Eigenvalue methods for calculating dominant poles of a transfer function and their applications in small-signal stability

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Abstract

In this paper we give a new short proof of the local quadratic convergence of the Dominant Pole Spectrum Eigensolver (DPSE). Also we introduce here the Diagonal Dominant Pole Spectrum Eigensolver (DDPSE), another fixed-point method that computes several eigenvalues of a matrix at a time, which also has local quadratic convergence. From results of some experiments with a large power system model, it is shown that DDPSE can also be used in small-signal stability studies to compute dominant poles of a transfer function of the type $c^T(A - sI)^{-1}b$, where b and c are vectors, by its own or combined with DPSE. Besides DDPSE is also effective in finding low damped modes of a large scale power system model.

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1. Introduction

A power system can be described as a coupled system of differential and algebraic equations. The following matrix equation is obtained by linearizing

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the system model at an operating point:

$$\begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where J_1, J_2, J_3, J_4 are matrices, x is the vector of dynamical variables and y is the vector of algebraic ones. The matrix $J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$ denotes the Jacobian matrix of the system. Since $y = -J_4^{-1} J_3 x$, we have

$$\dot{x} = \left(J_1 - J_2 J_4^{-1} J_3\right) x.$$

The matrix $A = J_1 - J_2 J_4^{-1} J_3$ is called the state matrix of the system. For large scale power systems J is very sparse, while A is not in general. We can observe that, given a vector x, we can easily compute $z = (A - \lambda I)^{-1} x$ by solving the following algebraic system

$$\begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}. \tag{1}$$

From that we note that we can compute eigenvalues of the state matrix A, even without explicitly calculating it.

Knowledge of rightmost eigenvalues of A is essential in the power system small-signal stability analysis. In the literature there are several papers that use standard methods to compute rightmost eigenvalues of a state matrix from Equation 1 [3, 10]. On the other hand, some authors prefer instead to deal with the generalized eigenvalue problem $Ju = \lambda Eu$, where

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

and I is the identity matrix [8]. However, this approach requires a non-obvious strategy to control instability caused by the spurious eigenvalue at infinity, for instance, if you use generalized Möbius transforms [1, 4]. The landscape of small-signal stability analysis has changed a little when methods based on transfer functions, like the Dominant Pole Algorithm (DPA) [5] and the Dominant Pole Spectrum Eigensolver (DPSE) [6], have arisen in literature. It was found that DPA, which computes a single eigenvalue at a time, is actually a Newton's method ([2], [9]). Furthermore, DPSE is a fixed-point method that can compute several eigenvalues at a time, and

it can be seen as a generalization of DPA in a certain way. In order to calculate p eigenvalues by DPSE, you should solve p linear systems of the type $(J-\mu_k E)z = w$, $k = 1:p$, at each step. On the other hand, in power system stability studies, a suitable pre-ordering of the Jacobian matrix equations and variables prevents large amounts of fill-in and thus its sparse LU factorization is done with lower computational complexity. Moreover, DPSE converges quadratically and a proof of its local quadratic convergence first appeared in [2]. Nevertheless, here we give an easier proof, which can be seen in §2. We will also show that a slight modification of DPSE yields a new fixed-point method, the Diagonal Dominant-Pole Spectrum Eigensolver (DDPSE), that also has local quadratic convergence, as discussed in §3. In the last section, we see results obtained from the implementation of those two methods in MATLAB. From these tests we verify that both methods really compute dominant poles of a transfer function of the type $c^T(A-sI)^{-1}b$, where b and c are vectors, besides being also effective in finding low damped modes of the system. Summarizing, our paper makes the following contributions: it shows a much easier proof of the local quadratic convergence of DPSE; it introduces DDPSE, a new eigensolver, that also has local quadratic convergence; it presents some performance results obtained from DPSE, DDPSE and SADPA (Subspace Accelerated Dominant Pole Algorithm) when applied to a large generalized eigenvalue problem. SADPA is part of a public software, which can be seen in `https://sites.google.com/site/rommes/software`.

