A Variational Formulation for the Relativistic Klein-Gordon Equation

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Abstract

This article develops a variational formulation for the relativistic Klein-Gordon equation. The main results are obtained through a connection between classical and quantum mechanics. Such a connection is established through the definition of normal field and its relation with the wave function concept.

1 Introduction

In this work we propose a variational formulation for the Klein-Gordon relativistic equation obtained through an extension of the classical mechanics approach to a more general context, which may include in some sense the quantum mechanics one. The main results are developed through the introduction of the normal field definition and concerning wave function concept.

The aim of introducing the energy part related to the normal field is to minimize and control, in a specific appropriate sense to be described in the next sections, the curvature field distribution along the concerned mechanical system.

About the references, this work is based on the book "A Classical Description of Variational Quantum Mechanics and Related Models" [5], published by Nova Science Publishers. In the first two sections we present a summary of the main introductory results presented in [5]. In the final section we develop in details the main result, namely, the establishment of the Klein-Gordon relativistic equation resulted from the respective variational formulation.

At this point we remark that details on the Sobolev Spaces involved may be found in [1, 4]. For standard references in quantum mechanics, we refer to [3, 6, 7] and the non-standard [2].

Finally, we emphasize this article is not about Bohmian mechanics, even though the David Bohm work has been always inspiring.

2 The Newtonian approach

In this section, specifically for a free particle context, we shall obtain a close relationship between classical and quantum mechanics.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$, on which we define a position field, in a free volume context, denoted by $\mathbf{r} : \Omega \times [0,T] \to \mathbb{R}^3$, where $[0,T]$ is a time interval.

Suppose also an associated density distribution scalar field is given by $(\rho \circ \mathbf{r}) : \Omega \times [0, T] \to [0, +\infty)$, so that the kinetics energy for such a system, denoted by $J : U \times V \to \mathbb{R}$, is defined as

$$J(\mathbf{r}, \rho) = \frac{1}{2}\int_0^T \int_\Omega \rho(\mathbf{r}(\mathbf{x}, t))\frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t} \cdot \frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t}\sqrt{g}\, d\mathbf{x}dt,$$

subject to

$$\int_\Omega \rho(\mathbf{r}(\mathbf{x}, t))\sqrt{g}\, d\mathbf{x} = m, \text{ on } [0, T],$$

where m is the total system mass, t denotes time and $d\mathbf{x} = dx_1\, dx_2\, dx_3$.

Here,

$$\begin{aligned} U &= \{\mathbf{r} \in W^{1,2}(\Omega \times [0, T]) \; : \; \mathbf{r}(\mathbf{x}, 0) = \mathbf{r}_0(\mathbf{x}) \\ &\quad \text{and } \mathbf{r}(\mathbf{x}, T) = \mathbf{r}_1(\mathbf{x}), \text{ in } \Omega\}, \end{aligned} \tag{1}$$

and

$$V = \{\rho(\mathbf{r}) \in L^2([0, T]; W^{1,2}(\Omega)) \; : \; \mathbf{r} \in U\}.$$

Also

$$\mathbf{g}_k = \frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial x_k},$$

$$g_{jk} = \mathbf{g}_j \cdot \mathbf{g}_k,$$

and

$$g = \det\{g_{jk}\}.$$

For such a standard Newtonian formulation, the kinetics energy takes into account just the tangential field given by the time derivative

$$\frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t}.$$

At this point, the idea is to complement such an energy with a new term which would consider also the variation of a normal field $\mathbf{n}$ and concerning distribution of curvature, such that

$$\mathbf{n} \cdot \frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t} = 0, \text{ in } \Omega \times [0, T].$$

