, Y, Z] + [Y, L_X Z], \ \forall X, Y, Z \in \tilde{\mathcal{X}}(M).$$

2. *Prove that*

$$L_X(L_Y)Z - L_Y(L_X)Z = L_{[X,Y]} Z, \ \forall X, Y, Z \in \tilde{\mathcal{X}}((M).$$

9.1 On the integral curve existence

In this section we present some well known results about the existence of solutions for a nonlinear system of ordinary differential equations. At this point we recall, in a more general fashion, the definition of integral curve.

Let $M \subset \mathbb{R}^n$ be a C^1 class m-dimensional surface, where $1 \leq m < n$, where

$$M = \{\mathbf{r}(\mathbf{u}) \ : \ \mathbf{u} \in D\},$$

where $D \subset \mathbb{R}^m$ is a connected set such that ∂D is of C^1 class.

Let $X \in \mathcal{X}(M)$, that is

$$X = X_i(\mathbf{u})\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}.$$

Let $\mathbf{u}_0 \in D^\circ$ and $p = \mathbf{r}(\mathbf{u}_0)$. An integral curve for X around p is a curve

$$(\mathbf{r} \circ u) : I \to M,$$

where I is a closed interval, such that $t_0 \in I^\circ$,

$$\mathbf{r}(\mathbf{u}(0)) = r(\mathbf{u}_0),$$

and

$$\frac{d\mathbf{r}(\mathbf{u}(t))}{dt} = X(\mathbf{u}(t)),$$

that is

$$\frac{\partial \mathbf{r}(\mathbf{u}(t))}{\partial u_i}\frac{du_i(t)}{\partial t} = X_i(\mathbf{u}(t))\frac{\partial \mathbf{r}(\mathbf{u}(t))}{\partial u_i},$$

so that, in a component wise fashion,

$$\frac{du_i(t)}{\partial t} = X_i(\mathbf{u}(t)), \ \forall i \in \{1, ...m\}.$$

We present now the results concerning the existence of solutions for the ordinary differential equation systems in question.

Teorema 9.7. *Let $V \subset \mathbb{R}^m$ be an open set. Let $\hat{X} : V \to \mathbb{R}^m$ be a continuous function such that*

$$|\hat{X}(\mathbf{u}) - \hat{X}(\mathbf{v})| \leq K|\mathbf{u} - \mathbf{v}|, \ \forall \mathbf{u}, \mathbf{v} \in V,$$

for some $K > 0$.

Let $\mathbf{u}_0 \in V$ and let $r > 0$ be such that $B_r(\mathbf{u}_0) \in V$, where

$$B_r(\mathbf{u}_0) = \{\mathbf{u} \in \mathbb{R}^m \ : \ |\mathbf{u} - \mathbf{u}_0| < r\}.$$

Let $C > 0$ be such that

$$|\hat{X}(\mathbf{u})| < C, \ \forall \mathbf{u} \in B_r(\mathbf{u}_0).$$

Let $t_0 \in \mathbb{R}$ and let $\alpha = \min\{1/K, r/C\}$. Under such hypotheses there exists a curve $\mathbf{u} : I_\alpha \to B_r(\mathbf{u}_0)$ such that

$$\begin{cases} \frac{d\mathbf{u}(t)}{dt} = \hat{X}(\mathbf{u}(t)) \\ \mathbf{u}(t_0) = \mathbf{u}_0, \end{cases} \tag{32}$$

where,

$$I_\alpha = [t_0 - \alpha, t_0 + \alpha].$$

Demonstração. Observe that

$$\begin{cases} \frac{d\mathbf{u}(t)}{dt} = \hat{X}(\mathbf{u}(t)) \\ \mathbf{u}(t_0) = \mathbf{u}_0, \end{cases} \tag{33}$$

is equivalent to

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_{t_0}^{t} \hat{X}(\mathbf{u}(s))\, ds.$$

Define the sequence of functions $\{\mathbf{u}_n : I_\alpha \to \mathbb{R}^m\}$ by

$$\mathbf{u}_1(t) = \mathbf{u}_0,$$

$$\mathbf{u}_{n+1}(t) = \mathbf{u}_0 + \int_{t_0}^{t} \hat{X}(\mathbf{u}_n(s))\, ds,\ \forall t \in I_\alpha,\ \forall n \in \mathbb{N}.$$

First, we shall prove by induction that

$$\mathbf{u}_n(t) \in B_r(\mathbf{u}_0),\ \forall n \in \mathbb{N},\ t \in I_\alpha.$$

Clearly

$$\mathbf{u}_1(t) \in B_r(\mathbf{u}_0), \forall t \in I_\alpha.$$

Suppose

$$\mathbf{u}_n(t) \in B_r(\mathbf{u}_0),\ \forall t \in I_\alpha.$$

Thus,

$$|\hat{X}(\mathbf{u}_n(t))| < C,\ \forall t \in I_\alpha,$$

so that

$$\begin{aligned} |\mathbf{u}_{n+1}(t) - \mathbf{u}_0| &\leq \left| \int_{t_0}^{t} |\hat{X}(\mathbf{u}_n(s))|\, ds \right| \\ &< C|t - t_0| \\ &\leq C\alpha \\ &\leq C\frac{r}{C} \\ &= r, \end{aligned} \tag{34}$$

so that

$$\mathbf{u}_{n+1}(t) \in B_r(\mathbf{u}_0),\ \forall t \in I_\alpha.$$

The induction is complete, that is,

$$\mathbf{u}_n(t) \in B_r(\mathbf{u}_0),\ \forall t \in I_\alpha,\ \forall n \in \mathbb{N}.$$

Moreover, for $t \geq t_0$ we have

$$\begin{aligned}
|\mathbf{u}_{n+1}(t) - \mathbf{u}_n(t)| &\leq \int_{t_0}^{t} |\hat{X}(\mathbf{u}_n(s)) - \hat{X}(\mathbf{u}_{n-1}(s))| \, ds \\
&\leq \int_{t_0}^{t} K|\mathbf{u}_n(s) - \mathbf{u}_{n-1}(s)| \, ds \\
&\leq K \int_{t_0}^{t} \int_{t_0}^{s} |\hat{X}(\mathbf{u}_{n-1}(s_1)) - \hat{X}(\mathbf{u}_{n-2}(s_1))| \, ds_1 \, ds \\
&\leq K^2 \int_{t_0}^{t} \int_{t_0}^{s} |\mathbf{u}_{n-1}(s_1) - \mathbf{u}_{n-2}(s_1)| \, ds_1 \, ds \\
&\leq K^2 \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{s_1} |\hat{X}(\mathbf{u}_{n-2}(s_2)) - \hat{X}(\mathbf{u}_{n-3}(s_2))| \, ds_2 \, ds_1 \, ds \\
&\leq K^3 \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{s_1} |\mathbf{u}_{n-2}(s_2) - \mathbf{u}_{n-3}(s_2)| \, ds_2 \, ds_1 \, ds. \qquad (35)
\end{aligned}$$

Proceeding inductively in this fashion, and recalling that point-wise $\mathbf{u}_1(t) = \mathbf{u}_0$, we finally would obtain

$$\begin{aligned}
|\mathbf{u}_{n+1}(t) - \mathbf{u}_n(t)| &\leq K^n \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{s_1} \cdots \int_{t_0}^{s_{n-1}} |\mathbf{u}_2(s_n) - \mathbf{u}_0| \, ds_n \, ds_{n_1} \cdots ds_1 \, ds \\
&\leq K^n r \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{s_1} \cdots \int_{t_0}^{s_{n-1}} ds_n \, ds_{n-1} \cdots ds_1 \, ds \\
&\leq K^n r \frac{|t - t_0|^{n+1}}{(n+1)!}. \qquad (36)
\end{aligned}$$

The same estimate is valid for $t \leq t_0$.

Therefore,

$$\begin{aligned}
|\mathbf{u}_{n+1}(t) - \mathbf{u}_n(t)| &\leq rK^n |t - t_0|^n \frac{|t - t_0|}{(n+1)!} \\
&\leq rK^n \alpha^n \frac{|t - t_0|}{(n+1)!} \\
&\leq r \frac{|t - t_0|}{(n+1)!} \\
&\to 0, \text{ uniformly as } n \to \infty. \qquad (37)
\end{aligned}$$

Fix $n, p \in \mathbb{N}$.

From the last inequality

$$
\begin{aligned}
& |\mathbf{u}_{n+p}(t) - \mathbf{u}_n(t)| \\
= \; & |\mathbf{u}_{n+p}(t) - \mathbf{u}_{n+p-1}(t) + \mathbf{u}_{n+p-1}(t) \\
& -\mathbf{u}_{n+p-2}(t) + \mathbf{u}_{n+p-2}(t) + \cdots + \mathbf{u}_{n+1}(t) - \mathbf{u}_n(t)| \\
\leq \; & |\mathbf{u}_{n+p}(t) - \mathbf{u}_{n+p-1}(t)| + |\mathbf{u}_{n+p-1}(t) - \mathbf{u}_{n+p-2}(t)| + \cdots + |\mathbf{u}_{n+1}(t) - \mathbf{u}_n(t)| \\
\leq \; & r|t - t_0| \sum_{k=1}^{p} \frac{1}{(n+k)!} \\
\leq \; & r|t - t_0| \sum_{k=(n+1)}^{\infty} \frac{1}{k!}. \qquad (38)
\end{aligned}
$$

Let us denote $\{a_k\} = \frac{1}{k!}$. Observe that

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{k!}{(k+1)!} = \lim_{k \to \infty} \frac{1}{k+1} = 0.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{k!}$$

is converging so that

$$\sum_{k=(n+1)}^{\infty} \frac{1}{k!} \to 0, \text{ as } n \to \infty.$$

From this and (38), $\{\mathbf{u}_n\}$ is a uniformly Cauchy sequence of continuous functions, which uniformly converges to some continuous $\mathbf{u} : I_\alpha \to \overline{B}_r(\mathbf{u}_0)$, such that

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_{t_0}^{t} \hat{X}(\mathbf{u}(s)) \, ds,$$

so that

$$\begin{cases} \frac{d\mathbf{u}(t)}{dt} = \hat{X}(\mathbf{u}(t)) \\ \mathbf{u}(t_0) = \mathbf{u}_0. \end{cases} \qquad (39)$$

The proof of uniqueness of $\mathbf{u}$ is left as an exercise. □

Teorema 9.8 (Gronwall's inequality)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous and nonnegative functions. Suppose there exists $A > 0$ such that*

$$f(t) \leq A + \int_a^t f(s) g(s) \, ds, \; \forall t \in [a, b].$$

Under such hypotheses,

$$f(t) \leq A e^{\int_a^t g(s) \, ds}, \; \forall t \in [a, b].$$

Demonstração. Define $h : [a,b] \to \mathbb{R}$ by

$$h(t) = A + \int_a^t f(s)g(s)\, ds.$$

From the hypotheses, $h(t) > 0$ and

$$f(t) \leq h(t), \ \forall t \in [a,b].$$

Moreover,

$$h'(t) = f(t)g(t) \leq h(t)g(t), \ \forall t \in [a,b].$$

Therefore,

$$\frac{h'(t)}{h(t)} \leq g(t), \ \forall t \in [a,b],$$

so that

$$\frac{d\ln(h(t))}{dt} \leq g(t), \ \forall t \in [a,b],$$

and hence

$$\ln(h(t)) - \ln(A) \leq \int_a^t g(s)\, ds, \ \forall t \in [a,b],$$

that is,

$$\ln(h(t)/A) \leq \int_a^t g(s)\, ds,$$

so that

$$\frac{h(t)}{A} \leq e^{\int_a^t g(s)\, ds},$$

that is,

$$f(t) \leq h(t) \leq Ae^{\int_a^t g(s)\, ds}, \ \forall t \in [a,b].$$

The proof is complete. □

Teorema 9.9. *Let $V \subset \mathbb{R}^m$ be an open set. Let $\hat{X} : V \to \mathbb{R}^m$ be a continuous function such that*

$$|\hat{X}(\mathbf{u}) - \hat{X}(\mathbf{v})| \leq K|\mathbf{u} - \mathbf{v}|, \ \forall \mathbf{u}, \mathbf{v} \in V,$$

for some $K > 0$.

