# PAM - H - Cálculo - 3 - Nona Lista de Exercícios

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1. Let $\mathbf{r} : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}^2$ be defined by
$$\mathbf{r}(t) = \frac{t}{t+1}\mathbf{i} + \ln(t^2+1)\mathbf{j}.$$
Find the equation of the tangent line to the graph of the curve defined by $\mathbf{r}$ at the point corresponding to $t = 1$.

2. Let $\mathbf{r}, \mathbf{s} : \mathbb{R} \rightarrow \mathbb{R}^2$ be defined by
$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$
and
$$\mathbf{s}(t) = (t^2 + t)\mathbf{i} + t^3\mathbf{j}.$$
Calculate the angle between $\mathbf{r}'(t)$ and $\mathbf{s}'(t)$ at the point corresponding to $t = 1$.

3. Let $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3$ be defined by
$$\mathbf{r}(t) = \frac{2t}{1+t^2}\mathbf{i} + \frac{1-t^2}{1+t^2}\mathbf{j} + \mathbf{k}.$$
Show that the angle between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ is constant.

4. Let $\mathbf{s} : [a, b] \rightarrow \mathbb{R}^3$ be a three times differentiable function.
Let $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ be defined by
$$\mathbf{r}(t) = \mathbf{s}(t) \times \mathbf{s}'(t).$$
Find
$$\mathbf{r}''(t)$$
on $[a, b]$.

5. Let $\mathbf{s} : [a, b] \rightarrow \mathbb{R}^3$ be a three times differentiable function.
Let $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ be defined by
$$\mathbf{r}(t) = \mathbf{s}(t) \cdot (\mathbf{s}'(t) \times \mathbf{s}''(t)).$$
Find
$$\mathbf{r}'(t)$$
on $[a, b]$.

6. A vectorial function $\mathbf{r}$ satisfies the equation,
$$t\mathbf{r}'(t) = \mathbf{r}(t) + t\mathbf{A},\ \forall t > 0$$
where
$$\mathbf{A} \in \mathbb{R}^3.$$
Suppose that $\mathbf{r}(1) = 2\mathbf{A}$. Calculate $\mathbf{r}''(1)$ and $\mathbf{r}(3)$ as functions of $\mathbf{A}$.

7. Find a function $\mathbf{r} : (0, +\infty) \to \mathbb{R}^3$ such that

$$\mathbf{r}(x) = xe^x \mathbf{A} + \frac{1}{x} \int_1^x \mathbf{r}(t)\, dt.$$

where $\mathbf{A} \in \mathbb{R}^3$, $\mathbf{A} \neq \mathbf{0}$.

8. Calculate the area of surface $S$, where

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1 \text{ and } \frac{1}{2} \leq z \leq \frac{\sqrt{3}}{2} \right\}.$$

9. Calculate the area of surface $S$, where

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1 \text{ and } \frac{-\sqrt{3}}{2} \leq z \leq \frac{1}{2} \right\}.$$

10. Caculate the area of surface $S$, where

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ z^2 = x^2 + y^2 \text{ and } x^2 + y^2 \leq 2ax \right\},$$

where $a \in \mathbb{R}$.

11. Calculate $I = \int\int_S x\, dS$, where

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = R^2 \text{ and } |z| \leq 1 \right\}.$$

12. Through the Divergence Theorem, calculate $I = \int\int_S (y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n}\, dS$, where

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ x = \sqrt{R^2 - y^2 - z^2} \text{ and } x \geq \frac{\sqrt{3}R}{2} \right\},$$

where $R > 0$.

13. Through the Divergence Theorem, calculate $I = \int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$ where

$$S = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 2R_0 x \text{ and } z \geq 0\}$$

and where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ e $R_0 > 0$.