So, with such statements in mind, we redefine the concerning energy, denoting it again by $J : U \times V \times V_1 \to \mathbb{R}$, as

$$\begin{aligned} J(\mathbf{r}, \mathbf{n}, \rho) &= -\frac{1}{2}\int_0^T \int_\Omega \rho(\mathbf{r}(\mathbf{x}, t))\frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t} \cdot \frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t}\sqrt{g}\, d\mathbf{x}dt \\ &\quad + \frac{\gamma}{2}\int_0^T \int_\Omega \hat{R}\sqrt{g}\, d\mathbf{x}dt, \end{aligned} \tag{2}$$

where $\gamma > 0$ is an appropriate constant,

$$\hat{R} = g^{ij}\hat{R}_{ij},$$

$$\hat{R}_{jk} = \hat{R}^i_{jik},$$

$$\hat{R}^i_{jkl} = b^l_i \, b_{jk},$$

$$b_{ij} = -\frac{1}{\sqrt{m}} \frac{\partial \left(\sqrt{\rho(\mathbf{r})} \mathbf{n}(\mathbf{r}) \right)}{\partial x_j} \cdot \mathbf{g}_i,$$

$$b^i_j = g^{il} b_{lj},$$

and,

$$\{g^{ij}\} = \{g_{ij}\}^{-1},$$

$\forall i, j, k, l \in \{1, 2, 3\}$.

subject to

$$\mathbf{n}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) = 1, \text{ in } \Omega \times [0, T],$$

$$\mathbf{n}(\mathbf{r}) \cdot \frac{\partial \mathbf{r}}{\partial t} = 0, \text{ in } \Omega \times [0, T],$$

and

$$\int_\Omega \rho(\mathbf{r}(\mathbf{x}, t)) \sqrt{g} \, d\mathbf{x} = m, \text{ on } [0, T].$$

Here

$$V_1 = \{\mathbf{n}(\mathbf{r}) \in L^2(\Omega \times [0, T]) \; : \; \mathbf{r} \in U\}.$$

Thus, defining ϕ such that

$$|\phi| = \sqrt{\frac{\rho}{m}}$$

and already including the Lagrange multipliers concerning the restrictions, the final expression for the energy, denoted by $J : U \times V \times V_1 \times V_2 \times [V_3]^2 \to \mathbb{R}$, would be given by

$$\begin{aligned} J(\mathbf{r}, \mathbf{n}, \phi, E, \lambda_1, \lambda_2) \;\; = \;\; & -\frac{1}{2} \int_0^T \int_\Omega m |\phi(\mathbf{r}(\mathbf{x}, t))|^2 \frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t} \cdot \frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t} \sqrt{g} \, d\mathbf{x} dt \\ & + \frac{\gamma}{2} \int_0^T \int_\Omega \hat{R} \sqrt{g} \, d\mathbf{x} dt \\ & - m \int_0^T E(t) \left(\int_\Omega |\phi(\mathbf{r})|^2 \, \sqrt{g} \, d\mathbf{x} - 1 \right) \, dt \\ & + \langle \lambda_1, \mathbf{n} \cdot \mathbf{n} - 1 \rangle_{L^2} \\ & + \left\langle \lambda_2, \mathbf{n} \cdot \frac{\partial \mathbf{r}}{\partial t} \right\rangle_{L^2}, \end{aligned} \tag{3}$$

where,

$$\begin{aligned} U \;\; = \;\; & \{\mathbf{r} \in W^{1,2}(\Omega \times [0, T]) \; : \; \mathbf{r}(\mathbf{x}, 0) = \mathbf{r}_0(\mathbf{x}) \\ & \text{ and } \mathbf{r}(\mathbf{x}, T) = \mathbf{r}_1(\mathbf{x}), \text{ in } \Omega\}, \end{aligned} \tag{4}$$

$$V = \{\phi(\mathbf{r}) \in L^2([0, T]; W^{1,2}(\Omega; \mathbb{C})) \; : \; \mathbf{r} \in U\},$$

$$V_1 = \{\mathbf{n}(\mathbf{r}) \in L^2(\Omega \times [0, T]) \; : \; \mathbf{r} \in U\},$$

$$V_2 = L^2([0, T]),$$

$$V_3 = L^2(\Omega \times [0,T]),$$

and generically

$$\langle f, h\rangle_{L^2} = \int_0^T \int_\Omega f h \sqrt{g}\, d\mathbf{x}\, dt, \forall f, h \in L^2(\Omega \times [0,T]).$$

Moreover,

$$\hat{R} = g^{ij} \hat{R}_{ij},$$

$$\hat{R}_{jk} = \hat{R}^i_{jik},$$

$$\hat{R}^i_{jkl} = b^l_i\, b^*_{jk},$$

$$b_{ij} = -\frac{\partial\left(\phi(\mathbf{r})\mathbf{n}(\mathbf{r})\right)}{\partial x_j} \cdot \mathbf{g}_i,$$

$$b^i_j = g^{il} b_{lj},$$

$\forall i, j, k, l \in \{1,2,3\}$.