2. DPSE

The motivation for the Dominant Pole Spectrum Eigensolver (DPSE) came from SISO dynamical systems (E, J, B, C, D) of the form

$$\begin{cases} E\dot{x}(t) = Jx(t) + Bu(t) \\ \quad y(t) = C^T x(t) + Du(t) \end{cases} \tag{2}$$

where $J, E \in \mathbb{R}^{N\times N}$, $E = diag([1, ..., 1, 0, ..., 0])$; $x(t) \in \mathbb{R}^{N\times 1}$ is composed by dynamical and algebraic variables, $x_d(t)$ and $x_a(t)$, which are respectively associated with the unit and the null diagonal entries of E; $B, C \in \mathbb{R}^{N\times 1}$, where $B^T = \begin{pmatrix} B_d^T & B_a^T \end{pmatrix}$ and $C^T = \begin{pmatrix} C_d^T & C_a^T \end{pmatrix}$; $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the output, and $D \in \mathbb{R}$. Suppose that there are n dynamical variables in the system. If $J_1 = J(1:n, 1:n)$, $J_2 = J(1:n, n{+}1:N)$, $J_3 = J(n{+}1:N, 1:n)$ and $J_4 = J(n+1:N, n+1:N)$, then that system is equivalent to the

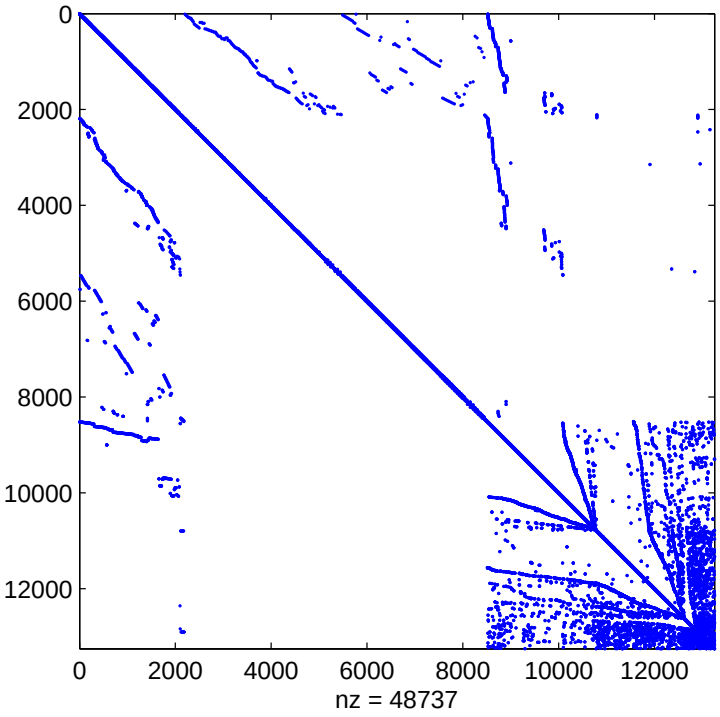

Figure 1: Sparse pattern of the Jacobian matrix that corresponds to a planning model of the Brazilian Interconnected Power System (nz is the number of its nonzero entries)

following one:

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = c^T x(t) + du(t) \end{cases}$$

where the matrix $A = J_1 - J_2 J_4^{-1} J_3$, which is called the state matrix of the system, $b = B_d - J_2 J_4^{-1} B_a$, $c = C_d - J_3^T J_4^{-T} C_a$ and $d = D - C_a^T J_4^{-1} B_a$. The corresponding transfer function $h : \mathbb{C} \to \mathbb{C}$ is defined as

$$h(s) = c^T \left(sI - \left(J_1 - J_2 J_4^{-1} J_3 \right) \right)^{-1} b + d. \tag{3}$$

