An Application of a Fixed Point Iteration Method to Object Reconstruction

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Abstract — Kirsch’s factorization method is a fast inversion technique for visualizing the profile of a scatterer from measurements of the far-field pattern. The mathematical basis of this method is given by the far-field equation, which is a Fredholm integral equation of the first kind in which the data function is a known analytic function and the integral kernel is the measured (and therefore noisy) far field pattern. We present a Tikhonov parameter choice approach based on a fast fixed point iteration method which constructs a regularization parameter associated with the corner of the L-curve in log-log scale. The performance of the method is evaluated by comparing our reconstructions with those obtained via the L-curve and we conclude that our method yields reliable reconstructions at a lower computational cost than the L-curve.

1. INTRODUCTION

For inverse acoustic scattering the original linear sampling method was introduced by Colton and Kirsch [5], and mathematically clarified in [6]. Kirsch improved the original version of the linear sampling method, leading to the so-called \((F^*F)^{1/4}\)-method shown in [9]. Recently, Arens presented a proof of convergence for this method for the case of acoustic scattering by sound soft obstacles in [1]. In most numerical applications of the linear sampling method, Tikhonov regularization has been employed and the regularization constant was computed via Morozov’s discrepancy principle [6], which involved the computation of the zeros of the discrepancy function at each point of the grid, a process that is time-consuming. In addition, the noise level in the data should be known \textit{a priori}, something that in real life applications is not the case in general. Several other strategies for selecting the regularization parameter have been proposed, one of them being the L-curve method [8], see also [10] for an application of the L-curve method in inverse elastic scattering. The L-curve method is a log-log plot of the norm of a regularized solution versus the norm of the corresponding residual norm. The L-curve method selects the parameter which maximizes the curvature of the L-curve. However, computing the point of maximum curvature in a robust way is not an easy task [7, 8, 11].

In this work we propose a fixed-point (FP) algorithm for selecting the regularization parameter based on an earlier work of Reginska [11] and Bazán [2, 3], which we use for the characterization of an object via Kirsch’s \((F^*F)^{1/4}\)-method. The FP algorithm only needs computation of the solution norm (or solution seminorm) and the residual norm, while the L-curve requires either the SVD (or GSVD) or the computation of the derivative of the solution norm with respect to the regularization parameter. Hence the FP algorithm is simpler and better suited for large-scale problems.

In what follows the method will be denoted by MKM-FP which stands for modified Kirsch’s method coupled with the FP-algorithm.

2. A DESCRIPTION OF THE FACTORIZATION METHOD

It is well known that the propagation of time-harmonic acoustic fields in a homogeneous medium, in the presence of a sound soft obstacle \(D\), is modeled by the exterior boundary value problem (direct obstacle scattering problem)

\begin{align}
\Delta_2 u(x) + k^2 u(x) &= 0, \quad x \in \mathbb{R}^2 \setminus D \quad (1) \\
u(x) + u'(x) &= 0, \quad x \in \partial D \quad (2)
\end{align}

where \(k\) is a real positive wavenumber and \(u'\) is a given incident field, that in the presence of \(D\) will generate the scattered field \(u\). In addition, the scattered field \(u\) will satisfy the Sommerfeld radiation condition. The Green formula
implies that the solution \( u \) of the direct obstacle scattering problem above has the asymptotic behaviour
\[
u(x) = u_\infty(\hat{x}) e^{ikr} + O(r^{-3/2})
\]
for some analytic function \( u_\infty \), called the far field-pattern of \( u \), and defined on the unit sphere \( \Omega \). In the case of the inverse problem, it represents the measured data. In particular, the inverse problem that will be considered here, is the problem of finding the shape of \( D \) from a complete knowledge of the far-field pattern.

