# An eigenvalue method to calculate dominant poles of a transfer function

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**Abstract**

Here we give a mathematical proof that a current method that calculates dominant poles is locally convergent. Moreover, the convergence is at least quadratic.

*Key words:* Eigenvalues, Newton's method, dominant poles, transfer function
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## 1 Introduction

Several methods have been proposed to solve the problem of finding dominant poles of a transfer function. Here we focus in methods arising in Power Systems Control, like those reported in [2], [3], [4].

We pointed out in a discussion enclosed in [4] that one of these methods, called Dominant Pole Method, had local quadratic convergence. This method intended to calculate one dominant pole of a transfer function $f$ given by $f(s) = c^T(A - sI)^{-1}b$, where $A \in \mathbb{R}^{N\times N}$, $b, c \in \mathbb{C}^N$. From a starting value $s^{(0)} \in \mathbb{C}$, this method generates a sequence of complex values from iterations of the following scheme:

$$s^{(k+1)} = \frac{y_k^T A x_k}{y_k^T x_k},$$

where $x_k = (A - s^{(k)}I)^{-1}b$ and $y_k^T = c^T(A - s^{(k)}I)^{-1}$. Simple calculations show that this expression corresponds to the classical Newton method for finding a zero of the function $g$ given by $g(s) = \dfrac{1}{c^T(A - sI)^{-1}b}$.

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A generalization of this method in order to calculate several dominant poles of that transfer function was proposed in [3] and was called the Dominant Pole Eigenvalue Solver (DPSE). There, it was expected that dominant poles would be yielded by iterating the following scheme:

$$\{s_1^{(k+1)}, ..., s_n^{(k+1)}\} = \lambda(\,(Y_k^T X_k)^{-1}(Y_k^T A X_k)\,)$$

where

$$X_k e_i = \frac{(A - s_i^{(k)} I)^{-1} b}{c^T (A - s_i^{(k)})^{-1} b}$$

and

$$e_i^T Y_k = \frac{c^T (A - s_i^{(k)} I)^{-1}}{c^T (A - s_i^{(k)})^{-1} b}.$$

In this scheme, and from now on, $\lambda(A)$ denotes the spectrum of the matrix $A$, and $\{e_1, ..., e_N\}$ is the canonical basis of $\mathbb{R}^N$.

The purpose of this paper is to present a view of DPSE from a slight different perspective. We will see that the eigenvalues of $A$ are superattracting fixed points of the scheme defined by this method, which leads to quadratic convergence to them.

## 2 Preliminary results

Let $A \in \mathbb{R}^{N \times N}$ be a matrix with $N$ distinct eigenvalues. Let $b, c \in \mathbb{R}^N$ be two unit vectors such that the transfer function

$$f(s) = c^T (A - sI)^{-1} b$$

has $p$ dominant poles ($1 \leq p << N$), that is, for a spectral decomposition of $A$, $A = PDP^{-1}$,

$$c^T (A - sI)^{-1} b = \sum_{i=1}^{N} \frac{r_i}{\lambda_i - s}, \;\; |r_1| \geq \cdots \geq |r_p| >> |r_{p+1}| \geq \cdots |r_N| \geq 0,$$

where $r_i = (e_i^T P^{-1} b)(c^T P e_i)$, $\lambda_i = D_{ii}$.

First of all, we will show that, for each $k$, the matrix $M_k = (Y_k^T X_k)^{-1}(Y_k^T A X_k)$ is well defined, if some conditions were satisfied. The local quadratic convergence of the method will be proved in the next section.

