Analysis of Optimized Quasi-Interpolators

Airton José Schmitt Junior[1], Leonardo Koller Sacht[1]

[1]Universidade Federal de Santa Catarina, SC, Brazil

Introduction

In computer graphics, image processing plays an important role and brings with it mathematical concepts such as Fourier transform and convolution. Thus, it becomes important to recognize the conditions under which an image can be reconstructed. Specifically, the error-free reconstruction of an input image (described by a function f) can be obtained from the so-called *Shannon-Whittaker Sampling Theorem*. However, most functions do not fulfill the hypotheses of this theorem.

For this reason, the modern sampling framework, illustrated by Fig. 1, is constituted by three generalized steps: A *continuous prefilter* ψ stage (or *antialiasing filter*); a discrete *digital filter* q; and the *generating function* φ, also called *reconstruction kernel*.

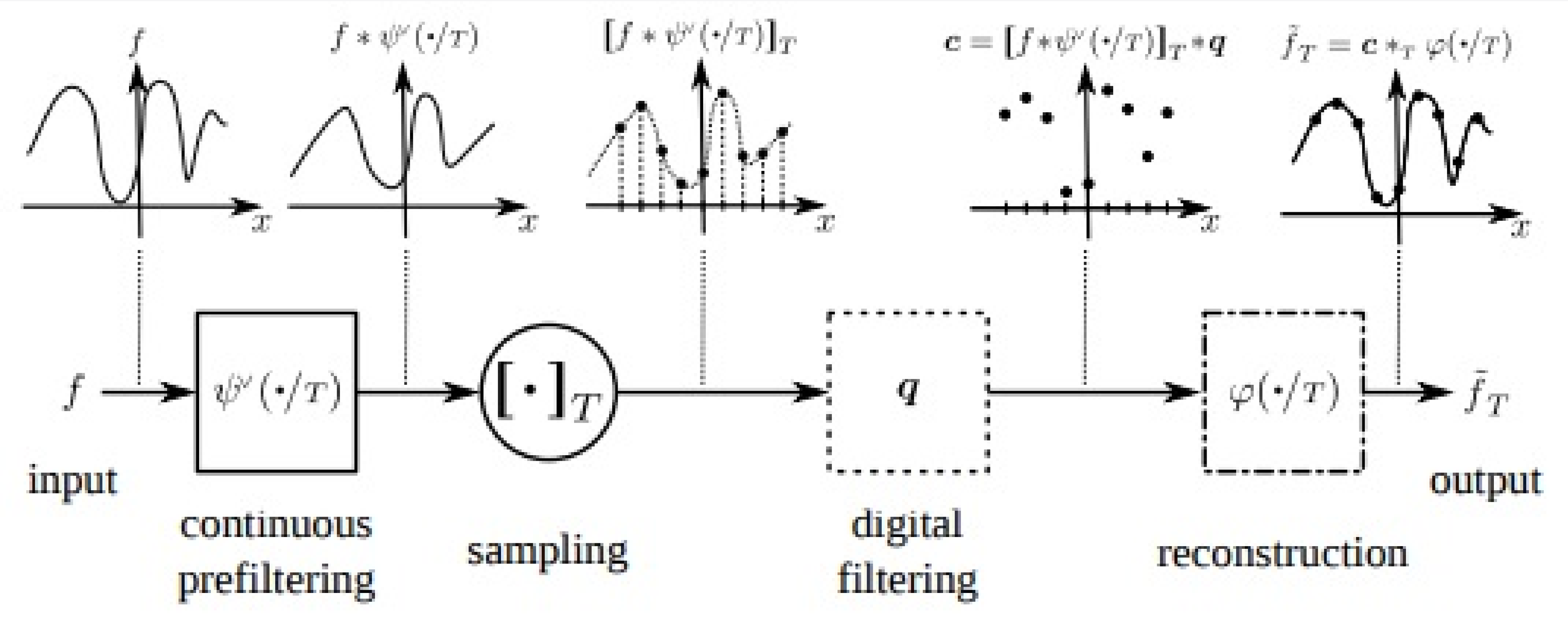

Figure 1. Modern sampling and reconstruction framework.

Notation

Let $f : \mathbb{R} \to \mathbb{C}$ be an input function and $q : \mathbb{Z} \to \mathbb{C}$ be a digital filter. Function scaling and reflection can be described, respectively, as $f_T(x) = f(x/T)$ and $f^\vee(x) = f(-x)$. Let continuous, discrete and mixed convolutions be denoted by:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, \mathrm{d}t,$$
$$(c * q)_n = \sum_{k\in\mathbb{Z}} c_k\, q_{n-k}, \quad \text{and}$$
$$(c *_T f)(x) = \sum_{i\in\mathbb{Z}} c_i\, f(x - iT).$$

Usually, uniform sampling of the input function at a sample spacing T (we consider $T = 1$, when there is no information about it) is given by $[f]_T = [\ldots, f(-T), f(0), f(T), \ldots]$.

These definitions can be used now to describe the reconstructed function $\tilde{f}$ by $\tilde{f} = \left[f * \psi_T^\vee\right]_T * q *_T \varphi_T$.

Previous Works

The main problem faced by several authors is related to the idea that the degrees of freedom of φ (usually described by polynomial functions by parts) and the digital filter q eventually vanish as the application impose specific requirements. Under these conditions, some important concepts were established for the quantification of the residual reconstruction error (or error kernel), such as approximation order and asymptotic constant [1]. Blu and Unser [2] quantified the residual as:

$$\|f - \tilde{f}\|_{L_2}^2 \approx \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 E(T\omega)\, \mathrm{d}\omega. \qquad (1)$$

Optimized Quasi-Interpolators

Based on the previous considerations, specially on Blu and Unser, Sacht and Nehab [3], in 2015, focused on the image reconstruction quality, directly involved with the degrees of freedom of ψ, q and φ. Thus, by quantifying the error between f and $\tilde{f}$, Sacht and Nehab came across with a minimization problem involving the kernel error and, consequently, the three steps of the image processing, which generate the quasi-interpolators optimization problem.

Constraint and Minimization Analysis

The proposal of this research is based on the analysis of the numerical results found by Sacht and Nehab, as well as minimizing the error kernel, initially considering the case of optimization in linear schemes. Such conditions, imposed on the parameters of the filtering and reconstruction stages, were chosen conveniently and in such a way that the minimization would be the best possible. By applying them to these parameters and using them to rewrite the error kernel, we have $E(\omega) = E(b_{0,0}, q_{1,0}, q_{1,1}, \omega)$. This equation denotes the dependence of the residual error with several terms related to the optimization problem. Thus, the residual error can be redefined now as:

$$F(b_{0,0}, q_{1,0}, q_{1,1}) := \int_0^{0.5} \frac{1}{\omega^2} E(b_{0,0}, q_{1,0}, q_{1,1}, \omega)\, \mathrm{d}\omega. \qquad (2)$$

In this context, it becomes necessary to verify that the minimization values obtained by Sacht and Nehab are really a local minimum and a solution to the problem. Indeed, when analyzing the gradient and the matrix of the second derivatives (hessian matrix) at these values, we have: $\nabla F(b_{0,0}, q_{0,0}, q_{1,1}) \approx \mathbf{0}$ and $\det(\mathrm{H}\,[F(b_{0,0}, q_{0,0}, q_{1,1})]) > 0$, and the result follows. In fact, it shows better results in the residual error minimization, when compared to other reconstruction schemes.

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References

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