Finally, in particular for the special case in which

$$\mathbf{r}(\mathbf{x}, t) \approx \mathbf{x},$$

so that

$$\frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t} \approx 0,$$

and

$$\mathbf{n} \cdot \frac{\partial \mathbf{r}}{\partial t} \approx 0,$$

we may set

$$\mathbf{n} = \mathbf{c},$$

where $\mathbf{c} \in \mathbb{R}^3$ is a constant such that

$$\mathbf{c} \cdot \mathbf{c} = 1,$$

and obtain

$$\mathbf{g}_k \approx \mathbf{e}_k,$$

where

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

is the canonical basis of $\mathbb{R}^3$.

Therefore, in such a case,

$$\frac{\gamma}{2} \int_0^T \int_\Omega \hat{R}\, \sqrt{g}\, d\mathbf{x} dt \approx \frac{\gamma T}{2} \sum_{k=1}^3 \int_\Omega \frac{\partial \phi}{\partial x_k} \frac{\partial \phi^*}{\partial x_k}\, d\mathbf{x}.$$

Hence, we would also obtain

$$\begin{aligned} J(\mathbf{r}, \mathbf{n}, \phi, E, \lambda_1, \lambda_2)/T &\approx \tilde{J}(\phi, E) \\ &= \frac{\gamma}{2} \sum_{k=1}^3 \int_\Omega \frac{\partial \phi}{\partial x_k} \frac{\partial \phi^*}{\partial x_k}\, d\mathbf{x} \\ &\quad - E\left(\int_\Omega |\phi|^2 d\mathbf{x} - 1\right). \end{aligned} \tag{5}$$

This last energy is just the standard Schrödinger one in a free particle context.

3 A brief note on the relativistic context, the Klein-Gordon equation

Denoting by c the speed of light and

$$d\bar{t}^2 = c^2 dt^2 - dX_1^2 - dX_2^2 - dX_3^2,$$

in a relativistic free particle context, the Hilbert variational formulation could be extended, for a motion in a pseudo Riemannian relativistic C^1 class manifold M, where locally

$$M = \{\mathbf{r}(\mathbf{u}) \ : \ \mathbf{u} \in \Omega\},$$

$$\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4,$$

and

$$\mathbf{r} : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^4$$

point-wise stands for,

$$\mathbf{r}(\mathbf{u}) = (ct(\mathbf{u}), X_1(\mathbf{u}), X_2(\mathbf{u}), X_3(\mathbf{u})),$$

to a functional J_1 where denoting $\rho(\mathbf{r}) = |R(\mathbf{r})|^2$, the mass differential is given by

$$dm = \frac{\rho(\mathbf{r})}{\sqrt{1 - v^2/c^2}} \sqrt{|g|}\ d\mathbf{u} = \frac{|R(\mathbf{r})|^2}{\sqrt{1 - v^2/c^2}} \sqrt{|g|}\ d\mathbf{u},$$

the semi-classical kinetics energy differential is given by

$$\begin{aligned} dE_c &= \frac{\partial \mathbf{r}(\mathbf{u})}{\partial t} \cdot \frac{\partial \mathbf{r}(\mathbf{u})}{\partial t}\ dm \\ &= -\left(\frac{d\bar{t}}{dt}\right)^2 dm \\ &= -(c^2 - v^2)\ dm, \end{aligned} \tag{6}$$