We now define the far-field equation
\[
(Fg)(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x} \cdot z}
\]
where the right hand side is the far-field pattern of the fundamental solution of the Helmholtz equation, \( z \in \mathbb{R}^2 \) and \( F : L^2(\Omega) \rightarrow L^2(\Omega) \) is given by
\[
(Fg)(\hat{x}) = \int_{\Omega} u_\infty(\hat{x}; \hat{d}) g(\hat{d}) \, ds(\hat{d}), \quad \hat{d} \in \Omega
\]
It is well known that the first version of the linear sampling method [5] solves the linear operator Equation (4) based on the numerical observation that its solution will have a large norm outside and close to \( \partial D \). Hence, reconstructions are obtained by plotting the norm of the solution. However, the problem is that the right-hand side does not in general belong to the range of the operator \( F \).

Kirsch [9] was able to overcome this difficulty with the introduction of a new version of the linear sampling method based on appropriate factorization of the far-field operator \( F \). In this method, Kirsch is elegantly using the spectral properties of the operator \( F \) to characterize the obstacle. In particular, the following far-field equation is now used in place of Equation (4)
\[
(F^* F)^{1/4} g_z = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x} \cdot z}
\]
and the spectral properties of \( F \) are used for the reconstructions. However, due to noisy data, the discretized version of the far field operator \( F \) is characterized by numerical instability which may result to false information about its singular system. In the next section, we will show that we can overcome this difficulty via the a priori determination of a Tikhonov regularization constant.

3. THE FIXED-POINT METHOD FOR THE SELECTION OF THE TIKHONOV PARAMETER

We start by describing some notation concerning Tikhonov method for problems of the form
\[
\min_{f \in \mathbb{R}^n} \| g - Af \|_2, \quad A \in \mathbb{R}^{m \times n} \ (m \geq n), \quad g \in \mathbb{R}^m,
\]
where \( A \) is ill-conditioned and has singular values decaying to zero without particular gap in the singular value spectrum. In its simplest form, Tikhonov’s method amounts to replacing the least squares problem (7) by
\[
\min_{f \in \mathbb{R}^n} \{\| g - Af \|_2^2 + \lambda^2 \| f \|_2^2\}
\]
where \( \lambda > 0 \) is the regularization parameter. Solving (7) is equivalent to solving the regularized normal equations
\[
(A^T A + \lambda^2 I_n) f = A^T g,
\]
whose solution is \( f_\lambda = (A^T A + \lambda^2 I_n)^{-1} A^T g \), where \( I_n \) is the \( n \times n \) identity matrix.

Let the SVD of \( A \) be
\[
A = U \Sigma V^*
\]
where \( U \) and \( V \) are square orthonormal matrices, and \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \), with \( \sigma_1 \geq \sigma_2 \cdots \geq \sigma_p > \sigma_{p+1} = \cdots = 0 \), \( p = \text{rank}(A) \leq n \).
Define \( \alpha_i = |u_i^T g|^2 \) (the squared Fourier coefficient of \( g \)), and \( \delta_0 = \|(I - UU^T)g\|_2 \) (the size of the incompatible component of \( g \) that lies outside the column space of \( A \)).

Define also \( y(\lambda) = \|f \|_2^2 \), and \( x(\lambda) = \|g - Af_\lambda\|_2^2 \). Then it is easy to see that

\[
x(\lambda) = \sum_{i=1}^{p} \frac{\lambda^2 \alpha_i}{(\sigma_i^2 + \lambda^2)^2} + \delta_0^2, \quad y(\lambda) = \sum_{i=1}^{p} \frac{\sigma_i^2 \alpha_i}{(\sigma_i^2 + \lambda^2)^2}.
\]

(10)

The fixed-point method can be regarded as a realization of a parameter choice rule devised by Regińska [11], who proposed as regularization parameter a local minimum of the function

\[
\Psi_\mu(\lambda) = x(\lambda) y(\lambda),
\]

(11)

for proper \( \mu > 0 \). The motivation for using this rule can be supported heuristically noting that when \( \lambda \) is small, the squared solution norm \( y(\lambda) \) gets large while \( x(\lambda) \) gets small, and \( \Psi_1(\lambda) \) is not minimized. Conversely, when \( \lambda \) is large, \( y(\lambda) \) gets small while \( x(\lambda) \) gets large, and once again \( \Psi_1(\lambda) \) is not minimized. This suggests that the minimizer of \( \Psi_1(\lambda) \) corresponds to a good balance between the size of the solution norm and the size of the corresponding residual norm. Regińska proved that if the curvature of the L-curve is maximized at \( \lambda = \lambda^* \), and if the tangent to the L-curve at \( (\log x(\lambda^*), \log y(\lambda^*)) \) has slope \( -1/\mu \), then \( \Psi_\mu(\lambda) \) is minimized at \( \lambda = \lambda^* \).