14. Let $u : V \to \mathbb{R}$ be a scalar field and let $\mathbf{F} : V \to \mathbb{R}^3$ be a vectorial one, where $V \subset \mathbb{R}^3$ is open $u, \mathbf{F}$ are of $C^1$ class. Show that

$$div(u\mathbf{F}) = (\nabla u) \cdot \mathbf{F} + u\ (div\mathbf{F}).$$

15. Let $u, v : V \to \mathbb{R}$ be $C^2$ class scalar fields, where $V \subset \mathbb{R}^3$ is open and its closure is simple. Defining

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

show that $div(\nabla u) = \nabla^2 u$ and prove the Green identities,

(a)

$$\int\int\int_V (v\nabla^2 u + \nabla v \cdot \nabla u)\, dV = \int\int_S v(\nabla u \cdot \mathbf{n})\, dS$$

where $S = \partial V$ (that is, $S$ is the boundary of $V$.)

(b)

$$\int\int\int_V (v\nabla^2 u - u\nabla^2 v)\, dV = \int\int_S \left( v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}} \right)\, dS,$$

where $S = \partial V$ and $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$.

16. Let $u : V \to \mathbb{R}$, $\mathbf{F} : V \to \mathbb{R}^3$ be $C^2$ class fields on the open set $V \subset \mathbb{R}^3$.

Prove that $curl(\nabla u) = \mathbf{0}$ and $div(curl(\mathbf{F})) = 0,$ on $V$.

17. Seja $D \subset \mathbb{R}^2$ uma região simples. Sejam $u, v \in C(\overline{D}) \cap C^1(D)$. Prove que

$$\int\int_D uv_x\, dxdy = \int_{\partial D} uv\, dy - \int\int_D u_x v\, dxdy,$$

e

$$\int\int_D uv_y\, dxdy = \int_{\partial D} uv\, dx - \int\int_D u_y v\, dxdy,$$

18. Seja $V \subset \mathbb{R}^3$ uma região aberta limitada por uma superfície de classe $C^1$ fechada. Utilize a primeira identidade de Green para provar a unicidade da solução do problema de Dirichlet,

$$\begin{cases} \nabla^2 u = f, & \text{em } V \\ u = u_0, & \text{em } \partial V, \end{cases} \tag{1}$$

onde $f : V \to \mathbb{R}$ é contínua e $u_0 : \partial V \to \mathbb{R}$ é contínua.

Também utilizando a primeira identidade de Green, prove que:

(a) para o prolema de Neumann

$$\begin{cases} \nabla^2 u = f, & \text{em } V \\ \frac{\partial u}{\partial \mathbf{n}} = u_0, & \text{em } \partial V, \end{cases} \tag{2}$$

ter uma solução, é necessário que

$$\int\int\int_V f\, dxdydz = \int\int_{\partial V} u_0\, dS.$$

Sugestão: considere $v \equiv 1$ apropriadamente na primeira identidade de Green.

19. Let $M \subset \mathbb{R}^n$ be a 3-dimensional $C^1$ class manifold, where $n \geq 4$,

$$M = \{\mathbf{r}(\mathbf{u}) = X_i(\mathbf{u})\mathbf{e}_i \ : \ \mathbf{u} \in D\},$$

$D \subset \mathbb{R}^3$ and $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is the canonical basis for $\mathbb{R}^n$,

Let $\omega = dX_1 \wedge dX_4 \wedge dX_3$ be a 3-form on $M$, where,

$$dX_1(\mathbf{u}) = \frac{\partial X_1(\mathbf{u})}{\partial u_1}\, du_1 + \frac{\partial X_1(\mathbf{u})}{\partial u_2}\, du_2 + \frac{\partial X_1(\mathbf{u})}{\partial u_3}\, du_3,$$

$$dX_4(\mathbf{u}) = \frac{\partial X_4(\mathbf{u})}{\partial u_1}\, du_1 + \frac{\partial X_4(\mathbf{u})}{\partial u_2}\, du_2 + \frac{\partial X_4(\mathbf{u})}{\partial u_3}\, du_3,$$

and

$$dX_3(\mathbf{u}) = \frac{\partial X_3(\mathbf{u})}{\partial u_1}\, du_1 + \frac{\partial X_3(\mathbf{u})}{\partial u_2}\, du_2 + \frac{\partial X_3(\mathbf{u})}{\partial u_3}\, du_3.$$