so that

$$dE_c = -c^2(\sqrt{1 - v^2/c^2})|R(\mathbf{r})|^2 \sqrt{|g|}\ d\mathbf{u},$$

and

$$\begin{aligned} J_1(\mathbf{r}, R, \mathbf{n}) &= -\int_\Omega dE_c + \frac{\gamma}{2} \int_\Omega \hat{R}\sqrt{|g|}\ d\mathbf{u} \\ &= c^2 \int_\Omega |R(\mathbf{r})|^2 \sqrt{1 - v^2/c^2}\ \sqrt{|g|}\ d\mathbf{u} \\ &\quad + \frac{\gamma}{2} \int_\Omega \hat{R}\sqrt{|g|}\ d\mathbf{u}, \end{aligned} \tag{7}$$

subject to

$$\int_\Omega |R(\mathbf{r})|^2\ \sqrt{|g|}\ d\mathbf{u} = m,$$

where m is the particle mass at rest.

Moreover,

$$\mathbf{n}(\mathbf{r}) \cdot \frac{\partial \mathbf{r}}{\partial \bar{t}} = 0, \quad \text{in } \Omega,$$

where

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \bar{t}} &= \frac{\partial \mathbf{r}}{\partial t}\frac{\partial t}{\partial \bar{t}} \\ &= \frac{\frac{\partial \mathbf{r}}{\partial t}}{\frac{\partial \bar{t}}{\partial t}} \\ &= \frac{\partial \mathbf{r}}{c\partial t}\frac{1}{\sqrt{1-v^2/c^2}}, \end{aligned} \tag{8}$$

and

$$\mathbf{n}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) = 1, \text{ in } \Omega.$$

Where γ is an appropriate positive constant to be specified.

Also,

$$\mathbf{g}_k = \frac{\partial \mathbf{r}(\mathbf{u})}{\partial u_k},$$

$$g = det\{g_{ij}\},$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

where here, in this subsection, such a product is given by

$$\mathbf{y} \cdot \mathbf{z} = -y_0 z_0 + \sum_{i=1}^{3} y_i z_i, \ \forall \mathbf{y} = (y_0, y_1, y_2, y_3), \ \mathbf{z} = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4,$$

$$\hat{R} = g^{ij}\hat{R}_{ij},$$

$$\hat{R}_{jk} = \hat{R}^i_{jik},$$

$$\hat{R}^i_{jkl} = b^l_i \, b^*_{jk},$$

$$b_{ij} = -\frac{1}{\sqrt{m}}\frac{\partial \left(R(\mathbf{r})\mathbf{n}(\mathbf{r})\right)}{\partial u_j} \cdot \mathbf{g}_i,$$

$$b^i_j = g^{il} b_{lj},$$

and,

$$\{g^{ij}\} = \{g_{ij}\}^{-1},$$

$\forall i, j, k, l \in \{1, 2, 3, 4\}$.

Finally,

$$v = \sqrt{\left(\frac{\partial X_1}{\partial t}\right)^2 + \left(\frac{\partial X_2}{\partial t}\right)^2 + \left(\frac{\partial X_3}{\partial t}\right)^2},$$

where,

$$\begin{aligned}\frac{\partial X_k(\mathbf{u})}{\partial t} &= \frac{\partial X_k(\mathbf{u})}{\partial u_j}\frac{\partial u_j}{\partial t} \\ &= \sum_{j=1}^{4} \frac{\frac{\partial X_k(\mathbf{u})}{\partial u_j}}{\frac{\partial t(\mathbf{u})}{\partial u_j}}, \ \forall k \in \{1,2,3\}. \end{aligned} \tag{9}$$

Here the Einstein sum convention holds.

Remark 3.1. *The role of the variable* $\mathbf{u}$ *concerns the idea of establishing a relation between* t, X_1, X_2 *and* X_3. *The dimension of* M *may vary with the problem in question.*

3.1 Obtaining the Klein-Gordon equation

Of particular interest is the case in which

$$\mathbf{u} = (t, x_1, x_2, x_3) = (t, \mathbf{x}) \in \mathbb{R}^4,$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

In such a case we could have, point-wise,

$$\mathbf{r}(\mathbf{x}, t) = (ct, X_1(t, \mathbf{x}), X_2(t, \mathbf{x}), X_3(t, \mathbf{x})),$$

and

$$M = \{\mathbf{r}(\mathbf{x}, t) \ : \ (\mathbf{x}, t) \in \Omega \times [0, T]\},$$

for an appropriate $\Omega \subset \mathbb{R}^3$.