Let $\mathbf{u}_0 \in V$ and let $r > 0$ be such that $B_r(\mathbf{u}_0) \in V$, where

$$B_r(\mathbf{u}_0) = \{\mathbf{u} \in \mathbb{R}^m \ : \ |\mathbf{u} - \mathbf{u}_0| < r\}.$$

Let $C > 0$ be such that

$$|\hat{X}(\mathbf{u})| < C, \ \forall \mathbf{u} \in B_r(\mathbf{u}_0).$$

Let $F_t(\mathbf{u}_0)$ denote the unique integral curve $\mathbf{u} : I_\alpha \to \mathbb{R}$ such that

$$\begin{cases} \frac{d\mathbf{u}(t)}{dt} = \hat{X}(\mathbf{u}(t)) \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{40}$$

where $\alpha = \min\{1/K, r/C\}$, and $I_\alpha = [-\alpha, \alpha]$.

Under such hypotheses there exists $r_1 > 0$ and $\varepsilon > 0$ such that for each $\mathbf{v}_0 \in B_r(\mathbf{u}_0)$ there exists an integral $\mathbf{v} : [-\varepsilon, \varepsilon] \to \mathbb{R}^m$ such that

$$\begin{cases} \frac{d\mathbf{v}(t)}{dt} = \hat{X}(\mathbf{v}(t)) \\ \mathbf{v}(0) = \mathbf{v}_0. \end{cases} \tag{41}$$

Moreover,

$$|\mathbf{u}(t) - \mathbf{v}(t)| \leq e^{K|t|}|\mathbf{u}_0 - \mathbf{v}_0|, \forall t \in [-\varepsilon, \varepsilon].$$

Demonstração. Define $r_1 = r/2$ and $\varepsilon = \min\{1/K, r/(2C)\}$.

Let $\mathbf{v}_0 \in B_r(\mathbf{u}_0)$. Thus $B_{r_1}(\mathbf{v}_0) \subset B_r(\mathbf{u}_0)$, so that

$$|\hat{X}(\mathbf{v})| \leq C, \ \forall \mathbf{v} \in B_{r_1}(\mathbf{u}_0).$$

From Theorem 9.7, with $\mathbf{v}_0$ in place of $\mathbf{u}_0$, r_1 in place of r, 0 in place of t_0 and ε in place of α, there exists a curve $\mathbf{v} : I_\varepsilon \to \mathbb{R}^m$, where $I_\varepsilon = [-\varepsilon, \varepsilon]$, such that

$$\begin{cases} \frac{d\mathbf{v}(t)}{dt} = \hat{X}(\mathbf{v}(t)) \\ \mathbf{v}(0) = \mathbf{v}_0. \end{cases} \tag{42}$$

Now define

$$f(t) = |\mathbf{u}(t) - \mathbf{v}(t)|.$$

Thus,

$$\begin{aligned} f(t) &= |\mathbf{u}(t) - \mathbf{v}(t)| \\ &= \left| \int_0^t (\hat{X}(\mathbf{u}(s)) - \hat{X}(\mathbf{v}(s)))\, ds + \mathbf{u}_0 - \mathbf{v}_0 \right| \\ &\leq \left| \int_0^t K|\mathbf{u}(s) - \mathbf{v}(s)|\, ds \right| + |\mathbf{u}_0 - \mathbf{v}_0| \\ &\leq |\mathbf{u}_0 - \mathbf{v}_0| + K \int_0^{|t|} f(s)\, ds. \end{aligned} \tag{43}$$

From the Gronwall's inequality with $A = |\mathbf{u}_0 - \mathbf{v}_0|$ and $g(t) = K$ we obtain,

$$\begin{aligned} |\mathbf{u}(t) - \mathbf{v}(t)| &= f(t) \\ &\leq h(t) \\ &= |\mathbf{u}_0 - \mathbf{v}_0| + K \int_0^{|t|} f(s)\, ds \\ &\leq Ae^{\int_0^{|t|} K\, ds} \\ &= |\mathbf{u}_0 - \mathbf{v}_0|e^{K|t|}. \end{aligned} \tag{44}$$

The proof is complete. □

10 Differential forms

We start with the following preliminary result.

Teorema 10.1 (Partition of unity). *Let $K \subset \mathbb{R}^n$ be a compact set such that*

$$K \subset \cup_{i=1}^m V_i,$$

where $V_i \subset \mathbb{R}^n$ is bounded and open, $\forall i \in \{1, \cdots m\}$.

Under such hypotheses, there exist functions $h_1, \cdots, h_m : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{i=1}^m h_i(u) = 1, \ \forall u \in K,$$

$$0 \leq h_i(u) \leq 1, \ \forall u \in \mathbb{R}^n$$

and

$$h_i \in C_c(V_i),$$

that is, h_i is continuous and with compact support contained in V_i, $\forall i \in \{1, \cdots, m\}$.

We recall that the support of h_i, denoted by supp h_i is defined by

$$supp\ h_i = \overline{\{u \in \mathbb{R}^n \ : \ h_i(u) \neq 0\}},$$

$\forall i \in \{1, \cdots, m\}$.

Demonstração. Let $u \in K \subset \cup_{i=1}^m V_i$.

Thus, there exists $j \in \{1, \cdots, m\}$ such that $u \in V_j$.

Since V_j is open, there exists $r_u > 0$ such that

$$\overline{B}_{r_u}(u) \subset V_j.$$

Observe that

$$K \subset \cup_{u \in K} B_{r_u}(u).$$

Since K is compact, there exists $u_1, \cdots u_N \in K$ such that

$$K \subset \cup_{j=1}^N B_{r_j}(u_j),$$

where we have denoted $r_{u_j} = r_j$, $\forall j \in \{1, \cdots, N\}$.

For each $i \in \{1, \cdots m\}$ define $\tilde{W}_i$ as the union of all $\overline{B}_{r_j}(u_j)$ which are contained in V_i.

For each $i \in \{1, \cdots, m\}$, select also an open set W_i such that

$$\overline{\tilde{W}}_i \subset W_i \subset \overline{W_i} \subset V_i.$$

At this point, define $g_i : \mathbb{R}^n \to \mathbb{R}$, by

$$g_i(u) = \frac{d(u, W_i^c)}{d(u, W_i^c) + d(u, \tilde{W}_i)},$$

where generically, for $B \subset \mathbb{R}^n$ we define

$$d(u, B) = \inf\{|u - v| \ : \ v \in B\}.$$

Observe that

$$g_i(u) = 1, \forall u \in \tilde{W}_i,$$

$$0 \leq g_i(u) \leq 1, \ \forall u \in \mathbb{R}^n$$

and

$$g_i(u) = 0, \forall u \in \overline{W_i^c},$$

so that

$$supp \ g_i \subset V_i, \ \forall i \in \{1, \cdots, m\}.$$

Define

$$h_1 = g_1,$$

$$h_2 = (1 - g_1)g_2,$$

$$h_3 = (1 - g_1)(1 - g_2)g_3,$$

$$\vdots$$

$$h_m = (1 - g_1)(1 - g_2) \cdots (1 - g_{m-1})g_m.$$

Hence

$$0 \leq h(u) \leq 1, \forall u \in \mathbb{R}^n,$$

and $h_i \in C_c(V_i), \ \forall i\{1, \cdots, m\}$.

Now we are going to show by induction that

$$h_1 + h_2 + \cdots + h_j = 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_j), \ \forall j \in \{1, \cdots m\}.$$

For $j = 1$, we have $h_1 = g_1 = 1 - (1 - g_1)$.

Suppose that for $2 \leq j < m$ we have

$$h_1 + h_2 + \cdots + h_j = 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_j).$$

Thus,

$$\begin{aligned} & h_1 + h_2 + \cdots + h_j + h_{j+1} \\ = \ & 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_j) + (1 - g_1)(1 - g_2) \cdots (1 - g_j)g_{j+1} \\ = \ & 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_{j+1}). \end{aligned} \tag{45}$$

The induction is complete so that, in particular, we have obtained

$$h_1 + h_2 + \cdots + h_m = 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_m). \tag{46}$$

Hence, if $u \in K$, then $u \in K \subset \cup_{j=1}^N B_{r_j}(u_j) = \cup_{i=1}^m \tilde{W}_i$ so that $u \in \tilde{W}_j$, for some $j \in \{1, \cdots, m\}$.

Thus, $g_j(u) = 0$, so that from this and (46), we obtain

$$h_1 + \cdots + h_m(u) = 1, \forall u \in K.$$

The proof is complete. □

Observação 10.2. *Concerning this last theorem, the set* $\{h_1, \cdots, h_m\}$ *is said to be a partition of unity subordinate to* $V_1, \cdots, V_m$ *and related to* K.

At this point, we recall to have already established that $\{du_i\}_{i=1}^m$ is the dual basis for $T_p(M)^*$ corresponding to the basis

$$\left\{\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^m,$$

of $T_p(M)$.

Observe that, for $f \in C^1(M)$ and

$$\mathbf{v} = v_i \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i} \in T_p(M),$$

we have,

$$df(\mathbf{v}) = \frac{\partial f(\mathbf{r}(\mathbf{u}))}{\partial X_j}\frac{\partial X_j(\mathbf{u})}{\partial u_i} v_i. \tag{47}$$

On the other hand, denoting

$$\tilde{\mathbf{e}}_j = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_j},$$

$$\begin{aligned} du_i(\mathbf{v}) &= du_i\left(\sum_{j=1}^m v_j \tilde{\mathbf{e}}_j\right) \\ &= v_j du_i(\tilde{\mathbf{e}}_j) = v_j \delta_{ij} = v_i. \end{aligned} \tag{48}$$

From these last result we obtain,

$$\begin{aligned} df(\mathbf{v}) &= \frac{\partial f(\mathbf{r}(\mathbf{u}))}{\partial X_j}\frac{\partial X_j(\mathbf{u})}{\partial u_i} v_i \\ &= \frac{\partial f(\mathbf{r}(\mathbf{u}))}{\partial X_j}\frac{\partial X_j(\mathbf{u})}{\partial u_i} du_i(\mathbf{v}) \\ &= \frac{\partial f(\mathbf{r}(\mathbf{u}))}{\partial u_i} du_i(\mathbf{v}). \end{aligned} \tag{49}$$

We have just got the differential expression

$$df = \frac{\partial f(\mathbf{r}(\mathbf{u}))}{\partial u_i} du_i,$$

that is,

$$df = \frac{\partial f(\mathbf{r}(\mathbf{u}))}{\partial X_j}\frac{\partial X_j(\mathbf{u})}{\partial u_i} du_i,$$

so that

$$df = \frac{\partial f(\mathbf{r}(\mathbf{u}))}{\partial X_j} dX_j(\mathbf{u}).$$

Definição 10.3. *Let $M \subset \mathbb{R}^n$ be a C^1 class m dimensional surface, where $1 \leq m \leq n$. For $p = \mathbf{r}(\mathbf{u}) \in M$, we define a 1-form in M, locally as an element ω of $(T_pM)^*$, and globally as a field of elements in $T(M)^* = \{(T_pM)^* \,:\, p \in M\}$, expressed locally by*

$$\omega = \sum_{k=1}^{m} \omega_k(\mathbf{u}) du_k.$$

where $\omega_k : D \to \mathbb{R}$ are functions of C^1 class, $\forall k \in \{1, \cdots, m\}$.