Simple calculation yields that

$$h(s) = C^T \left(sE - J \right)^{-1} B + D.$$

Note that, for any $\mu \notin \lambda(A)$ and for any $b \in \mathbb{C}^n$,

$$\begin{pmatrix} (A - \mu I)^{-1} b \\ 0 \end{pmatrix} = E \left(J - \mu E \right)^{-1} \begin{pmatrix} b \\ 0 \end{pmatrix}. \tag{4}$$

Suppose that $A \in \mathbb{R}^{n \times n}$ is diagonalizable, that is, $A = PDP^{-1}$, where P is an invertible matrix and D, a diagonal matrix. So, the spectrum of A, denoted by $\lambda(A)$, is the set of the diagonal entries of D. From now on we suppose that every eigenvalue of A is simple.

Let $b, c \in \mathbb{R}^n$ such that $c^T (A - sI)^{-1} b \neq 0$ for all $s \in \mathbb{C} - \lambda(A)$, and $c^T P e_k e_k^T P^{-1} b \neq 0$ for $k = 1 : n$. If $d = 0$, from Equation 3,

$$h(s) = \sum_{k=1}^{n} \frac{R_k}{d_{k,k} - s},$$

where $R_k = c^T P e_k e_k^T P^{-1} b$, $k = 1 : n$.

Definition 1. A pole $d_{k,k}$ is called a dominant pole if it corresponds to a relatively large $m_k = \dfrac{|R_k|}{|Re(d_{k,k})|}$. m_k is the measure for dominance of the pole $d_{k,k}$.

Remark 1. Suppose that, for some $k \in \{1, ...n\}$, d_{kk} is a converged eigenvalue at step r. Then, the corresponding right and left vectors, x_r and y_r, are such that

$$y_r^T x_r \approx \frac{e_k^T P^{-1}}{e_k^T P^{-1} b} \frac{P e_k}{c^T P e_k} = \frac{1}{c^T P e_k e_k^T P^{-1} b} = \frac{1}{R_k}.$$

From that we can easily calculate the corresponding measure m_k for dominance of the pole.

Now, since

$$\frac{(A-sI)^{-1}b}{c^T(A-sI)^{-1}b} = \frac{Adj\,(A-sI)b}{c^T Adj\,(A-sI)b} \text{ and } \frac{(A^T-sI)^{-1}c}{c^T(A-sI)^{-1}b} = \frac{Adj\,(A^T-sI)c}{c^T Adj\,(A-sI)b},$$

we conclude that the functions $f : \mathbb{C} \to \mathbb{C}^n$ and $g : \mathbb{C} \to \mathbb{C}^n$ defined respectively by

$$f(s) = \begin{cases} \frac{(A-sI)^{-1}b}{c^T(A-sI)^{-1}b} & \text{for } s \in \mathbb{C} - \lambda(A); \\ \frac{Pe_j}{c^T Pe_j} & \text{for } s = d_{jj}, j = 1:n. \end{cases} \tag{5}$$

and

$$g(s) = \begin{cases} \frac{(A^T-sI)^{-1}c}{c^T(A-sI)^{-1}b} & \text{for } s \in \mathbb{C} - \lambda(A); \\ \frac{P^{-T}e_j}{b^T P^{-T}e_j} & \text{for } s = d_{jj}, j = 1:n. \end{cases} \tag{6}$$

are entire functions (bear in mind that any entry of the Classical Adjoint of $(A - sI)$ is a sum of products of its elements).

Let $S = (s_1 \dots s_p)^T \in \mathbb{C}^p$, $p \leq n$, and suppose that $X(S), Y(S) \in C^{n\times p}$ are defined by $X(S)e_k = f(s_k)$ and $Y(S)e_k = g(s_k)$, for $k = 1 : p$, where $e_1, ..., e_n$ are the canonical vectors.