Let $S = (s_1 \cdots s_p)^T \in \mathbb{C}^p$, such that $(\forall i, j)$ $s_i \neq s_j$, if $i \neq j$. Suppose that $(\forall k)$ $s_k$ is not an eigenvalue of $A$. For $k = 1, ..., p$ let

$$x_k = \frac{(A - s_k I)^{-1} b}{c^T (A - s_k I)^{-1} b}, \ y_k^T = \frac{c^T (A - s_k I)^{-1}}{c^T (A - s_k I)^{-1} b}.$$

**Remark 1** *Note that*

$$x_k = \frac{Adj\,(A - s_k I) b}{c^T Adj\,(A - s_k I) b}, \ y_k^T = \frac{c^T Adj\,(A - s_k I)}{c^T Adj\,(A - s_k I) b}.$$

*Therefore, the definition of $x_k$ and $y_k$ would make sense even when $s_k$ is a simple eigenvalue of $A$:*

$$x_k = \frac{P e_j}{c^T P e_j}, \ y_k^T = \frac{e_j^T P^{-1}}{e_j^T P^{-1} b},$$

*where $1 \leq j \leq N$ is such that $d_{jj} = s_k$. Moreover, $x_k$ and $y_k$ are entire functions of $s$.*

**Lemma 2** *Let $A \in \mathbb{R}^{N \times N}$ be a matrix with $N$ distinct eigenvalues. Let $S = (s_1 \cdots s_p)^T \in \mathbb{C}^p$, such that $(\forall i, j)$ $s_i \neq s_j$, if $i \neq j$. Let $X = [x_1 \cdots x_p]$ and $Y = [y_1 \cdots y_p]$, where $(\forall k)$ $x_k$ and $y_k$ are defined as above. Then the matrix $Y^T X$ is symmetric and non-singular.*

***PROOF.*** *First, suppose that $(\forall k)$ $s_k$ is not an eigenvalue of $A$. Therefore,*

$$(Y^T X)_{ij} = e_i^T Y^T X e_j = \frac{c^T (A - s_i I)^{-1}}{c^T (A - s_i I)^{-1} b} \frac{(A - s_j I)^{-1} b}{c^T (A - s_j I)^{-1} b} =$$

$$= \frac{c^T (A - s_i I)^{-1}}{c^T (A - s_i I)^{-1} b} \frac{(A - s_j I)^{-1} b}{c^T (A - s_j I)^{-1} b} = \frac{c^T P (D - s_i I)^{-1}}{c^T (A - s_i I)^{-1} b} \frac{(D - s_j I)^{-1} P^{-1} b}{c^T (A - s_j I)^{-1} b} =$$

$$= \frac{(\lambda_1 - s_i, ..., \lambda_N - s_i) diag(P^T c)}{c^T (A - s_i I)^{-1} b} \frac{diag(P^{-1} b)(\lambda_1 - s_j, ..., \lambda_N - s_j)^T}{c^T (A - s_j I)^{-1} b}.$$

*Let $G \in \mathbb{C}^{N \times p}$ be defined by $G_{ij} = (\lambda_i - s_j)^{-1}$. Let $R$ be the $N \times N$ diagonal matrix such that $R_{ii} = (c^T P e_i)(e_i^T P^{-1} b)$, that is, the residual $r_i$. Let $E$ be the $N \times N$ diagonal matrix defined by $F_{ii} = [c^T (A - s_i I)^{-1} b]^{-1}$. Then, $Y^T X = E G^T R G E$.*

*Since $|r_1|, ..., |r_p|$ are nonzero, the rank of $R$ is at least $p$. $(\forall i)$ $E_{ii} \neq 0$ for $s_i$ is not an eigenvalue. Moreover, since the submatrix $G(1 : p, 1 : p)$ is a non-singular Cauchy matrix [1], the rank of $R G G^T$ is $p$. Therefore, the rank of $Y^T X$ is $p$.*

*If for some $k$ $s_k$ is an eigenvalue of $A$, let $E_{kk} = R_{kk}$ and $Ge_k = e_k$, and the proof works with slight modifications.*

□

**Remark 3** *In practice, the columns vectors $x_k, y_k$ are obtained after normalizing the vectors $(A - s_k I)^{-1}b$ and $(A - s_k I)^{-T}c$, and dividing both the yielding vectors $z_k, w_k$, respectively, by $c^T z_k$ (which is equal to $b^T w_k$). Notice that the result is the same in exact arithmetic. This procedure has diminished numerical errors near convergence.*

## 3 DPSE has local quadratic convergence

Let $M = (Y^T X)^{-1} Y^T AX$. Note that if $s_i$ is an eigenvalue of $A$ then, by **Remark 1**, $Y^T AXe_i = s_i Y^T Xe_i$. Thus $s_i$ is also an eigenvalue of $M$.