We define also a $k-form$ in M, locally by

$$\omega = \sum_{I} \omega_I(\mathbf{u}) du_I,$$

that is, locally ω is an element of $(T_pM)^* \times \cdots \times (T_pM)^* = [(T_pM)^*]^k$, where $1 \leq k \leq m$, ω_I is a C^1 class function, $du_I = du_{i_1} \wedge \cdots \wedge du_{i_k}$, and

$$I = (i_1, ..., i_k)$$

is any collection of k indices $i_j \in \{1, \cdots, m\}$.

Definição 10.4 (Differential of a form)**.** *For a 1-form*

$$\omega = \sum_{k=1}^{m} \omega_k(\mathbf{u}) du_k,$$

we define its exterior differential, denoted by $d\omega$, locally by

$$d\omega = \sum_{k=1}^{m} d\omega_k(\mathbf{u}) \wedge du_k,$$

that is,

$$d\omega = \sum_{k=1}^{m} \sum_{j=1}^{m} \frac{\partial \omega_k(\mathbf{u})}{\partial u_j} du_j \wedge du_k.$$

11 Integration of differential forms

In this section first we present a discussion about the subject, before presenting the main result, namely, the Stokes Theorem on its general form.

Let $M \subset \mathbb{R}^n$ be a m-dimensional C^1 class surface with a boundary ∂M, where generically,

$$M = \{\mathbf{r}(\mathbf{u}) \,:\, \mathbf{u} \in D\},$$

and

$$\partial M = \mathbf{r}(\partial D).$$

Here we assume $\overline{D}$ to be compact and therefore $\overline{M}$ is compact. Let $\omega = \omega_I \, du_I$ be a k-form and consider the problem of calculating the integral:

$$I = \int_M \omega_I \, du_I.$$

At this point, we suppose the manifold M is oriented, in the specific sense that, considering a canonical system

$$\{\mathbf{e}_1, \cdots, \mathbf{e}_n\} \subset \mathbb{R}^n,$$

and

$$\left\{\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^m,$$

we may select normal continuous fields, point-wise denoted by

$$\{\mathbf{n}_1(\mathbf{r}(\mathbf{u})), \mathbf{n}_2(\mathbf{r}(\mathbf{u})), \ldots, \mathbf{n}_{n-m}(\mathbf{r}(\mathbf{u}))\},$$

where each $\mathbf{n}_j(\mathbf{r}(\mathbf{u}))$ is orthogonal to

$$\left\{\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^m,$$

and such that, for each $\mathbf{u} \in D$, $\mathbb{R}^n$ is spanned by

$$\left\{\left\{\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^m, \mathbf{n}_1(\mathbf{r}(\mathbf{u})), \mathbf{n}_2(\mathbf{r}(\mathbf{u})), \ldots, \mathbf{n}_{n-m}(\mathbf{r}(\mathbf{u}))\right\}.$$

Moreover, we assume the matrix to change from this local basis

$$\left\{\left\{\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}\right\}_{i=1}^m, \mathbf{n}_1(\mathbf{r}(\mathbf{u})), \mathbf{n}_2(\mathbf{r}(\mathbf{u})), \ldots, \mathbf{n}_{n-m}(\mathbf{r}(\mathbf{u}))\right\}.$$

to the canonical one of $\mathbb{R}^n$ has always positive determinant, $\forall \mathbf{u} \in D$. In such a case we say that M is positively oriented concerning the canonical base of $\mathbb{R}^n$ (the other possibility for a oriented surface, is that the mentioned determinant be negative $\forall p \in M$. In this case we say that M is negatively oriented related to the canonical base of $\mathbb{R}^n$).

Let $p \in \partial M$. Thus $\mathbf{u}_p = \mathbf{r}^{-1}(p) \in \partial D$.

We also assume that, for an appropriate local system of coordinates compatible with the orientation, there exists a C^1 class function point-wise indicated by $g^p(u_1, \cdots, u_{m-1})$ such that for a rectangle $B_p = \prod_{k=1}^m [\alpha_k^p, \beta_k^p]$ we have,

$$\begin{aligned} D \cap B_p &= \{(u_1, ..., u_m) \in \mathbb{R}^m \; : \; \alpha_k^p \leq u_k \leq \beta_k^p, \; \forall k \in \{1, .., m-1\}, \\ & \quad \text{and } \alpha_m^p \leq u_m \leq g^p(u_1, \cdots, u_{m-1}) \leq \beta_m^p\}. \end{aligned} \tag{50}$$

Thus,

$$\cup_{p \in \partial M} B_p^\circ \supset \partial D,$$

and since ∂D is compact, There exists, $p_1, \cdots, p_s \in M$ such that

$$\partial D \subset \cup_{l=1}^s B_{p_l}^\circ.$$

Define

$$B_{p_0} = D^\circ \setminus \cup_{l=1}^s B_{p_l}.$$

Select a partition of unit

$$\{\rho_{p_l}\}_{l=0}^s$$

subordinate to

$$\{B^\circ_{p_0}, B^\circ_{p_1}, \cdots, B^\circ_{p_s}\}.$$

Observe that

$$\text{supp}\{\rho_{p_l}\} \subset B^\circ_{p_l},\ \forall l \in \{0, \cdots, s\},$$

$$0 \leq \rho_{p_l}(\mathbf{u}) \leq 1,\ \forall l \in \{0, \cdots, s\},\ \mathbf{u} \in D,$$

$$\sum_{l=0}^{s} \rho_{p_l} = 1, \forall \mathbf{u} \in D.$$

Let us first consider the specific general case in which:

$$\omega = \sum_{j=1}^{m} (-1)^{j+1} f_j(\mathbf{u})\ du_1 \wedge du_2 \cdots \wedge \hat{du_j} \wedge ... \wedge du_m,$$

where $f_j : D \to \mathbb{R}$ are C^1 class functions $\forall j \in \{1, \cdots, m\}$, and where $\hat{du_j}$ means that the term du_j is absent in the products in question.

Observe that

$$d\omega = \sum_{j=1}^{m} \frac{\partial f_j(\mathbf{u})}{\partial u_j} du_1 \wedge \cdots \wedge du_m.$$

Denote also,

$$\mathbf{s}_k = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_k}\ du_k,$$

$$\mathbf{S}_k = (\mathbf{s}_1, \mathbf{s}_2, \cdots, \hat{\mathbf{s}}_k, \cdots, \mathbf{s}_m),$$

$$\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_m),$$

Thus,

$$\int_M d\omega = \int_D \sum_{j=1}^{m} \frac{\partial f_j(\mathbf{u})}{\partial u_j} (du_1 \wedge \cdots \wedge du_m)(\mathbf{S}), \tag{51}$$

and hence,

$$
\begin{aligned}
& \int_M d\omega \\
= \;& \int_D \sum_{j=1}^{m} (-1)^{j+1} \frac{\partial f_j(\mathbf{u})}{\partial u_j} (du_j \wedge (du_1 \wedge du_2 \cdots \wedge \hat{du_j} \wedge \cdots \wedge du_m))(\mathbf{S}) \\
= \;& \int_D \sum_{j=1}^{m} (-1)^{j+1} \frac{\partial [(\sum_{l=1}^{s} \rho_{p_l})\, f_j(\mathbf{u})]}{\partial u_j} (du_j \wedge (du_1 \wedge du_2 \cdots \wedge \hat{du_j} \wedge \cdots \wedge du_m))(\mathbf{S}) \\
= \;& \sum_{l=1}^{s} \int_D \sum_{j=1}^{m} \left[(-1)^{j+1} \frac{\partial (\rho_{p_l}(\mathbf{u}) f_j(\mathbf{u}))}{\partial u_j} du_j \right] du_1 du_2 \cdots \hat{du_j} \cdots du_m \\
= \;& \sum_{l=1}^{s} \int_{\partial B_l} \sum_{j=1}^{m-1} (-1)^{j+1} [(\rho_{p_l} f_j)(u_1, \cdots, \beta_j^l, \cdots, u_m) \\
& -(\rho_{p_l} f_j)(u_1, \cdots, \alpha_j^l, \cdots, u_m)] du_1 du_2 \cdots \hat{du_j} \cdots du_m \\
& + \int_{\partial D \cap B_l} (-1)^{m+1} (\rho_{p_l} f_m)(u_1, \cdots, u_{m-1}, g^l(u_1, \cdots, u_{m-1})\; du_1 \cdots du_{m-1} \\
& - \int_{\partial B_l} (-1)^{m+1} (\rho_{p_l} f_m)(u_1, \cdots, u_{m-1}, \alpha_m^l)\; du_1 \cdots du_{m-1} \\
= \;& \sum_{l=1}^{s} \int_{\partial D \cap B_l} (-1)^{m+1} (\rho_{p_l} f_m)(u_1, \cdots, u_{m-1}, g^l(u_1, \cdots, u_{m-1}))\; du_1 \cdots du_{m-1} \\
= \;& \int_{\partial D} (-1)^{m+1} \left(\sum_{l=1}^{s} \rho_{p_l} f_m \right) (u_1, \cdots, u_{m-1}, g^l(u_1, \cdots, u_{m-1}))\; du_1 \cdots du_{m-1} \\
= \;& \int_{\partial D} \left(\sum_{l=1}^{s} \rho_{p_l}(\mathbf{u}) \right) \sum_{j=1}^{m} (-1)^{j+1} f_j(\mathbf{u}) du_1 du_2 \cdots \hat{du_j} \cdots du_m \\
= \;& \int_{\partial D} \sum_{j=1}^{m} (-1)^{j+1} f_j(\mathbf{u}) du_1 du_2 \cdots \hat{du_j} \cdots du_m, \qquad (52)
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_M d\omega &= \int_{\partial D} \sum_{j=1}^{m} (-1)^{j+1} f_j(\mathbf{u}) du_1 du_2 \cdots \hat{du_j} \cdots du_m \\
&= \int_{\partial D} \sum_{j=1}^{m} (-1)^{j+1} f_j(\mathbf{u}) (du_1 \wedge du_2 \cdots \wedge \hat{du_j} \wedge \cdots \wedge du_m)(\mathbf{S}_j) \\
&= \int_{\partial M} \omega. \qquad (53)
\end{aligned}
$$

12 A simple example to illustrate the integration process

We would emphasize, in the final of last section, when the wedge product is absent, the differential forms and integrations are relating the usual calculus sense.