Compute

$$(dX_1(\mathbf{u}) \wedge dX_4(\mathbf{u}) \wedge dX_3(\mathbf{u}))(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3),$$

where

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1}\,\Delta u_1,$$

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_2}\,\Delta u_2$$

and

$$\mathbf{s}_3 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_3}\,\Delta u_3.$$

20. Let $V \subset \mathbb{R}^3$ be a simple region. Let $\mathbf{F} : V \to \mathbb{R}^3$ be a vectorial field of $C^1$ class.
    Let $\mathbf{x}_0 \in V^\circ$. Show that

$$div(\mathbf{F}(\mathbf{x}_0)) = \lim_{r\to 0} \frac{\int\int_{\partial B_r(\mathbf{x}_0)} \mathbf{F}\cdot\mathbf{n}\, dS}{Vol(B_r(\mathbf{x}_0))},$$

    where $\mathbf{n}$ denotes unit outward normal field to $B_r(\mathbf{x}_0)$.

21. Let $V \subset \mathbb{R}^3$ be a simple region. Let $f : V \to \mathbb{R}$ be a scalar field of $C^2$ class.
    Let $\mathbf{x}_0 \in V^\circ$. Through the first Green identity, show that

$$\nabla^2 f(\mathbf{x}_0) = \lim_{r\to 0} \frac{\int\int_{\partial B_r(\mathbf{x}_0)} \frac{\partial f}{\partial \mathbf{n}}\, dS}{Vol(B_r(\mathbf{x}_0))},$$

    where $\mathbf{n}$ denotes the unit outward normal field to $B_r(\mathbf{x}_0)$.

22. Let $M \subset \mathbb{R}^n$ be a m-dimensional surface of $C^2$ class, where $1 \leq m < n$.
    Let $X, Y, Z \in \tilde{\mathcal{X}}(M)$, where $\tilde{\mathcal{X}}(M)$ denotes the set of tangential vector fields of $C^\infty$ class defined on $M$.
    Show that

    (a) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z],\ \forall \alpha, \beta \in \mathbb{R}$,

    (b) $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z],\ \forall \alpha, \beta \in \mathbb{R}$.

    (c) Anti-symmetry:
    $$[X, Y] = -[Y, X],$$

    (d) Jacob Identity:
    $$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

    (e) Leibnitz rule:
    $$[fX, gY] = fg[X, Y] + f(X\cdot g)Y - g(Y\cdot f)X,\ \forall f, g \in C^2(M).$$

    (f) Recalling that $L_X Y = [X, Y]$, show that

    i.
    $$L_X[Y, Z] = [L_X Y, Z] + [Y, L_X Z],$$

    ii.
    $$L_X(L_Y)Z - L_Y(L_X)Z = L_{[X,Y]}Z.$$

23. Consider a 3-dimensional surface $M \subset \mathbb{R}^4$ defined by

$$M = \{(x, y, z, w) \in \mathbb{R}^4 \ :\ x^2 + y^2 + z^2 + \ln(w) = 1\}.$$

    (a) Defining $\mathbf{u} = (u_1, u_2, u_3) = (x, y, z)$, write $M$ in the form,

$$M = \{\mathbf{r}(\mathbf{u}) \in \mathbb{R}^4 \ :\ \mathbf{u} \in \mathbb{R}^3\}.$$

(b) Let $p = \mathbf{r}(\mathbf{u})$. Obtain the tangent space and equation of the hyper-plan tangent to $M$ at $p$.