Also, denoting $d\mathbf{x} = dx_1 dx_2 dx_3$, the mass differential would be given by

$$dm = \frac{\rho(\mathbf{r})}{\sqrt{1 - v^2/c^2}}\sqrt{-g}\ d\mathbf{x} = \frac{|R(\mathbf{r})|^2}{\sqrt{1 - v^2/c^2}}\sqrt{-g}\ d\mathbf{x},$$

the semi-classical kinetics energy differential would be expressed by

$$\begin{aligned} dE_c &= \frac{\partial \mathbf{r}(t, \mathbf{x})}{\partial t} \cdot \frac{\partial \mathbf{r}(t, \mathbf{x})}{\partial t}\ dm \\ &= -\left(\frac{d\bar{t}}{dt}\right)^2 \ dm \\ &= -(c^2 - v^2)\ dm, \end{aligned} \tag{10}$$

so that

$$dE_c = -c^2(\sqrt{1 - v^2/c^2})|R(\mathbf{r})|^2\sqrt{-g}\ d\mathbf{x},$$

where

$$d\bar{t}^2 = c^2 dt^2 - dX_1(t, \mathbf{x})^2 - dX_2(t, \mathbf{x})^2 - dX_3(t, \mathbf{x})^2,$$

and

$$
\begin{aligned}
J_1(\mathbf{r},R,\mathbf{n}) &= -\int_0^T\int_\Omega dE_c\,dt+\frac{\gamma}{2}\int_0^T\int_\Omega \hat{R}\sqrt{-g}\,d\mathbf{x}\,dt \\
&= c^2\int_0^T\int_\Omega |R(\mathbf{r})|^2\sqrt{1-v^2/c^2}\,\sqrt{-g}\,d\mathbf{x}\,dt \\
&\quad +\frac{\gamma}{2}\int_0^T\int_\Omega \hat{R}\sqrt{-g}\,d\mathbf{x}\,dt, \qquad (11)
\end{aligned}
$$

subject to

$$R(\mathbf{r}(\mathbf{x},0))=R_0(\mathbf{x})$$

$$R(\mathbf{r}(\mathbf{x},T))=R_1(\mathbf{x})$$

and

$$R(\mathbf{r}(\mathbf{x},t))=0,\ \text{ on } \partial\Omega\times[0,T],$$

$$\int_\Omega |R(\mathbf{r})|^2\,\sqrt{-g}\,d\mathbf{x}=m,\ \text{ on } [0,T],$$

$$\mathbf{n}(\mathbf{r})\cdot\frac{\partial\mathbf{r}}{\partial\bar{t}}=0,\ \text{ in } \Omega\times[0,T],$$

where

$$
\begin{aligned}
\frac{\partial\mathbf{r}}{\partial\bar{t}} &= \frac{\partial\mathbf{r}}{\partial t}\frac{\partial t}{\partial\bar{t}} \\
&= \frac{\frac{\partial\mathbf{r}}{\partial t}}{\frac{\partial\bar{t}}{\partial t}} \\
&= \frac{\partial\mathbf{r}}{c\partial t}\frac{1}{\sqrt{1-v^2/c^2}}, \qquad (12)
\end{aligned}
$$

and

$$\mathbf{n}(\mathbf{r})\cdot\mathbf{n}(\mathbf{r})=1,\ \text{ in } \Omega\times[0,T].$$

Also, we have denoted

$$x_0=ct,$$

$$(x_0,\mathbf{x})=(x_0,x_1,x_2,x_3),$$

$$\mathbf{g}_k=\frac{\partial\mathbf{r}(t,\mathbf{x})}{\partial x_k},$$

$$g=det\{g_{ij}\},$$

$$g_{ij}=\mathbf{g}_i\cdot\mathbf{g}_j,$$

where here again, such a product is given by

$$\mathbf{y}\cdot\mathbf{z}=-y_0z_0+\sum_{i=1}^3 y_iz_i,\ \forall\mathbf{y}=(y_0,y_1,y_2,y_3),\ \mathbf{z}=(z_0,z_1,z_2,z_3)\in\mathbb{R}^4,$$