In the next example, we see how to connect the general differential form concept to a more usual one, in the ordinary calculus sense.

So, let $M \subset \mathbb{R}^n$ be C^1 class surface given by:

$$M = \{\mathbf{r}(\mathbf{u}) \; : \; \mathbf{u} \in D\},$$

where, for simplicity $D = [a, b] \times [c, d]$.

Consider the integral

$$\begin{aligned} I &= \int_M f \, dX_1 \wedge dX_2 \\ &= \int_D f(\mathbf{u})(dX_1(\mathbf{u}) \wedge dX_2(\mathbf{u})) \left(\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1} \, du_1, \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_2} \, du_2 \right) \end{aligned} \quad (54)$$

where, $X_1 : D \to \mathbb{R}$ and $X_2 : D \to \mathbb{R}$ are C^1 class functions.

Observe that

$$dX_1(\mathbf{u}) = \frac{\partial X_1(\mathbf{u})}{\partial u_1} \, du_1 + \frac{\partial X_1(\mathbf{u})}{\partial u_2} \, du_2$$

and

$$dX_2(\mathbf{u}) = \frac{\partial X_2(\mathbf{u})}{\partial u_1} \, du_1 + \frac{\partial X_2(\mathbf{u})}{\partial u_2} \, du_2$$

Consider the elementary rectangle

$$(u_1, u_1 + \Delta u_1) \times (u_2, u_2 + \Delta u_2),$$

which has as its sides the vectors

$$(\Delta u_1, 0) \in \mathbb{R}^2,$$

and

$$(0, \Delta u_2) \in \mathbb{R}^2.$$

Define

$$\tilde{\mathbf{s}}_1 = \frac{\partial \mathbf{r}(u)}{\partial u_1} \Delta u_1,$$

and

$$\tilde{\mathbf{s}}_2 = \frac{\partial \mathbf{r}(u)}{\partial u_2} \Delta u_2,$$

Considering the basis

$$\left\{ \frac{\partial \mathbf{r}(u)}{\partial u_1}, \frac{\partial \mathbf{r}(u)}{\partial u_2} \right\},$$

all to simplify the notation we shall denote

$$\tilde{\mathbf{s}}_1 = (\Delta u_1, 0),$$

and

$$\tilde{\mathbf{s}}_2 = (0, \Delta u_2).$$

Recall that, for the general case, for

$$\mathbf{v} = v_i \frac{\partial \mathbf{r}(u)}{\partial u_i}.$$

we have

$$du_j(\mathbf{v}) = v_j, \forall j \in \{1, \cdots, m\}.$$

Recall also that, from Definition 8.2,

$$(F_1 \wedge \cdots \wedge F_k)(\mathbf{v}_1, \cdots, \mathbf{v}_k) = \det\{F_i(\mathbf{v}_j)\}_{i,j=1}^k.$$

Let us evaluate

$$\begin{aligned}
& (dX_1(\mathbf{u}) \wedge dX_2(\mathbf{u})) \left(\frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1} \Delta u_1, \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_2} \Delta u_2 \right) \\
= & (dX_1(\mathbf{u}) \wedge dX_2(\mathbf{u}))[(\tilde{\mathbf{s}}_1, \tilde{\mathbf{s}}_2)] \\
= & (dX_1(\mathbf{u}) \wedge dX_2(\mathbf{u}))[(\Delta u_1, 0), (0, \Delta u_2)] \\
= & \left[\left(\frac{\partial X_1(\mathbf{u})}{\partial u_1} \, du_1 + \frac{\partial X_1(\mathbf{u})}{\partial u_2} \, du_2 \right) \wedge \left(\frac{\partial X_2(\mathbf{u})}{\partial u_1} \, du_1 + \frac{\partial X_2(\mathbf{u})}{\partial u_2} \, du_2 \right) \right] \\
& ((\Delta u_1, 0), (0, \Delta u_2)) \\
= & \left[\left(\frac{\partial X_1(\mathbf{u})}{\partial u_1} \frac{\partial X_2(\mathbf{u})}{\partial u_2} du_1 \wedge du_2 \right) + \left(\frac{\partial X_1(\mathbf{u})}{\partial u_2} \frac{\partial X_2(\mathbf{u})}{\partial u_1} du_2 \wedge du_1 \right) \right] \\
& ((\Delta u_1, 0), (0, \Delta u_2)) \\
= & \left(\frac{\partial X_1(\mathbf{u})}{\partial u_1} \frac{\partial X_2(\mathbf{u})}{\partial u_2} du_1 \wedge du_2 \right) ((\Delta u_1, 0), (0, \Delta u_2)) \\
& + \left(\frac{\partial X_1(\mathbf{u})}{\partial u_2} \frac{\partial X_2(\mathbf{u})}{\partial u_1} du_2 \wedge du_1 \right) ((\Delta u_1, 0), (0, \Delta u_2)) \\
= & \frac{\partial X_1(\mathbf{u})}{\partial u_1} \frac{\partial X_2(\mathbf{u})}{\partial u_2} \begin{vmatrix} du_1(\Delta u_1, 0) & du_1(0, \Delta u_2) \\ du_2(\Delta u_1, 0) & du_2(0, \Delta u_2) \end{vmatrix} \\
& + \frac{\partial X_1(\mathbf{u})}{\partial u_2} \frac{\partial X_2(\mathbf{u})}{\partial u_1} \begin{vmatrix} du_2(\Delta u_1, 0) & du_2(0, \Delta u_2) \\ du_1(\Delta u_1, 0) & du_1(0, \Delta u_2) \end{vmatrix} \\
= & \frac{\partial X_1(\mathbf{u})}{\partial u_1} \frac{\partial X_2(\mathbf{u})}{\partial u_2} \begin{vmatrix} \Delta u_1 & 0 \\ 0 & \Delta u_2 \end{vmatrix} \\
& + \frac{\partial X_1(\mathbf{u})}{\partial u_2} \frac{\partial X_2(\mathbf{u})}{\partial u_1} \begin{vmatrix} 0 & \Delta u_2 \\ \Delta u_1 & 0 \end{vmatrix} \\
= & \left(\frac{\partial X_1(\mathbf{u})}{\partial u_1} \frac{\partial X_2(\mathbf{u})}{\partial u_2} - \frac{\partial X_1(\mathbf{u})}{\partial u_2} \frac{\partial X_2(\mathbf{u})}{\partial u_1} \right) \Delta u_1 \Delta u_2. \qquad (55)
\end{aligned}$$

So, to summarize,

$$\begin{aligned}
& (dX_1(\mathbf{u}) \wedge dX_2(\mathbf{u}))[(\Delta u_1, 0), (0, \Delta u_2)] \\
= & \left(\frac{\partial X_1(\mathbf{u})}{\partial u_1} \frac{\partial X_2(\mathbf{u})}{\partial u_2} - \frac{\partial X_1(\mathbf{u})}{\partial u_2} \frac{\partial X_2(\mathbf{u})}{\partial u_1} \right) \Delta u_1 \Delta u_2, \qquad (56)
\end{aligned}$$

or in its differential form:

$$\begin{aligned}
&(dX_1(\mathbf{u}) \wedge dX_2(\mathbf{u}))[(du_1, 0), (0, du_2)] \\
= &\left(\frac{\partial X_1(\mathbf{u})}{\partial u_1} \frac{\partial X_2(\mathbf{u})}{\partial u_2} - \frac{\partial X_1(\mathbf{u})}{\partial u_2} \frac{\partial X_2(\mathbf{u})}{\partial u_1} \right) du_1 du_2.
\end{aligned} \tag{57}$$

Observe that this last differential form is one in the usual sense calculus, which is also used in the final usual integration process.

13 Volume (area) of a surface

Consider the problem of calculating the volume defined by the vectors

$$\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m \in \mathbb{R}^n$$

where $1 \leq m \leq n$.

Definição 13.1. *Given $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ we shall define the volume (in fact area) defined by $\mathbf{v}_1$ and $\mathbf{v}_2$, denoted by V_2, by*

$$V_2 = |\mathbf{v}_1||\mathbf{v}_2|| \sin \theta|,$$

where θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, which is given by:

$$\cos \theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|},$$

so that

$$\begin{aligned}
V_2 &= |\mathbf{v}_1||\mathbf{v}_2| \sqrt{1 - \cos^2 \theta} \\
&= |\mathbf{v}_1||\mathbf{v}_2| \sqrt{1 - \frac{(\mathbf{v}_1 \cdot \mathbf{v}_2)^2}{|\mathbf{v}_1|^2 |\mathbf{v}_2|^2}} \\
&= \sqrt{|\mathbf{v}_1|^2 |\mathbf{v}_2|^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2},
\end{aligned} \tag{58}$$

that is,

$$\begin{aligned}
V_2^2 &= |\mathbf{v}_1|^2 |\mathbf{v}_2|^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 \\
&= \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{vmatrix} \\
&\equiv g_2,
\end{aligned} \tag{59}$$

so that,

$$V_2 = \sqrt{g_2},$$

where

$$g_2 = \det\{\mathbf{v}_i \cdot \mathbf{v}_j\}_{i,j=1}^2.$$

Definição 13.2. *Let* $\mathbf{v}_1, \cdots, \mathbf{v}_m \subset \mathbb{R}^n$ *be non zero vectors, where* $2 \leq m \leq n$.

For $2 \leq s < m$*, we shall define inductively the volume defined by* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{s+1}$*, by*

$$V_2^2 = g_2,$$

and

$$V_{k+1}^2 = V_k^2 |\tilde{\mathbf{v}}_{k+1}|^2, \forall k \in \{2, \cdots, s\},$$

where

$$\tilde{\mathbf{v}}_{k+1} = \mathbf{v}_{k+1} - \sum_{j=1}^{k} \tilde{a}_j \mathbf{v}_j,$$

and where

$$\{\tilde{a}_j\} = arg\min \left\{ \left| \mathbf{v}_{k+1} - \sum_{j=1}^{k} a_j \mathbf{v}_j \right|^2 \; : \; a_1, \cdots, a_k \in \mathbb{R} \right\}$$

Teorema 13.3. *Under the statements of last definition, we have:*

$$V_k^2 = g_k, \forall k \in \{2, \cdots, s+1\},$$

where

$$g_k = \det\{\mathbf{v}_i \cdot \mathbf{v}_j\}_{i,j=1}^{k}.$$

Demonstração. We prove the result by induction.