Now, since

$$AXe_i = A\frac{(A - s_i I)^{-1}b}{c^T(A - s_i I)^{-1}b} = s_i Xe_i + \frac{b}{c^T(A - s_i I)^{-1}b},$$

it follows that

$$M = diag(S) + (Y^T X)^{-1} Y^T B,$$

where $diag(S)$, as above, is the diagonal matrix defined with the elements of the vector $S$ in the diagonal, and $Be_i = b/c^T(A - s_i I)^{-1}b$ (if $s_i$ is an eigenvalue of $A$, then $AXe_i = s_i Xe_i$ and $Be_i = 0$). Hence $M$ is a rank-one perturbation of the diagonal matrix $diag(S)$.

**Definition 4** *Suppose $X$ is a topological space and $\sim$ is an equivalence relation on $X$. We define a topology on the quotient set $X/\sim$ (the set consisting of all equivalence classes of $\sim$) as follows: a subset $V \subseteq X/\sim$ is open in $X/\sim$ if and only if its preimage $q^{-1}(V)$ is open in $X$, where $q : X \to X/\sim$ is the projection map which sends each element of $X$ to its equivalence class. The quotient set $X/\sim$ with this topology is said to be a quotient space.*

Now, let $P\mathbb{C}^p$ be the quotient space $\mathbb{C}^p/\sim$, where $x \sim y$ iff there is a permutation matrix $P$ such that $y = Px$. Let $\Omega = \{(s_1, ..., s_p) | s_i \neq s_j, \text{ if } i \neq j\}/\sim$. $\Omega$ is an open set of $P\mathbb{C}^p$.

Let $\lambda_1, ..., \lambda_p$ be distinct eigenvalues of $A$. Since the spectrum of a matrix depends continuosly on its entries, let $\mathcal{O}$ be an open set of $\Omega$ which contains $(\lambda_1, ..., \lambda_p)/\sim$ such that the eigenvalues of $M$ are simple.

Let $F : \mathcal{O} \to P\mathbb{C}^p$ be the function defined as follows: if $M = VTV^{-1}$ is a spectral decomposition of $M$, $F(S)$ is the class of equivalence of $(F_1(S), ..., F_p(S))$ where

$$F_i(S) = s_i + \frac{c^T XVe_i}{c^T(A - s_iI)^{-1}XVe_i}$$

Since $c^T(A - s_iI)^{-1}A = c^T + s_ic^T(A - s_iI)^{-1}$, and $c^T(A - s_iI)^{-1}AXVe_i = t_{ii}c^T(A - s_iI)^{-1}XVe_i$, it follows that

$$c^T XVe_i = (t_{ii} - s_i)c^T(A - s_iI)^{-1}XVe_i.$$

Therefore, $F_i(S) = t_{ii}$. Thus, $F$ is well defined. Moreover, since the eigenvalues of $M$ are simple, $F$ is a differentiable function.