(c) For $f : M \to \mathbb{R}$, $X, Y \in \mathcal{X}(M)$ such that

$$f(\mathbf{r}(\mathbf{u})) = x^2 + e^{x^2 y} + (\sin((x^2 + z^2))^3 + w(x, y, z),$$

$$X(x, y, z) = e^x \frac{\partial \mathbf{r}(x, y, z)}{\partial x} + y \frac{\partial \mathbf{r}(x, y, z)}{\partial y} + (x + z^2)^2 \frac{\partial \mathbf{r}(x, y, z)}{\partial z},$$

and

$$Y(x, y, z) = (\sin(xy))^2 \frac{\partial \mathbf{r}(x, y, z)}{\partial x} + (\cos(x^2 + y^2))^3 \frac{\partial \mathbf{r}(x, y, z)}{\partial y} + e^{x+z^2} \frac{\partial \mathbf{r}(x, y, z)}{\partial z},$$

for $p = \mathbf{r}(x_0, y_0, z_0)$, calculate

i. $(X \cdot f)(p)$,
ii. $(D_X Y)(p)$,
iii. $[X, Y](p)$,
iv. $([X, Y] \cdot f)(p)$
v. Compute numerically the results obtained the 4 last items at the point $p_0 = \mathbf{r}(x_0, y_0, z_0) = \mathbf{r}(\pi, 0, 1)$.

24. Obtain the differential $dM(x, y, z)$ to calculate the area of the surface $M \subset \mathbb{R}^4$, where

$$M = \{(x, y, z, w) \in \mathbb{R}^4 \ : \ e^w = \{x + z^2 + 5 \text{ and } x^2 + y^2 + z^2 \leq 1\}.$$

Remark: You do not need to calculate the area, just write the concerning integral.

25. Consider the surface $M \subset \mathbb{R}^4$ defined by

$$M = \{(x, y, z) \in \mathbb{R}^4 \ : \ e^w - x^2 - y^2 - z^2 = 1\}.$$

Write its equation in the form,

$$M = \{\mathbf{r}(x, y, z) \ : \ (x, y, z) \in \mathbb{R}^3\},$$

where

$$\mathbf{r}(x, y, z) = X_1(x, y, z)\mathbf{e}_1 + \cdots + X_4(x, y, z)\mathbf{e}_4.$$

Let

$$dX_1 = \frac{\partial X_1}{\partial x} dx + \frac{\partial X_1}{\partial y} dy + \frac{\partial X_1}{\partial z} dz,$$

and

$$dX_4 = \frac{\partial X_4}{\partial x} dx + \frac{\partial X_4}{\partial y} dy + \frac{\partial X_4}{\partial z} dz.$$

(a) Calculate

$$(dX_1 \wedge dX_4)(\mathbf{s}_1, \mathbf{s}_2),$$

where

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(x, y, z)}{\partial x} \Delta x,$$

and

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(x, y, z)}{\partial y} \Delta y,$$

and where $\Delta x, \ \Delta y \in \mathbb{R}$.

(b) Consider the differential form

$$\omega = (w(x,y,z) + x^2y + z)dX_1 \wedge dX_4 + (w(x,y,z)^2 + \sin(x^2+y) - z^2)dX_1 \wedge dX_2,$$

where

$$dX_2 = \frac{\partial X_2}{\partial x}\,dx + \frac{\partial X_2}{\partial y}\,dy + \frac{\partial X_2}{\partial z}\,dz.$$

Obtain the exterior differential $d\omega$ of $\omega$ at $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$, where

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(x,y,z)}{\partial x}\,\Delta x,$$

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(x,y,z)}{\partial y}\,\Delta y,$$

and

$$\mathbf{s}_3 = \frac{\partial \mathbf{r}(x,y,z)}{\partial z}\,\Delta z,$$

and where $\Delta x,\ \Delta y,\ \Delta z \in \mathbb{R}$.