Bazán in [2] investigated the properties of \( \Psi_\mu(\lambda) \) and concluded that its minimizers are converging values of a sequence defined by

\[
\lambda_{j+1} = \phi_\mu(\lambda_j), \quad j \geq 0, \quad \phi_\mu(\lambda) = \sqrt{\mu \|g - Af_\lambda\|_2} \quad \mu > 0,
\]

(12)

which gave rise to the fixed-point (FP) algorithm. One of the main advantages of the FP method is that it does not exploit any knowledge of the norm of the error in \( g \). The main steps of FP can be described as follows.

- Given a proper initial guess, FP starts with \( \mu = 1 \) as a default value, and then proceeds with the iterates (12) until the largest-fixed point of \( \phi_1 \) is reached. The choice \( \mu = 1 \) is because in most problems this value yields a regularization parameter that leads to a point on the L-curve near the corner of maximum curvature (geometrically, \( \mu = 1 \) means that the tangent line to the L-curve at the L-corner forms an angle of 135 degrees with the horizontal axis).

- If \( \mu = 1 \) does not work (which means \( \phi_1(\lambda > \lambda) \) for all \( \lambda > 0 \)), \( \mu \) is adjusted and the iterations restart; see [2, 3] for details.

Numerical experiments have shown that FP performs well when the L-curve is well-behaved (i.e., when there is a unique and sharp convex L-corner), and when the convexity regions of the L-curve are well-defined, as often seen in most practical problems.

However, due to the fact that minimizers of \( \Psi_\mu \) (when they exist) are points where the L-curve is convex, it results intuitively clear that the FP-method can fail when the curve is concave for all \( \lambda > 0 \). We conclude by pointing out the following convexity result due to Regińska [11]: “When \( \delta_0 \neq 0 \) the curve is always convex on \( (0, \epsilon) \) for \( \epsilon \) sufficiently small and concave on \( (\sigma_1, \infty) \).” More informative results on convexity properties of the L-curve can be found in [3].

4. NUMERICAL APPLICATIONS

In this section we shall describe a method for selecting the Tikhonov regularization parameter in connection with the \((F^*F)^{1/4}\) method by Kirsch [9]. Recall that the goal of Kirsch’s method is to solve the linear equation

\[
(F^*F)^{1/4} g = r_z,
\]

for each \( z \) in a grid, with \( F \) replaced by a discrete counterpart that is corrupted by noise: \( \tilde{F}_d = F_d + E \in \mathbb{C}^{n \times n} \), where \( n \) denotes the number of observed incident fields. We also observe that because the far-field operator is compact, the singular values of \( F_d \) decay quickly to zero and in general there is no particular gap in the singular spectrum. Taking all these under consideration, the problem we are interested in is to select the Tikhonov regularization parameter for

\[
\min_{g \in \mathbb{R}^n} \{ \|r_z - \tilde{A}_d\|_2^2 + \lambda^2 \|g\|_2^2 \},
\]

(13)
where for simplicity we denote
\[
\tilde{A}_d = (\tilde{F}_d^* \tilde{F}_d^*)^{1/4}.
\]
The choice of \( \lambda \) for the above problem has been done via Morozov’s discrepancy principle \([5, 6, 9]\), adapted to the case when the data matrix is corrupted by noise. The strategy is reliable but the noise level (or some estimate) must be known in advance. However, the norm of the error \( A_d - \tilde{A}_d \) is rarely available in practice (making the discrepancy principle of little use), and rules that do not exploit this information are highly desirable.