Lemma 1. *Let $S_0 = \left(d_{k_1,k_1} \dots d_{k_p,k_p}\right)$, where $d_{k_1,k_1} \dots d_{k_p,k_p}$ is a p-uple of distinct eigenvalues, $1 \leq k_1 < ... < k_p \leq n$. Then, there is an open neighborhood $\mathcal{O}$ of S_0 so that $Y(S)^T X(S)$ is invertible for any S belonging to $\mathcal{O}$.*

Proof. The lemma follows because

$$Y(S_0)^T X(S_0) = \operatorname{diag}\left(\left[\frac{1}{c^T Pe_{k_1} e_{k_1}^T P^{-1}b}, ..., \frac{1}{c^T Pe_{k_p} e_{k_p}^T P^{-1}b}\right]\right), \tag{7}$$

that is, $Y(S_0)^T X(S_0)$ is invertible. □

Let $F : \mathcal{O} \to \mathbb{C}^{n\times n}$ defined by $F(S) = \left(Y(S)^T X(S)\right)^{-1}\left(Y(S)^T AX(S)\right)$. We see that F is analytic. Since

$$F\left(\left(d_{k_1,k_1} \dots d_{k_p,k_p}\right)^T\right) = \operatorname{diag}\left(\left[d_{k_1,k_1}, ..., d_{k_p,k_p}\right]\right),$$

every eigenvalue of F is simple for S belonging to an open subset $\mathcal{Z}$ of $\mathcal{O}$. Let $G : \mathcal{Z} \to \mathbb{C}^p$ be the function defined by $G(S) = (\lambda_1(F(S)) \dots \lambda_p(F(S)))$, where $\lambda_1(F(S)) < ... < \lambda_p(F(S))$ are the eigenvalues of $F(S)$ (for some order

on the complex numbers). Observe that $\left(d_{k_1,k_1} \dots d_{k_p,k_p}\right)^T$ is a fixed point of G. On the other hand, if s_i is not an eigenvalue for $i = 1 : p$, we conclude that $F(S)$ is equal to

$$\text{diag}(S) + \left(Y(S)^T X(S)\right)^{-1} ee^T \text{diag}\left(\left[\frac{1}{c^T(A - s_1 I)^{-1}b}, ..., \frac{1}{c^T(A - s_p I)^{-1}b}\right]\right),$$

for $Y(S)b = e$, where $e = ones(p, 1)$. In order to calculate the derivative of G, we first see that

$$\frac{\partial F}{\partial s_i}\left(d_{k_1,k_1}, ..., d_{k_p,k_p}\right) = e_i e_i^T - \frac{1}{c^T P e_{k_i} e_{k_i}^T P^{-1} b} v e_i^T,$$

where $v^T = \left(c^T P e_{k_1} e_{k_1}^T P^{-1} b \dots c^T P e_{k_p} e_{k_p}^T P^{-1} b\right)$. Therefore,

$$\frac{\partial \lambda_k}{\partial s_i}\left(d_{k_1,k_1}, ..., d_{k_p,k_p}\right) = \langle \frac{\partial \lambda_k}{\partial a_{rs}}\left(F\left(d_{k_1,k_1}, ..., d_{k_p,k_p}\right)\right), \frac{\partial F}{\partial s_i}\left(d_{k_1,k_1}, ..., d_{k_p,k_p}\right) \rangle.$$

Notice that

$$\frac{\partial \lambda_k}{\partial a_{rs}}\left(F\left(d_{k_1,k_1}, ..., d_{k_p,k_p}\right)\right) = e_k e_k^T.$$

Hence,

$$\frac{\partial \lambda_k}{\partial s_i}\left(d_{k_1,k_1}, ..., d_{k_p,k_p}\right) = e_i^T e_k - \frac{1}{c^T P e_{k_i} e_{k_i}^T P^{-1} b} e_i^T e_k e_k^T v = 0.$$

Definition 2 (DPSE). The fixed-point iteration applied to the function

$$G(S) = \left(\lambda_1(F(S)) \dots \lambda_p(F(S))\right),$$

where $F(S) = \left(Y(S)^T X(S)\right)^{-1}\left(Y(S)^T A X(S)\right)$, defines the Dominant Pole Spectrum Eigensolver.