For $k = 2$ we have already:

$$V_2^2 = g_2.$$

For $2 < k < s+1$ assume

$$V_k^2 = g_k.$$

It suffices, to complete the induction, to show that

$$V_{k+1}^2 = g_{k+1}.$$

By definition,

$$V_{k+1}^2 = V_k^2 |\tilde{\mathbf{v}}_{k+1}|^2,$$

where

$$\tilde{\mathbf{v}}_{k+1} = \mathbf{v}_{k+1} - \sum_{j=1}^{k} \tilde{a}_j \mathbf{v}_j,$$

and where

$$\{\tilde{a}_j\} = \text{ arg } \min \left\{ \left| \mathbf{v}_{k+1} - \sum_{j=1}^{k} a_j \mathbf{v}_j \right|^2 \; : \; a_1, \cdots, a_k \in \mathbb{R} \right\}$$

For this last optimization problem, the extremal necessary conditions are given by:

$$(\mathbf{v}_{k+1} - \sum_{j=1}^{k} \tilde{a}_j \mathbf{v}_j) \cdot \mathbf{v}_s = 0, \forall s \in \{1, \cdots, k\},$$

so that

$$\{\mathbf{v}_s \cdot \mathbf{v}_j\}\{\tilde{a}_j\} = \{\mathbf{v}_{k+1} \cdot \mathbf{v}_s\},$$

that is,

$$\{\tilde{a}_j\} = \{\mathbf{v}_s \cdot \mathbf{v}_j\}^{-1}\{\mathbf{v}_{k+1} \cdot \mathbf{v}_s\}.$$

Observe that

$$\begin{aligned} g_{k+1} &= \{\mathbf{v}_i \cdot \mathbf{v}_j\}_{i,j=1}^{k+1} \\ &= \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{v}_{k+1} \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{v}_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_{k+1} \cdot \mathbf{v}_1 & \mathbf{v}_{k+1} \cdot \mathbf{v}_2 & \dots & \mathbf{v}_{k+1} \cdot \mathbf{v}_{k+1} \end{vmatrix} \end{aligned} \tag{60}$$

so that,

$$g_{k+1} = \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \dots & \mathbf{v}_1 \cdot (\mathbf{v}_{k+1} - \sum_{j=1}^k \tilde{a}_j \mathbf{v}_j) \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \dots & \mathbf{v}_2 \cdot (\mathbf{v}_{k+1} - \sum_{j=1}^k \tilde{a}_j \mathbf{v}_j) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_{k+1} \cdot \mathbf{v}_1 & \mathbf{v}_{k+1} \cdot \mathbf{v}_2 & \dots & \mathbf{v}_{k+1} \cdot (\mathbf{v}_{k+1} - \sum_{j=1}^k \tilde{a}_j \mathbf{v}_j) \end{vmatrix} \tag{61}$$

that is,

$$g_{k+1} = \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \dots & 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_{k+1} \cdot \mathbf{v}_1 & \mathbf{v}_{k+1} \cdot \mathbf{v}_2 & \dots & \mathbf{v}_{k+1} \cdot (\mathbf{v}_{k+1} - \sum_{j=1}^k \tilde{a}_j \mathbf{v}_j) \end{vmatrix}. \tag{62}$$

From this, since

$$\mathbf{v}_s \cdot (\mathbf{v}_{k+1} - \sum_{j=1}^{k} \tilde{a}_j \mathbf{v}_j) = 0, \forall s \in \{1, .., k\},$$

we obtain,

$$g_{k+1} = \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \dots & 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_{k+1} \cdot \mathbf{v}_1 & \mathbf{v}_{k+1} \cdot \mathbf{v}_2 & \dots & (\mathbf{v}_{k+1} - \sum_{j=1}^k \tilde{a}_j \mathbf{v}_j) \cdot (\mathbf{v}_{k+1} - \sum_{j=1}^k \tilde{a}_j \mathbf{v}_j) \end{vmatrix}$$

so that

$$g_{k+1} = \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \dots & 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_{k+1} \cdot \mathbf{v}_1 & \mathbf{v}_{k+1} \cdot \mathbf{v}_2 & \dots & \tilde{\mathbf{v}}_{k+1} \cdot \tilde{\mathbf{v}}_{k+1} \end{vmatrix} \tag{63}$$

from which, calculating the determinant through the last column, we have,

$$g_{k+1} = g_k|\tilde{\mathbf{v}}_{k+1}|^2.$$

This completes the induction.

The proof is complete.

$\square$

Let us turn our attention to the problem of calculating the volume of a manifold.

Consider a m-dimensional C^1 class surface $M \subset \mathbb{R}^n$, where, $1 \leq m \leq n$ and

$$M = \{\mathbf{r}(\mathbf{u}) \ : \ \mathbf{u} \in D\}.$$

We define, the volume of M, denoted by V, by:

$$V = \int_M dM,$$

where dM will be specified in the next lines.

We may also denote,

$$V = \int_D dM(\mathbf{u}).$$

At this point we address the problem of finding $dM(\mathbf{u})$.

Let $\mathbf{u} = (u_1, \cdots, u_m) \in D$, and let $p = \mathbf{r}(\mathbf{u})$..

Consider the $m-$ dimensional elementary rectangle

$$(u_1, u_1 + \Delta u_1) \times (u_2, u_2 + \Delta u_2) \times \cdots \times (u_m, u_m + \Delta u_m).$$

We are going to obtain the corresponding approximate volume in the surface, which we denote by

$$\Delta M(u).$$

Observe that,

$$\begin{aligned}
\Delta \mathbf{r}_{u_j} &= \mathbf{r}(u_1, \cdots, u_j + \Delta u_j, \cdots, u_m) - \mathbf{r}(u_1, \cdots, u_j, \cdots, u_m) \\
&= \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_j} \Delta u_j + o(\Delta u_j) \\
&\approx \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_j} \Delta u_j, \qquad (64)
\end{aligned}$$

for Δu_j sufficiently small.

Hence, $\Delta M(\mathbf{u})$ is approximately defined by the set of vectors

$$\{\Delta \mathbf{r}_{u_1}, \cdots, \Delta \mathbf{r}_{u_m}\} \subset \mathbb{R}^n,$$

where, $\Delta \mathbf{r}_{u_j} \approx \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_j} \Delta u_j$, from the last theorem, we obtain,

$$\Delta M(\mathbf{u}) \approx \sqrt{g} \Delta u_1 \Delta u_2 \cdots \Delta u_m,$$

where

$$g = \det\{g_{ij}\}_{i,j=1}^{m},$$

and where

$$\mathbf{g}_i \equiv \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_j}, \forall i \in \{1, \cdots, m\},$$

and finally

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \ \forall i, j \in \{1, \cdots, m\}.$$

So, in its differential form, we could write,

$$dM(\mathbf{u}) = \sqrt{g}\ du_1 \cdots du_m,$$

so that up to local coordinates and concerning partition of unity, we have

$$\begin{aligned} \int_M dM &= \int_D dM(\mathbf{u}) \\ &= \int_D \sqrt{g}\ du_1 \cdots du_m. \end{aligned} \tag{65}$$

14 Change of variables, the general case

Let $D \subset \mathbb{R}^n$ be a compact block (or even a more general compact region analogous to a simple one in $\mathbb{R}^3$). Let $f : D \to \mathbb{R}$ be a continuous function.

Consider the integral I, where

$$I = \int_D f(\mathbf{x})\ d\mathbf{x} = \int_D f(x_1, \cdots, x_n)\ dx_1 \cdots dx_n.$$

Consider also the change in variables, given by the C^1 class functions $X_1, \cdots, X_n : D_0 \to \mathbb{R}$, where we denote

$$\mathbf{r}(\mathbf{u}) = X_1(\mathbf{u})\mathbf{e}_1 + \cdots + X_n(\mathbf{u})\mathbf{e}_n.$$

We assume $\mathbf{r} : D_0 \to D$ to be a bijection, where $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ denotes the canonical basis for $\mathbb{R}^n$ and

$$\mathbf{u} = (u_1, \cdots, u_n) \in D_0 \subset \mathbb{R}^n.$$

More specifically, the change of variables is given by,

$$x_1 = X_1(\mathbf{u}), \cdots, x_n = X_n(\mathbf{u}).$$

At this point we shall show that

$$\left|\det\left\{\frac{\partial X_i(\mathbf{u})}{\partial u_j}\right\}\right|^2 = g,$$

where

$$g = \det\{g_{ij}\}$$

and

$$g_{ij} = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i} \cdot \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_j},$$

so that

$$\begin{aligned} g_{ij} &= \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i} \cdot \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_j} \\ &= \sum_{k=1}^{n} \frac{\partial X_k}{\partial u_i} \frac{\partial X_k}{\partial u_j} \\ &= \sum_{k=1}^{n} (X_k)_{u_i} (X_k)_{u_j}. \end{aligned} \tag{66}$$

Observe that

$$\begin{aligned} &\left\{ \frac{\partial X_i(\mathbf{u})}{\partial u_j} \right\}^T \left\{ \frac{\partial X_i(\mathbf{u})}{\partial u_j} \right\} \\ &= \left\{ \begin{array}{cccc} (X_1)_{u_1} & (X_2)_{u_1} & \cdots & (X_n)_{u_1} \\ (X_1)_{u_2} & (X_2)_{u_2} & \cdots & (X_n)_{u_2} \\ \vdots & \vdots & \ddots & \vdots \\ (X_1)_{u_n} & (X_2)_{u_n} & \cdots & (X_n)_{u_n} \end{array} \right\} \cdot \left\{ \begin{array}{cccc} (X_1)_{u_1} & (X_1)_{u_2} & \cdots & (X_1)_{u_n} \\ (X_2)_{u_1} & (X_2)_{u_2} & \cdots & (X_2)_{u_n} \\ \vdots & \vdots & \ddots & \vdots \\ (X_n)_{u_1} & (X_n)_{u_2} & \cdots & (X_n)_{u_n} \end{array} \right\} \end{aligned} \tag{67}$$

so that

$$\begin{aligned} &\left\{ \frac{\partial X_i(\mathbf{u})}{\partial u_j} \right\}^T \left\{ \frac{\partial X_i(\mathbf{u})}{\partial u_j} \right\} \\ &= \left\{ \sum_{k=1}^{n} (X_k)_{u_i} (X_k)_{u_j} \right\} \\ &= \left\{ \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i} \cdot \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_j} \right\} \\ &= \{g_{ij}\}. \end{aligned} \tag{68}$$

Hence,

$$\left| \det \left\{ \frac{\partial X_i(\mathbf{u})}{\partial u_j} \right\} \right|^2 = \det\{g_{ij}\} = g.$$

Thus,

$$\left| \det \left\{ \frac{\partial X_i(\mathbf{u})}{\partial u_j} \right\} \right| = \sqrt{g},$$

so that, from the exposed in the last two sections, we have,

$$\begin{aligned} I &= \int_D f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \\ &= \int_{D_0} f(X_1(\mathbf{u}), \cdots, X_n(\mathbf{u}))\, \sqrt{g}\, du_1 \cdots du_n \\ &= \int_{D_0} f(X_1(\mathbf{u}), \cdots, X_n(\mathbf{u})) \left| \det \left\{ \frac{\partial X_i(\mathbf{u})}{\partial u_j} \right\} \right| du_1 \cdots du_n. \end{aligned} \tag{69}$$

15 The Stokes Theorem

In this subsection we present the Stokes theorem:

Teorema 15.1. *Let $M \subset \mathbb{R}^n$ be a C^1 class oriented compact m-dimensional surface with a boundary ∂M, where $1 \leq m \leq n$.*

Let $\omega = \sum_I \omega_I \; du_I$ be a (m-1)-form. Under such hypotheses,

$$\int_M d\omega = \int_{\partial M} \omega.$$

Demonstração. The proof follows from the discussion at section 11.