$$\hat{R} = g^{ij}\hat{R}_{ij},$$

$$\hat{R}_{jk} = \hat{R}^i_{jik},$$

$$\hat{R}^i_{jkl} = b^l_i \, b^*_{jk},$$

$$b_{ij} = -\frac{1}{\sqrt{m}} \frac{\partial \left(R(\mathbf{r})\mathbf{n}(\mathbf{r}) \right)}{\partial x_j} \cdot \mathbf{g}_i,$$

$$b^i_j = g^{il} b_{lj},$$

and,

$$\{g^{ij}\} = \{g_{ij}\}^{-1},$$

$\forall i, j, k, l \in \{0, 1, 2, 3\}$.

Finally, we would also have

$$v = \sqrt{\left(\frac{\partial X_1}{\partial t}\right)^2 + \left(\frac{\partial X_2}{\partial t}\right)^2 + \left(\frac{\partial X_3}{\partial t}\right)^2}.$$

In particular for the special case in which

$$\mathbf{r}(\mathbf{x}, t) \approx (ct, \mathbf{x}),$$

so that

$$\frac{\partial \mathbf{r}(\mathbf{x}, t)}{\partial t} \approx (c, 0, 0, 0),$$

and

$$\mathbf{n} \cdot \frac{\partial \mathbf{r}}{\partial t} \approx 0,$$

where we have set

$$\mathbf{n} = \mathbf{c} = (0, c_1, c_2, c_3).$$

Here $\mathbf{c} \in \mathbb{R}^4$ is a constant such that

$$\mathbf{c} \cdot \mathbf{c} = 1,$$

and thus we would obtain

$$\mathbf{g}_0 \approx (1, 0, 0, 0), \;\; \mathbf{g}_1 \approx (0, 1, 0, 0), \;\; \mathbf{g}_2 \approx (0, 0, 1, 0) \text{ and } \mathbf{g}_3 \approx (0, 0, 0, 1) \in \mathbb{R}^4.$$

Therefore, defining $\phi \in W^{1,2}(\Omega \times [0, T]; \mathbb{C})$ as

$$\phi(\mathbf{x}, t) = \frac{R(ct, \mathbf{x})}{\sqrt{m}},$$

we have

$$\begin{aligned} \frac{\gamma}{2} \int_0^T \int_\Omega \hat{R} \, \sqrt{g} \, d\mathbf{x} dt \;\; &\approx \;\; \frac{\gamma}{2} \int_0^T \int_\Omega \left(-\frac{1}{c^2} \frac{\partial \phi(\mathbf{x}, t)}{\partial t} \frac{\partial \phi^*(\mathbf{x}, t)}{\partial t} \right. \\ &\quad \left. + \sum_{k=1}^{3} \frac{\partial \phi(\mathbf{x}, t)}{\partial x_k} \frac{\partial \phi^*(\mathbf{x}, t)}{\partial x_k} \right) \; d\mathbf{x} dt, \end{aligned} \tag{13}$$

and

$$c^2 \int_0^T \int_\Omega |R(\mathbf{r})|^2 \sqrt{1 - v^2/c^2}\ \sqrt{-g}\ d\mathbf{x}\ dt \approx mc^2 \int_0^T \int_\Omega |\phi(\mathbf{x}, t)|^2\ d\mathbf{x}dt.$$

Hence, we would also obtain

$$\begin{aligned} J(\mathbf{r}, \mathbf{n}, \phi, E, \lambda_1, \lambda_2) \quad &\approx \quad \frac{\gamma}{2} \left(\int_0^T \int_\Omega -\frac{1}{c^2} \frac{\partial \phi(\mathbf{x}, t)}{\partial t} \frac{\partial \phi^*(\mathbf{x}, t)}{\partial t}\ d\mathbf{x}dt \right. \\ & \left. + \sum_{k=1}^{3} \int_\Omega \int_0^T \frac{\partial \phi(\mathbf{x}, t)}{\partial x_k} \frac{\partial \phi^*(\mathbf{x}, t)}{\partial x_k}\ d\mathbf{x}dt \right) \\ & + mc^2 \int_0^T \int_\Omega |\phi(\mathbf{x}, t)|^2\ d\mathbf{x}dt \\ & - m \int_0^T E(t) \left(\int_\Omega |\phi(\mathbf{x}, t)|^2 d\mathbf{x} - 1 \right)\ dt. \end{aligned} \tag{14}$$