26. Let $M \subset \mathbb{R}^n$ be a 3-dimensional $C^1$ class surface, where $n \geq 4$,

$$M = \{\mathbf{r}(\mathbf{u}) = X_i(\mathbf{u})\mathbf{e}_i \ : \ \mathbf{u} \in D\},$$

$D \subset \mathbb{R}^3$ and $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is the canonical basis for $\mathbb{R}^n$,
Let $\omega = dX_1 \wedge dX_4 \wedge dX_3$ be a 3-form on $M$, where,

$$dX_1(\mathbf{u}) = \frac{\partial X_1(\mathbf{u})}{\partial u_1}\,du_1 + \frac{\partial X_1(\mathbf{u})}{\partial u_2}\,du_2 + \frac{\partial X_1(\mathbf{u})}{\partial u_3}\,du_3,$$

$$dX_4(\mathbf{u}) = \frac{\partial X_4(\mathbf{u})}{\partial u_1}\,du_1 + \frac{\partial X_4(\mathbf{u})}{\partial u_2}\,du_2 + \frac{\partial X_4(\mathbf{u})}{\partial u_3}\,du_3,$$

and

$$dX_3(\mathbf{u}) = \frac{\partial X_3(\mathbf{u})}{\partial u_1}\,du_1 + \frac{\partial X_3(\mathbf{u})}{\partial u_2}\,du_2 + \frac{\partial X_3(\mathbf{u})}{\partial u_3}\,du_3.$$

Compute

$$(dX_1(\mathbf{u}) \wedge dX_4(\mathbf{u}) \wedge dX_3(\mathbf{u}))(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3),$$

where

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_1}\,\Delta u_1,$$

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_2}\,\Delta u_2$$

and

$$\mathbf{s}_3 = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_3}\,\Delta u_3.$$

27. Consider the vectorial field $F : \mathbb{R}^3 \to \mathbb{R}^3$ where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + (z - x^2)\mathbf{k}$.
Through the Stokes Theorem, calculate

$$I = \int\int_S curl(\mathbf{F}) \cdot \mathbf{n}\, dS$$

where

$$S = \{(x,y,z) \in \mathbb{R}^3 \ : \ z = 8 - x^2 - 2y^2 \text{ and } 2 \leq z \leq 4\}.$$

28. Consider the vectorial field $F:\mathbb{R}^3\to\mathbb{R}^3$ where $\mathbf{F}=y\mathbf{i}+y\mathbf{j}+5\mathbf{k}$.

Through the Stokes theorem, calculate

$$I=\int\int_S curl(\mathbf{F})\cdot\mathbf{n}\,dS$$

where

$$S=\{(x,y,z)\in\mathbb{R}^3\ :\ z=16-x^2-3y^2 \text{ and } z\geq y^2+2x+y\}.$$

29. Let $D\subset\mathbb{R}^n$ be an open set and let $f:D\to\mathbb{R}$ be a function of $C^1$ class (therefore differentiable on $D$). Let $\mathbf{x}_0\in D$ and $\varepsilon>0$. Prove that there exists $\delta>0$ such that if $\mathbf{x}\in D$ and $|\mathbf{x}-\mathbf{x_0}|<\delta$, then

(a)

$$f(\mathbf{x})-f(\mathbf{x}_0)=f'(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)+r(\mathbf{x}),$$

where

$$|r(\mathbf{x})|\leq\varepsilon|\mathbf{x}-\mathbf{x}_0|.$$

(b) Use the previous item to show that if $\mathbf{x},\ \mathbf{y}\in D$, $|\mathbf{x}-\mathbf{x_0}|<\delta$ and $|\mathbf{y}-\mathbf{x_0}|<\delta$ then

$$f(\mathbf{y})-f(\mathbf{x})=f'(\mathbf{x}_0)\cdot(\mathbf{y}-\mathbf{x})+r_1(\mathbf{x},\mathbf{y}),$$

where

$$|r_1(\mathbf{x},\mathbf{y})|\leq 2\varepsilon\delta.$$

30. Let $M\subset\mathbb{R}^n$ be a m-dimensional surface of $C^1$ class, where $1\leq m<n$, where

$$M=\{\mathbf{r}(\mathbf{u})\ :\ \mathbf{u}\in D\subset\mathbb{R}^m\}.$$

Let $f\in C^2(M)$ and $X,Y\in\mathcal{X}(M)$.