Indeed, denoting

$$\mathbf{s}_k = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_k} \; du_k, \; \forall k \in \{1, \cdots, m\},$$

and

$$\mathbf{S}_j = (\mathbf{s}_1, \cdots, \hat{\mathbf{s}}_j, \cdots, \mathbf{s}_m),$$

where $\hat{\mathbf{s}}_j$ means the absence of such a term in the concerning list of vectors, we may infer that the general form

$$\sum_{j=1}^m \sum_I (\omega_I(\mathbf{u}) du_I(\mathbf{u}))(\mathbf{S}_j)$$

would stand for:

$$\begin{aligned} & \sum_{j=1}^m (-1)^{j+1} h_j(\mathbf{u}) \; (du_1 \wedge \cdots \wedge \hat{du_j} \wedge \cdots \wedge du_m)(\mathbf{S}_j), \\ = & \sum_{j=1}^m (-1)^{j+1} h_j(\mathbf{u}) \; du_1 \cdots \hat{du_j} \cdots du_m, \end{aligned} \tag{70}$$

for appropriate C^1 class functions $h_j, \forall j \in \{1, \cdots, m\}$.

□

15.1 Recovering the classical results on vector calculus in $\mathbb{R}^3$ from the general Stokes Theorem

- Recovering the standard stokes Theorem in $\mathbb{R}^3$. Let $S \subset \mathbb{R}^3$ be a C^1 class surface with a boundary $C = \partial S$, where

  $$S = \{(x, y, z) \in \mathbb{R}^3 \; : \; z = z(x, y) \text{ and } (x, y) \in D\},$$

  where $D \subset \mathbb{R}^2$ is a simple region.

  Consider the form $\omega = P \; dx + Q \; dy + R \; dz$, where we denote $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{z}$, and where $P, Q, R : V \supset S \to \mathbb{R}$ are C^1 class scalar functions.

  From the Stokes Theorem 15.1 we have

$$\int_C \omega = \int_S d\omega,$$

that is,

$$\begin{aligned}
\int_C P\,dx + Q\,dy + R\,dz &= \int_S d\omega \\
&= \int_S (dP \wedge dx + dQ \wedge dy + dR \wedge dz) \\
&= \int_S \left(\frac{\partial P}{\partial dx}\,dx + \frac{\partial P}{\partial y}\,dy + \frac{\partial P}{\partial z}\,dz\right) \wedge dx \\
&\quad + \left(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy + \frac{\partial Q}{\partial z}\,dz\right) \wedge dy \\
&\quad + \left(\frac{\partial R}{\partial x}\,dx + \frac{\partial R}{\partial y}\,dy + \frac{\partial R}{\partial z}\,dz\right) \wedge dz \\
&= \int_S (Q_x - P_y)\,dx \wedge dy + \int_S (R_y - Q_z)\,dy \wedge dz \\
&\quad + (P_z - R_x)\,dz \wedge dx. \qquad (71)
\end{aligned}$$

We recall that in S, $z = z(x, y)$, so that, in the context of informal calculus language, we have

$$dz = z_x\,dx + z_y\,dy.$$

Also,

$$dx \wedge dy = dxdy,$$

$$dy \wedge dz = dy \wedge (z_x dx + z_y dy) = -z_x dxdy,$$

$$dz \wedge dx = (z_x dx + z_y dy) \wedge dx = -z_y dxdy,$$

so that from (72), we obtain,

$$\begin{aligned}
&\int_C P\,dx + Q\,dy + R\,dz \\
= &\int_D (Q_x - P_y)\,dxdy + \int_D (R_y - Q_z)\,(-z_x)\,dxdy \wedge dz \\
&+ (P_z - R_x)\,(-z_y)\,dxdy \\
= &\int_D (R_y - Q_z)\,(-z_x)\,dxdy + (P_z - R_x)\,(-z_y)\,dxdy + \int_D (Q_x - P_y)\,dxdy \\
= &\int_D curl(\mathbf{F}) \cdot [-z_x\mathbf{i} - z_{\mathbf{y}}\mathbf{j} + k]\,dxdy \\
= &\int_D curl(\mathbf{F}) \cdot \frac{[-z_x\mathbf{i} - z_{\mathbf{y}}\mathbf{j} + k]}{\sqrt{z_x^2 + z_y^2 + 1}} \sqrt{z_x^2 + z_y^2 + 1}\,dxdy \\
= &\int_D curl(\mathbf{F}) \cdot \mathbf{n} \sqrt{z_x^2 + z_y^2 + 1}\,dxdy \\
= &\int_S curl(\mathbf{F}) \cdot \mathbf{n}\,dS, \qquad (72)
\end{aligned}$$

where $\mathbf{n}$ is unit outward normal relating S.

So, to summarize we have obtained,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S curl(\mathbf{F}) \cdot \mathbf{n}\, dS,$$

which is the standard Stokes Theorem result in $\mathbb{R}^3$.

Observação 15.2. *We have used some informality here.*
In fact, in a more rigorous fashion, we should have:

$$\mathbf{r}(x, y) = (x, y, z(x, y)),$$

and

$$\begin{aligned} & (dx \wedge dy)(\mathbf{r}_x(x,y)\; dx, \mathbf{r}_y(x,y)\; dy) \\ = \; & (dx \wedge dy)((1,0,z_x)\; dx, (0,1,z_y)\; dy) \\ = \; & dx\; dy, \end{aligned} \tag{73}$$

and with more details,

$$\begin{aligned} & (dy \wedge dz)(\mathbf{r}_x(x,y)\; dx,\; \mathbf{r}_y(x,y)\; dy) \\ = \; & (dy \wedge dz)((1,0,z_x)\; dx, (0,1,z_y)\; dy) \\ = \; & (dy \wedge (z_x dx + z_y dy))((1,0,z_x)\; dx,\; (0,1,z_y)\; dy) \\ = \; & z_x(dy \wedge dx)((1,0,z_x)\; dx, (0,1,z_y)\; dy) \\ & + z_y(dy \wedge dy)((1,0,z_x)\; dx, (0,1,z_y)\; dy) \\ = \; & z_x \begin{vmatrix} dy(dx, 0, z_x dx) & dy(0, dy, z_y\; dy) \\ dx(dx, 0, z_x dx) & dx(0, dy, z_y\; dy) \end{vmatrix} \\ = \; & z_x \begin{vmatrix} 0 & dy \\ dx & 0 \end{vmatrix} \\ = \; & -z_x dx dy. \end{aligned} \tag{74}$$

A similar remark is valid for

$$dz \wedge dx.$$

- Recovering the Divergence Theorem in $\mathbb{R}^3$:
  Let $V \subset \mathbb{R}^3$ be a simple volume with a boundary $S = \partial V$. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ where $P, Q, R : V \to \mathbb{R}$ are C^1 class scalar functions.
  From the last item, we have

$$(dx \wedge dy)((1,0,z_x)\; dx, (0,1,z_y)\; dy) = dxdy,$$

$$(dy \wedge dz)(((1,0,z_x)\; dx, (0,1,z_y)\; dy)) = -z_x dxdy,$$

$$(dz \wedge dx)(((1,0,z_x)\; dx, (0,1,z_y)\; dy)) = -z_y dxdy,$$

so that,

$$\begin{aligned}
& (P\ dy \wedge dz + Q dz \wedge dx + R dx \wedge dy)((dx,0),(0,dy)) \\
= \ & P(-z_x)\ dxdy + Q(-z_y)\ dxdy + R\ dxdy \\
= \ & (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (-z_x\mathbf{i} - z_y\mathbf{j} + \mathbf{k})\ dxdy \\
= \ & \mathbf{F} \cdot \frac{(-z_x\mathbf{i} - z_y\mathbf{j} + \mathbf{k})}{\sqrt{z_x^2 + z_y^2 + 1}} \sqrt{z_x^2 + z_y^2 + 1}\ dxdy \\
= \ & \mathbf{F} \cdot \mathbf{n} \sqrt{z_x^2 + z_y^2 + 1}\ dxdy \\
= \ & \mathbf{F} \cdot \mathbf{n}\ dS, \qquad (75)
\end{aligned}$$

where

$$\mathbf{n} = \frac{(-z_x\mathbf{i} - z_y\mathbf{j} + \mathbf{k})}{\sqrt{z_x^2 + z_y^2 + 1}},$$

is the unit outward normal relating S and

$$dS = \sqrt{z_x^2 + z_y^2 + 1}\ dxdy.$$

Consider, with some informality here, the form

$$\omega = P\ dy \wedge dz + Q dz \wedge dx + R dx \wedge dy = \mathbf{F} \cdot \mathbf{n}\ dS.$$

From the Stokes Theorem 15.1, we have

$$\int_S \omega = \int_V d\omega.$$

Observe that

$$\begin{aligned}
\int_V d\omega \ = \ & \int_V (dP \wedge (dy \wedge dz)) + (dQ \wedge (dz \wedge dx)) + (dR \wedge (dx \wedge dy)) \\
= \ & \int_V \left(\frac{\partial P}{\partial x}\ dx + \frac{\partial P}{\partial y}\ dy + \frac{\partial P}{\partial z}\ dz \right) \wedge (dy \wedge dz) \\
& + \left(\frac{\partial Q}{\partial x}\ dx + \frac{\partial Q}{\partial y}\ dy + \frac{\partial Q}{\partial z}\ dz \right) \wedge (dz \wedge dx) \\
& + \left(\frac{\partial R}{\partial x}\ dx + \frac{\partial R}{\partial y}\ dy + \frac{\partial R}{\partial z}\ dz \right) \wedge (dx \wedge dy) \\
= \ & \int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)\ dx \wedge dy \wedge dz \\
= \ & \int_V div(\mathbf{F})\ dx\ dy\ dz. \qquad (76)
\end{aligned}$$

Joining the pieces we obtain

$$\begin{aligned}\int_S \omega &= \int_S (\mathbf{F}\cdot\mathbf{n})\, dS \\ &= \int_S P\, dy\wedge dz + Qdz\wedge dx + Rdx\wedge dy \\ &= \int_V div(\mathbf{F})\, dx\, dy\, dz, \end{aligned} \tag{77}$$

which is the standard Gauss Divergence Theorem.

Exercícios 15.3.

1. *Find the domain of the one variable vectorial functions* $\mathbf{r}$ *below indicated.*

   (a)
   $$\mathbf{r}(t) = \frac{1}{t^2+1}\mathbf{i} + \sqrt{(t-1)(t+3)}\mathbf{j},$$

   (b)
   $$\mathbf{r}(t) = \ln(t^2-16)\mathbf{i} + \sqrt{t^2+2t-15}\mathbf{j} + \tan(t+1)\mathbf{k},$$

   (c)
   $$\mathbf{r}(t) = \sqrt{25-t^2}\mathbf{i} + \sqrt{t^2+2t-8}\mathbf{j}.$$

2. *Through the formula*
   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$
   calculate the derivatives of the functions defined by the parametric equations indicated,

   (a)
   $$\mathbf{r}(t) = \frac{e^t}{1+e^t}\mathbf{i} + t^2\ln(t)\mathbf{j},$$

   (b)
   $$\mathbf{r}(t) = \frac{\cos(t)}{5+\sin(t)}\mathbf{i} + \ln(\sqrt{t^4+t^2})\mathbf{j}.$$

3. *Let* $\mathbf{r} : \mathbb{R}\setminus\{-1\} \to \mathbb{R}^2$ *be defined by*
   $$\mathbf{r}(t) = \frac{t}{t+1}\mathbf{i} + \ln(t^2+1)\mathbf{j}.$$
   Find the equation of the tangent line to the graph of the curve defined by $\mathbf{r}$ *at the point corresponding to* $t = 1$.