In what follows we will discuss how to choose the Tikhonov regularization parameter by using the FP-method.

We start with the crucial observation that due to the fact that \( \tilde{F}_d \) is square and nonsingular (since \( A_d \) so is), the linear system
\[
\tilde{A}_d g = r_z
\]
is always compatible and hence the important constant \( \delta_0 \) introduced in the previous section satisfies \( \delta_0 = 0 \). To make matters worse, numerical examples with several reconstruction problems showed that the iteration function \( \phi_1 \) has a unique fixed point that is a maximizer of \( \Psi_1 \), making the FP-method unpractical for solving the reconstruction problem. In order to overcome this difficulty we propose to replace \( \tilde{A}_d \) by another matrix, say \( \tilde{B}_d \), with \( \tilde{B}_d \) close to \( A_d \) in some sense and such that \( \delta_0 \neq 0 \).

To be precise, let the SVD of \( \tilde{F}_d \) be
\[
\tilde{F}_d = \tilde{U} \tilde{\Sigma} \tilde{V}^* = \sum_{i=1}^{n} \tilde{\sigma}_i \tilde{u}_i \tilde{v}_i^*
\]
such that \( \tilde{A}_d = \sum_{i=1}^{n} \sqrt{\tilde{\sigma}_i} \tilde{v}_i \tilde{v}_i^* \). Then if we take \( \tilde{B}_d \) to be
\[
\tilde{B}_d = \sum_{i=1}^{m} \sqrt{\tilde{\sigma}_i} \tilde{v}_i \tilde{v}_i^*,
\]
with a chosen cut-off index \( m < p \), our proposal is to reconstruct the object by solving (13) with \( \tilde{B}_d \) in place of \( \tilde{A}_d \) with the Tikhonov parameter being chosen by the FP-method. In what follows this method will be denoted by MKM-FP which stands for modified Kirsch’s method coupled with the FP-algorithm.

In order to explain why MKM-FP is expected to work, we first observe that the constant \( \delta_0 \) for the new problem is given by

\[
\delta_0 = \sum_{k=m+1}^{p} |v_k^{*}r_z|^2
\]

and that this value is expected to be very small but nonzero. This is an important observation since when this is so the L-curve is convex in a vicinity of the origin, in which case at least one minimizer of \( \Psi_1 \) is always guaranteed to exist [1]. In order to illustrate the fundamental difference of using \( \tilde{B}_d \) instead of \( \tilde{A}_d \), we have computed the functions \( \phi_1 \) and \( \Psi_1 \) associated with (13) as well as the corresponding functions associated with MKM-FP which we denote by \( \phi_1^{(m)} \) and \( \Psi_1^{(m)} \) respectively. The problem considered in this illustration is that of reconstructing the profile of a kite. All these functions are depicted in Figure 1. Observe from this figure that while \( \phi_1 \) has a unique fixed point that maximizes \( \Psi_1 \), the function \( \phi_1^{(m)} \) has two fixed-points, the smallest one being a minimizer of \( \Psi_1^{(m)} \) and the largest one being a maximizer of \( \Psi_1^{(m)} \). Note that the cut-off index \( m \) is chosen to be the largest index in which the quantity \( \sum_{k=m+1}^{p} |v_k^{*}r_z|^2 < \epsilon \) for a given tolerance \( \epsilon \). In our image reconstruction experiment, tolerance \( \epsilon \) of 1.0e-12 is used.

The image reconstructions of a kite via L-curve and MKM-FP are displayed in Figure 2. In this example the far-field matrix \( \tilde{F}_d \) is \( 21 \times 21 \) (i.e., we use 21 incident observed directions), the noise level (in a relative sense) is 3% and the object is located in a grid of \( 64 \times 64 \) points. For MKM-FP, the cut-off index \( m \) is 20, meaning that MKM-FP uses only the first 20 singular values and corresponding singular vectors for the image reconstruction.

REFERENCES