In this chapter, we have denoted

$$X\cdot f=df(X)=\frac{\partial(f\circ\mathbf{r})(\mathbf{u})}{\partial u_i}X_i(\mathbf{u}).$$

(a) Calculate

$$X\cdot(Y\cdot f).$$

(b) Show that

$$X\cdot(Y\cdot f)-Y\cdot(X\cdot f)=[X,Y]\cdot f,$$

where

$$[X,Y]=(dY_i(X)-dX_i(Y))\frac{\partial\mathbf{r}(\mathbf{u})}{\partial u_i}.$$

(c) Consider the 3-dimensional manifold $M\subset\mathbb{R}^4$ defined by

$$M=\{(x,y,z,w)\in\mathbb{R}^4\ :\ e^w-x^6-y^2-z^4=5\}.$$

i. Defining $\mathbf{u}=(u_1,u_2,u_3)=(x,y,z)$, write $M$ in the form,

$$M=\{\mathbf{r}(\mathbf{u})\in\mathbb{R}^4\ :\ \mathbf{u}\in\mathbb{R}^3\}.$$

ii. For $f : M \to \mathbb{R}$, $X, Y \in \mathcal{X}(M)$ such that

$$f(\mathbf{r}(\mathbf{u})) = 5y^2 + w(x,y,z),$$

$$X(x,y,z) = \cos(x-y)\frac{\partial \mathbf{r}(x,y,z)}{\partial x} + y^3\frac{\partial \mathbf{r}(x,y,z)}{\partial y} + (x+z^2)^3\frac{\partial \mathbf{r}(x,y,z)}{\partial z},$$

and

$$Y(x,y,z) = e^x\frac{\partial \mathbf{r}(x,y,z)}{\partial x} + (\sin(x^2+y^3))^4\frac{\partial \mathbf{r}(x,y,z)}{\partial y} + e^{x^3 z}\frac{\partial \mathbf{r}(x,y,z)}{\partial z},$$

for $p = \mathbf{r}(x,y,z)$, calculate

$$([X,Y]\cdot f)(p)$$

31. Consider a 3-dimensional surface $M \subset \mathbb{R}^4$ defined by

$$M = \{(x,y,z) \in \mathbb{R}^4 \; : \; \ln(w) - x + 2y^2 - z^3 = 1\}.$$

Write the equation of $M$ in the form,

$$M = \{\mathbf{r}(x,y,z) \; : \; (x,y,z) \in \mathbb{R}^3\},$$

where

$$\mathbf{r}(x,y,z) = X_1(x,y,z)\mathbf{e}_1 + \cdots + X_4(x,y,z)\mathbf{e}_4,$$

(a) Obtain $(dX_1 \wedge dX_2 \wedge dX_4)(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$.

(b) Consider the differential form

$$\omega = e^{x^2+y^5}\, dX_2 + \sin(xy^2)\, dX_1.$$

Obtain the exterior differential $d\omega$ of $\omega$ at $(\mathbf{s}_1, \mathbf{s}_2)$, where for the last two sub-items,

$$\mathbf{s}_1 = \frac{\partial \mathbf{r}(x,y,z)}{\partial x}\,\Delta x,$$

$$\mathbf{s}_2 = \frac{\partial \mathbf{r}(x,y,z)}{\partial y}\,\Delta y,$$

and

$$\mathbf{s}_3 = \frac{\partial \mathbf{r}(x,y,z)}{\partial z}\,\Delta z,$$

and where $\Delta x,\ \Delta y,\ \Delta z \in \mathbb{R}$.