4. *Let* $\mathbf{r}, \mathbf{s} : \mathbb{R} \to \mathbb{R}^2$ *be defined by*
   $$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$
   and
   $$\mathbf{s}(t) = (t^2+t)\mathbf{i} + t^3\mathbf{j}.$$
   Calculate the angle between $\mathbf{r}'(t)$ *and* $\mathbf{s}'(t)$ *at the point corresponding to* $t = 1$.

5. *Let* $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ *be defined by*

$$\mathbf{r}(t) = \frac{2t}{1+t^2}\mathbf{i} + \frac{1-t^2}{1+t^2}\mathbf{j} + \mathbf{k}.$$

Show that the angle between $\mathbf{r}(t)$ *and* $\mathbf{r}'(t)$ *is constant.*

6. *A vectorial function* $\mathbf{r}$ *satisfies the equation,*

$$t\mathbf{r}'(t) = \mathbf{r}(t) + t\mathbf{A},\ \forall t > 0$$

where

$$\mathbf{A} \in \mathbb{R}^3.$$

Suppose that $\mathbf{r}(1) = 2\mathbf{A}$. *Calculate* $\mathbf{r}''(1)$ *and* $\mathbf{r}(3)$ *as functions of* $\mathbf{A}$.

7. *Find a function* $\mathbf{r} : (0, +\infty) \to \mathbb{R}^3$ *such that*

$$\mathbf{r}(x) = xe^x\mathbf{A} + \frac{1}{x}\int_1^x \mathbf{r}(t)\ dt.$$

where $\mathbf{A} \in \mathbb{R}^3$, $\mathbf{A} \neq \mathbf{0}$.

8. *Through the Green Theorem, calculate the areas of the regions* D, *where,*

(a)

$$D = \{(x,y) \in \mathbb{R}^2\ :\ x^2 + y^2 \leq 1 \text{ and } y \geq 1/2\}.$$

(b)

$$D = \{(x,y) \in \mathbb{R}^2\ :\ x^2 + y^2 \leq 1 \text{ and } -1/2 \leq y \leq \sqrt{3}/2\}.$$

(c)

$$D = \{(x,y) \in \mathbb{R}^2\ :\ x^2 + y^2 \leq 1 \text{ and } 0 \leq x \leq 1/2\}.$$

9. *Calculate the area of surface* S, *where*

$$S = \left\{(x,y,z) \in \mathbb{R}^3\ :\ x^2 + y^2 + z^2 = 1 \text{ and } \frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2}\right\}.$$

10. *Calculate the area of surface* S, *where*

$$S = \left\{(x,y,z) \in \mathbb{R}^3\ :\ x^2 + y^2 + z^2 = 1 \text{ and } \frac{-\sqrt{3}}{2} \leq z \leq \frac{1}{2}\right\}.$$

11. *Calculate the area of surface* S, *where*

$$S = \left\{(x,y,z) \in \mathbb{R}^3\ :\ z^2 = x^2 + y^2 \text{ and } x^2 + y^2 \leq 2ax\right\},$$

where $a \in \mathbb{R}$.

12. *Calculate* $I = \int\int_S x\ dS$, *where*

$$S = \left\{(x,y,z) \in \mathbb{R}^3\ :\ x^2 + y^2 = R^2 \text{ and } |z| \leq 1\right\}.$$

13. *Through the Divergence Theorem, calculate* $I = \int\int_S (y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n}\, dS$, *where*

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \ : \ x = \sqrt{R^2 - y^2 - z^2} \text{ and } x \geq \frac{\sqrt{3}R}{2} \right\},$$

where $R > 0$.

14. *Through the Divergence Theorem, calculate* $I = \int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$ *where*

$$S = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 2R_0 x \text{ and } z \geq 0\}$$

and where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ *e* $R_0 > 0$.

15. *Let* $u : V \to \mathbb{R}$ *be a scalar field and let* $\mathbf{F} : V \to \mathbb{R}^3$ *be a vectorial one, where* $V \subset \mathbb{R}^3$ *is open* $u, \mathbf{F}$ *are of* C^1 *class. Show that*

$$div(u\mathbf{F}) = (\nabla u) \cdot \mathbf{F} + u\,(div\mathbf{F}).$$

16. *Let* $u, v : V \to \mathbb{R}$ *be* C^2 *class scalar fields, where* $V \subset \mathbb{R}^3$ *is open and its closure is simple. Defining*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

show that $div(\nabla u) = \nabla^2 u$ *and prove the Green identities,*

(a)

$$\int\int\int_V (v\nabla^2 u + \nabla v \cdot \nabla u)\, dV = \int\int_S v(\nabla u \cdot \mathbf{n})\, dS$$

where $S = \partial V$ *(that is,* S *is the boundary of* V*.)*

(b)

$$\int\int\int_V (v\nabla^2 u - u\nabla^2 v)\, dV = \int\int_S \left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}} \right)\, dS,$$

where $S = \partial V$ *and* $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$.

17. *Let* $u : V \to \mathbb{R}$, $\mathbf{F} : V \to \mathbb{R}^3$ *be* C^2 *class fields on the open set* $V \subset \mathbb{R}^3$.
Prove that $curl(\nabla u) = \mathbf{0}$ *and* $div(curl(\mathbf{F})) = 0$, *on* V.

18. *Let* $D \subset \mathbb{R}^2$ *be a simple region. Let* $u, v \in C(\overline{D}) \cap C^1(D)$. *Prove that*

$$\int\int_D uv_x\, dxdy = \int_{\partial D} uv\, dy - \int\int_D u_x v\, dxdy,$$

and

$$\int\int_D uv_y\, dxdy = \int_{\partial D} uv\, dx - \int\int_D u_y v\, dxdy,$$

19. *Let* $V \subset \mathbb{R}^3$ *be an open region bounded by a closed surface of* C^1 *class. Through the first Green identity, prove the uniqueness of solution of the Dirichlet problem,*

$$\begin{cases} \nabla^2 u = f, & in V \\ u = u_0, & on\ \partial V, \end{cases} \tag{78}$$

where $f : V \to \mathbb{R}$ *is continuous and* $u_0 : \partial V \to \mathbb{R}$ *is also continuous.*
Also through the first Green identity, prove that

(a) *for the Neumann problem*

$$\begin{cases} \nabla^2 u = f, & \text{in } V \\ \frac{\partial u}{\partial \mathbf{n}} = u_0, & \text{on } \partial V, \end{cases} \tag{79}$$

to have a solution, it is necessary that

$$\int\int\int_V f \, dxdydz = \int\int_{\partial V} u_0 \, dS.$$

Hint: Consider $v \equiv 1$ *in the first Green identity.*

(b) *Prove that any two solutions of the Neumann problem differ by a constant.*

20. *Let* $V \subset \mathbb{R}^3$ *be a simple region. Let* $\mathbf{F} : V \to \mathbb{R}^3$ *be a vectorial field of* C^1 *class. Let* $\mathbf{x}_0 \in V^\circ$. *Show that*

$$div(\mathbf{F}(\mathbf{x}_0)) = \lim_{r \to 0} \frac{\int\int_{\partial B_r(\mathbf{x}_0)} \mathbf{F} \cdot \mathbf{n} \, dS}{Vol(B_r(\mathbf{x}_0))},$$

where $\mathbf{n}$ *denotes unit outward normal field to* $B_r(\mathbf{x}_0)$.

21. *Let* $V \subset \mathbb{R}^3$ *be a simple region. Let* $f : V \to \mathbb{R}$ *be a scalar field of* C^2 *class. Let* $\mathbf{x}_0 \in V^\circ$. *Through the first Green identity, show that*

$$\nabla^2 f(\mathbf{x}_0) = \lim_{r \to 0} \frac{\int\int_{\partial B_r(\mathbf{x}_0)} \frac{\partial f}{\partial \mathbf{n}} \, dS}{Vol(B_r(\mathbf{x}_0))},$$

where $\mathbf{n}$ *denotes the unit outward normal field to* $B_r(\mathbf{x}_0)$.

22. *Let* $M \subset \mathbb{R}^n$ *be a m-dimensional surface of* C^2 *class, where* $1 \leq m < n$. *Let* $X, Y, Z \in \tilde{\mathcal{X}}(M)$, *where* $\tilde{\mathcal{X}}(M)$ *denotes the set of tangential vector fields of* C^∞ *class defined on* M.
Show that

(a) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z], \ \forall \alpha, \beta \in \mathbb{R}$,

(b) $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z], \ \forall \alpha, \beta \in \mathbb{R}$.

(c) *Anti-symmetry:*

$$[X, Y] = -[Y, X],$$

(d) *Jacob Identity:*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

(e) *Leibnitz rule:*

$$[fX, gY] = fg[X, Y] + f(X \cdot g)Y - g(Y \cdot f)X, \ \forall f, g \in C^2(M).$$

(f) *Recalling that* $L_X Y = [X, Y]$, *show that*

i.

$$L_X[Y, Z] = [L_X Y, Z] + [Y, L_X Z],$$

ii.

$$L_X(L_Y)Z - L_Y(L_X)Z = L_{[X,Y]}Z.$$

23. *Consider a 3-dimensional surface $M \subset \mathbb{R}^4$ defined by*

$$M = \{(x, y, z, w) \in \mathbb{R}^4 \ : \ x^2 + y^2 + z^2 + \ln(w) = 1\}.$$

(a) *Defining $\mathbf{u} = (u_1, u_2, u_3) = (x, y, z)$, write M in the form,*

$$M = \{\mathbf{r}(\mathbf{u}) \in \mathbb{R}^4 \ : \ \mathbf{u} \in \mathbb{R}^3\}.$$

(b) *Let $p = \mathbf{r}(\mathbf{u})$. Obtain the tangent space and equation of the hyper-plan tangent to M at p.*

(c) *For $f : M \to \mathbb{R}$, $X, Y \in \mathcal{X}(M)$ such that*

$$f(\mathbf{r}(\mathbf{u})) = x^2 + e^{x^2 y} + (\sin((x^2 + z^2))^3 + w(x, y, z),$$

$$X(x, y, z) = e^x \frac{\partial \mathbf{r}(x, y, z)}{\partial x} + y \frac{\partial \mathbf{r}(x, y, z)}{\partial y} + (x + z^2)^2 \frac{\partial \mathbf{r}(x, y, z)}{\partial z},$$

and

$$Y(x, y, z) = (\sin(xy))^2 \frac{\partial \mathbf{r}(x, y, z)}{\partial x} + (\cos(x^2 + y^2))^3 \frac{\partial \mathbf{r}(x, y, z)}{\partial y} + e^{x+z^2} \frac{\partial \mathbf{r}(x, y, z)}{\partial z},$$

for $p = \mathbf{r}(x_0, y_0, z_0)$, calculate

i. $(X \cdot f)(p)$,

ii. $(D_X Y)(p)$,

iii. $[X, Y](p)$,

iv. $([X, Y] \cdot f)(p)$

v. *Compute numerically the results obtained the 4 last items at the point $p_0 = \mathbf{r}(x_0, y_0, z_0) = \mathbf{r}(\pi, 0, 1)$.*

(d) *Let $Z \in \mathcal{X}(M)$, where*

$$Z(x, y, z) = (x + y + z) \frac{\partial \mathbf{r}(x, y, z)}{\partial x} + (2x + y + z) \frac{\partial \mathbf{r}(x, y, z)}{\partial y} + (-y + z) \frac{\partial \mathbf{r}(x, y, z)}{\partial z}.$$

Obtain the integral curve $r(\mathbf{u}(t))$ of Z, such that $\mathbf{u}(0) = (1, -1, 0)$.