The Euler Lagrange equations for such an energy are given by

$$\begin{aligned} & \frac{\gamma}{2} \left(\frac{1}{c^2} \frac{\partial^2 \phi(\mathbf{x}, t)}{\partial t^2} - \sum_{k=1}^{3} \frac{\partial^2 \phi(\mathbf{x}, t)}{\partial x_k^2} \right) \\ & + mc^2 \phi(\mathbf{x}, t) - E_1(t) \phi(\mathbf{x}, t) = 0,\ \text{ in } \Omega, \end{aligned} \tag{15}$$

where,

$$\begin{aligned} \phi(\mathbf{x}, 0) &= \phi_0(\mathbf{x}),\ \text{ in } \Omega, \\ \phi(\mathbf{x}, T) &= \phi_1(\mathbf{x}),\ \text{ in } \Omega, \\ \phi(\mathbf{x}, t) &= 0,\ \text{ on } \partial\Omega \times [0, T] \end{aligned}$$

and $E_1(t) = mE(t)$.

Equation (15) is the relativistic Klein-Gordon one.

For $E_1(t) = E_1 \in \mathbb{R}$ (not time dependent), at this point we suggest a solution (and implicitly related time boundary conditions) $\phi(\mathbf{x}, t) = e^{-\frac{iE_1 t}{\hbar}} \phi_2(\mathbf{x})$, where

$$\phi_2(\mathbf{x}) = 0,\ \text{ on } \partial\Omega.$$

Therefore, replacing this solution into equation (15), we would obtain

$$\left(\frac{\gamma}{2} \left(-\frac{E_1^2}{c^2 \hbar^2} \phi_2(\mathbf{x}) - \sum_{k=1}^{3} \frac{\partial^2 \phi_2(\mathbf{x})}{\partial x_k^2} \right) + mc^2 \phi_2(\mathbf{x}) - E_1 \phi_2(\mathbf{x}) \right) e^{-\frac{iE_1 t}{\hbar}} = 0,$$

in Ω.

Denoting

$$E_2 = -\frac{\gamma E_1^2}{2c^2 \hbar^2} + mc^2 - E_1,$$

the final eigenvalue problem would stand for

$$-\frac{\gamma}{2} \sum_{k=1}^{3} \frac{\partial^2 \phi_2(\mathbf{x})}{\partial x_k^2} + E_2 \phi_2(\mathbf{x}) = 0,\ \text{ in } \Omega$$

where E_1 is such that

$$\int_\Omega |\phi_2(\mathbf{x})|^2 \, d\mathbf{x} = 1.$$

Moreover, from (15), such a solution $\phi(\mathbf{x},t) = e^{-\frac{iE_1 t}{\hbar}}\phi_2(\mathbf{x})$ is also such that

$$\frac{\gamma}{2}\left(\frac{1}{c^2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} - \sum_{k=1}^{3}\frac{\partial^2\phi(\mathbf{x},t)}{\partial x_k^2}\right)$$
$$+mc^2\phi(\mathbf{x},t) = i\hbar\frac{\partial\phi(\mathbf{x},t)}{\partial t}, \text{ in } \Omega. \tag{16}$$

At this point, we recall that in quantum mechanics,

$$\gamma = \hbar^2/m.$$

Finally, we remark this last equation (16) is a kind of relativistic Schrödinger-Klein-Gordon equation.

4 Conclusion

In this article we have developed a variational formulation for the relativistic Klein-Gordon equation by extending the standard classical mechanics energy to a more general functional, obtained thorough the introduction of the normal field concept.

We believe the results here presented may be applied and extended to other models in mechanics, including the quantum and relativistic approaches for the study of atoms and molecules.

Finally, we have the objective and interest in applying such results also for the case in which electromagnetic fields are included, however we postpone such developments for a future research.

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