It is not difficult to see that (the class of) a $p$-uple of distinct eigenvalues $(\lambda_1, ..., \lambda_p)$ is a fixed point of $F$. For if $A = PDP^{-1}$ and $S = (\lambda_1, ..., \lambda_p)$, then

$$e_i^T Y X e_j = \frac{e_{k_i}^T P^{-1} P e_{k_j}}{e_{k_i}^T P^{-1} bc^T P e_{k_j}} = \frac{e_{k_i}^T e_{k_j}}{e_{k_i}^T P^{-1} bc^T P e_{k_j}}$$

and

$$e_i^T Y A X e_j = \frac{e_{k_i}^T P^{-1} A P e_{k_j}}{e_{k_i}^T P^{-1} bc^T P e_{k_j}} = \frac{e_{k_i}^T D e_{k_j}}{e_{k_i}^T P^{-1} bc^T P e_{k_j}}$$

where $1 \leq k_r \leq N$ is such that $d_{k_r k_r} = \lambda_r$. Hence $M = diag(d_{k_1k_1}, ..., d_{k_pk_p})$, and so $F(S/\sim) = S/\sim$.

The following theorem is the main goal of this paper.

**Theorem 5** *If for each iteration $k$ the $s_i$ are distinct then the DPSE scheme has local quadratic convergence.*

***PROOF.*** *The result follows once it is established that $DF(\Lambda) = 0$, where $\Lambda = (\lambda_1, ..., \lambda_p)$. To this end, one computes that*

$$\frac{\partial F_i}{\partial s_j}(\Lambda) = \delta_{ij} + \det(A - \lambda_i I)\frac{c^T \frac{\partial}{\partial s_j}(XVe_i)}{c^T Adj\,(A - \lambda_i I)XVe_i}$$

$$-\delta_{ij}\frac{c^T XVe_i}{(c^T Adj\,(A - \lambda_i I)XVe_i)^2} c^T(Adj\,(A - \lambda_i I))^2 XVe_i$$

$$-\det(A - \lambda_i I)\frac{c^T XVe_i}{(c^T Adj\,(A - \lambda_i I)XVe_i)^2} c^T Adj\,(A - \lambda_i I)\frac{\partial}{\partial s_j}(XVe_i) =$$

$$= \delta_{ij} - \delta_{ij}\frac{c^T XVe_i}{(c^T Adj\,(A - \lambda_i I)XVe_i)^2} c^T(Adj\,(A - \lambda_i I))^2 XVe_i$$

*where* $\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$

*For $S = \Lambda$, it follows that $V$ is the identity matrix, and so,*

$$\frac{(c^T XVe_i)(c^T(Adj\,(A-\lambda_i I))^2 XVe_i)}{(c^T Adj\,(A-\lambda_i I)\,XVe_i)^2} =$$

$$= \frac{(c^T Xe_i)(c^T(Pe_{k_i}e_{k_i}^T P^{-1})^2 Xe_i)}{(c^T Pe_{k_i}e_{k_i}^T P^{-1}Xe_i)^2} = \frac{(c^T Xe_i)(c^T Pe_{k_i}e_{k_i}^T P^{-1}Xe_i)}{(c^T Pe_{k_i}e_{k_i}^T P^{-1}Xe_i)^2} =$$

$$= \frac{c^T Xe_i}{c^T Pe_{k_i}e_{k_i}^T P^{-1}Xe_i} = \frac{c^T Xe_i}{c^T Pe_{k_i}e_{k_i}^T P^{-1}Pe_{k_i}} =$$

$$\frac{c^T Xe_i}{c^T Pe_{k_i}} = 1$$

*Therefore,* $\dfrac{\partial F_i}{\partial s_j}(\Lambda) = 0.$ □

## 4 Conclusions

DPSE has been extensively applied and become popular in model reductions of large linear systems (see [3], [4]). In this paper it was proven that DPSE is like a Newton's method: its convergence is quadratic once one of its iterates is attracted by a fixed point. However, there is no proof whatsoever that DPSE converges to dominant poles in the sense of [3]. This will depend on the basins of attraction of the fixed points and on the initial point selection. Moreover, in practice it has been seing that the performance of DPSE as an eigensolver depends enormously of the vectors $b$ and $c$, besides the initial vector for its scheme. Some experiments with DPSE as a general eigensolver are reported in [2].

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