24. *Obtain the differential $dM(x, y, z)$ to calculate the area of the surface $M \subset \mathbb{R}^4$, where*

$$M = \{(x, y, z, w) \in \mathbb{R}^4 \ : \ e^w = [\sin(x^2 + y)]^3 + z^2 + 5 \text{ and } x^2 + y^2 + z^2 \leq 1\}.$$

25. *Consider the surface $M \subset \mathbb{R}^4$ defined by*

$$M = \{(x, y, z) \in \mathbb{R}^4 \ : \ e^w - x^2 - y^2 - z^2 = 1\}.$$

Write its equation in the form,

$$M = \{\mathbf{r}(x, y, z) \ : \ (x, y, z) \in \mathbb{R}^3\},$$

where

$$\mathbf{r}(x, y, z) = X_1(x, y, z)\mathbf{e}_1 + \cdots + X_4(x, y, z)\mathbf{e}_4.$$

Let

$$dX_1 = \frac{\partial X_1}{\partial x}\, dx + \frac{\partial X_1}{\partial y}\, dy + \frac{\partial X_1}{\partial z}\, dz,$$

and

$$dX_4 = \frac{\partial X_4}{\partial x}\, dx + \frac{\partial X_4}{\partial y}\, dy + \frac{\partial X_4}{\partial z}\, dz.$$

(a) Calculate

$$(dX_1 \wedge dX_4)(\mathbf{s}_1, \mathbf{s}_2),$$

where

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(x, y, z)}{\partial x}\, \Delta x,$$

and

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(x, y, z)}{\partial y}\, \Delta y,$$

and where $\Delta x,\ \Delta y \in \mathbb{R}$.

(b) Consider the differential form

$$\omega = (w(x, y, z) + x^2 y + z) dX_1 \wedge dX_4 + (w(x, y, z)^2 + \sin(x^2 + y) - z^2) dX_1 \wedge dX_2,$$

where

$$dX_2 = \frac{\partial X_2}{\partial x}\, dx + \frac{\partial X_2}{\partial y}\, dy + \frac{\partial X_2}{\partial z}\, dz.$$

Obtain the exterior differential $d\omega$ *of* ω *at* $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$, *where*

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(x, y, z)}{\partial x}\, \Delta x,$$

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(x, y, z)}{\partial y}\, \Delta y,$$

and

$$\mathbf{s}_3 = \frac{\partial \mathbf{r}(x, y, z)}{\partial z}\, \Delta z,$$

and where $\Delta x,\ \Delta y,\ \Delta z \in \mathbb{R}$.

26. *Let* $M \subset \mathbb{R}^n$ *be a 3-dimensional* C^1 *class surface, where* $n \geq 4$,

$$M = \{\mathbf{r}(\mathbf{u}) = X_i(\mathbf{u})\mathbf{e}_i \ : \ \mathbf{u} \in D\},$$

$D \subset \mathbb{R}^3$ *and* $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ *is the canonical basis for* $\mathbb{R}^n$,
Let $\omega = dX_1 \wedge dX_4 \wedge dX_3$ *be a 3-form on* M, *where,*

$$dX_1(\mathbf{u}) = \frac{\partial X_1(\mathbf{u})}{\partial u_1}\, du_1 + \frac{\partial X_1(\mathbf{u})}{\partial u_2}\, du_2 + \frac{\partial X_1(\mathbf{u})}{\partial u_3}\, du_3,$$

$$dX_4(\mathbf{u}) = \frac{\partial X_4(\mathbf{u})}{\partial u_1}\, du_1 + \frac{\partial X_4(\mathbf{u})}{\partial u_2}\, du_2 + \frac{\partial X_4(\mathbf{u})}{\partial u_3}\, du_3,$$

and

$$dX_3(\mathbf{u}) = \frac{\partial X_3(\mathbf{u})}{\partial u_1}\, du_1 + \frac{\partial X_3(\mathbf{u})}{\partial u_2}\, du_2 + \frac{\partial X_3(\mathbf{u})}{\partial u_3}\, du_3.$$

Compute

$$(dX_1(\mathbf{u}) \wedge dX_4(\mathbf{u}) \wedge dX_3(\mathbf{u}))(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3),$$

where

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1}\, \Delta u_1,$$

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_2}\, \Delta u_2$$

and

$$\mathbf{s}_3 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_3}\, \Delta u_3.$$

27. *Consider the vectorial field* $F : \mathbb{R}^3 \to \mathbb{R}^3$ *where* $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + (z - x^2)\mathbf{k}$.
Through the Stokes Theorem, calculate

$$I = \int\int_S curl(\mathbf{F}) \cdot \mathbf{n}\, dS$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 \; : \; z = 8 - x^2 - 2y^2 \text{ and } 2 \leq z \leq 4\}.$$

28. *Consider the vectorial field* $F : \mathbb{R}^3 \to \mathbb{R}^3$ *where* $\mathbf{F} = y\mathbf{i} + y\mathbf{j} + 5\mathbf{k}$.
Through the Stokes theorem, calculate

$$I = \int\int_S curl(\mathbf{F}) \cdot \mathbf{n}\, dS$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 \; : \; z = 16 - x^2 - 3y^2 \text{ and } z \geq y^2 + 2x + y\}.$$

29. *Let* $D \subset \mathbb{R}^n$ *be an open set and let* $f : D \to \mathbb{R}$ *be a function of* C^1 *class (therefore differentiable on* D*).*
Let $\mathbf{x}_0 \in D$ *and* $\varepsilon > 0$*. Prove that there exists* $\delta > 0$ *such that if* $\mathbf{x} \in D$ *and* $|\mathbf{x} - \mathbf{x_0}| < \delta$*, then*

   (a)

   $$f(\mathbf{x}) - f(\mathbf{x}_0) = f'(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + r(\mathbf{x}),$$

   where

   $$|r(\mathbf{x})| \leq \varepsilon |\mathbf{x} - \mathbf{x}_0|.$$

   (b) *Use the previous item to show that if* $\mathbf{x},\ \mathbf{y} \in D$, $|\mathbf{x} - \mathbf{x_0}| < \delta$ *and* $|\mathbf{y} - \mathbf{x_0}| < \delta$ *then*

   $$f(\mathbf{y}) - f(\mathbf{x}) = f'(\mathbf{x}_0) \cdot (\mathbf{y} - \mathbf{x}) + r_1(\mathbf{x}, \mathbf{y}),$$

   where

   $$|r_1(\mathbf{x}, \mathbf{y})| \leq 2\varepsilon\delta.$$

30. *Let $M \subset \mathbb{R}^n$ be a m-dimensional surface of C^1 class, where $1 \le m < n$, where*

$$M = \{\mathbf{r}(\mathbf{u}) \;:\; \mathbf{u} \in D \subset \mathbb{R}^m\}.$$

Let $f \in C^2(M)$ and $X, Y \in \mathcal{X}(M)$.
In this chapter, we have denoted

$$X \cdot f = df(X) = \frac{\partial (f \circ \mathbf{r})(\mathbf{u})}{\partial u_i} X_i(\mathbf{u}).$$

(a) *Calculate*

$$X \cdot (Y \cdot f).$$

(b) *Show that*

$$X \cdot (Y \cdot f) - Y \cdot (X \cdot f) = [X, Y] \cdot f,$$

where

$$[X, Y] = (dY_i(X) - dX_i(Y)) \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_i}.$$

(c) *Consider the 3-dimensional manifold $M \subset \mathbb{R}^4$ defined by*

$$M = \{(x, y, z, w) \in \mathbb{R}^4 \;:\; e^w - x^6 - y^2 - z^4 = 5\}.$$

i. *Defining $\mathbf{u} = (u_1, u_2, u_3) = (x, y, z)$, write M in the form,*

$$M = \{\mathbf{r}(\mathbf{u}) \in \mathbb{R}^4 \;:\; \mathbf{u} \in \mathbb{R}^3\}.$$

ii. *For $f : M \to \mathbb{R}$, $X, Y \in \mathcal{X}(M)$ such that*

$$f(\mathbf{r}(\mathbf{u})) = 5y^2 + w(x, y, z),$$

$$X(x, y, z) = \cos(x - y) \frac{\partial \mathbf{r}(x, y, z)}{\partial x} + y^3 \frac{\partial \mathbf{r}(x, y, z)}{\partial y} + (x + z^2)^3 \frac{\partial \mathbf{r}(x, y, z)}{\partial z},$$

and

$$Y(x, y, z) = e^x \frac{\partial \mathbf{r}(x, y, z)}{\partial x} + (\sin(x^2 + y^3))^4 \frac{\partial \mathbf{r}(x, y, z)}{\partial y} + e^{x^3 z} \frac{\partial \mathbf{r}(x, y, z)}{\partial z},$$

for $p = \mathbf{r}(x, y, z)$, calculate

$$([X, Y] \cdot f)(p)$$

31. *Consider a 3-dimensional surface $M \subset \mathbb{R}^4$ defined by*

$$M = \{(x, y, z) \in \mathbb{R}^4 \;:\; \ln(w) - x + 2y^2 - z^3 = 1\}.$$

Write the equation of M in the form,

$$M = \{\mathbf{r}(x, y, z) \;:\; (x, y, z) \in \mathbb{R}^3\},$$

where

$$\mathbf{r}(x, y, z) = X_1(x, y, z)\mathbf{e}_1 + \cdots + X_4(x, y, z)\mathbf{e}_4,$$

(a) Obtain $(dX_1 \wedge dX_2 \wedge dX_4)(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$.

(b) Consider the differential form

$$\omega = e^{x^2+y^5}\, dX_2 + \sin(xy^2)\, dX_1.$$

Obtain the exterior differential $d\omega$ *of* ω *at* $(\mathbf{s}_1, \mathbf{s}_2)$, *where for the last two sub-items,*

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(x,y,z)}{\partial x}\, \Delta x,$$

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(x,y,z)}{\partial y}\, \Delta y,$$

and

$$\mathbf{s}_3 = \frac{\partial \mathbf{r}(x,y,z)}{\partial z}\, \Delta z,$$

and where $\Delta x,\ \Delta y,\ \Delta z \in \mathbb{R}$.

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