With this definition, we have just proved the following proposition:

Proposition 2 (DPSE converges at least quadratically)**.** *Let λ_1, ..., λ_p be p distinct eigenvalues of A. Then, there is a neighborhood V of $(\lambda_1 \dots \lambda_p)^T$ such that DPSE converges at least quadratically to $(\lambda_1 \dots \lambda_p)^T$ for any $S_0 \in V$.*

Remark 2. Suppose that DPSE has just calculated $S^{(r)} = \left(s_1^{(r)} \dots s_p^{(r)} \right)^T$ from $S^{(r-1)}$, and $X(S^{(r)})$ has not been computed yet. For any $k = 1 : p$, we have that

$$AX(S^{(r-1)})e_k =$$

$$= (A - s_k^{(r-1)} I + s_k^{(r-1)} I) \frac{(A - s_k^{(r-1)})^{-1} b}{c^T (A - s_k^{(r-1)})^{-1} b} =$$

$$= \frac{b}{c^T (A - s_k^{(r-1)})^{-1} b} + s_k^{(r-1)} X(S^{(r-1)})e_k.$$

Therefore, the relative error, $||(A - s_k^{(r)})X(S^{(r-1)})e_k||/||X(S^{(r-1)})e_k||$, becomes as follows:

$$\frac{||\frac{b}{c^T (A - s_k^{(r-1)})^{-1} b} + (s_k^{(r-1)} - s_k^{(r)})\, X(S^{(r-1)})e_k||}{||X(S^{(r-1)})e_k||}. \tag{8}$$

Note that, if $s_k^{(r-1)}$ tends to an eigenvalue, then $c^T (A - s_k^{(r-1)})^{-1} b$ tends to zero, and so, $X(S^{(r-1)})e_k$ is an approximation of an associated eigenvector.

3. DDPSE

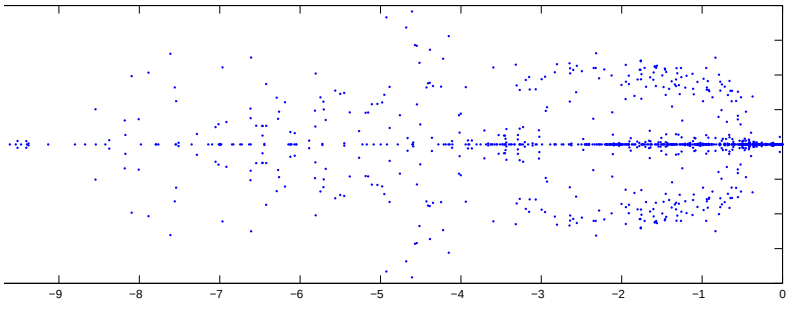

Figure 2: Partial spectrum of the state matrix

If the fixed-point method is only applied to the diagonal of the matrix $F(S)$, then a variant of the DPSE is created, which will be called here as the Diagonal Dominant Pole Spectrum Eigensolver (DDPSE):

$$H\left((s_1 \dots s_p)^T \right) \stackrel{\text{def}}{=} \text{diag}\left(F(s_1 \dots s_p) \right) = (s_1 \dots s_p)^T +$$

$$+\text{diag}\left(\left[\frac{1}{c^T(A-s_1I)^{-1}b},...,\frac{1}{c^T(A-s_pI)^{-1}b}\right]\right)\left(Y(S)^TX(S)\right)^{-1}e$$

where $e = ones(p,1)$ and $s_i \notin \lambda(A)$ for all $i = 1:p$. Note that for $i = 1:p$

$$\frac{1}{c^T(A-s_iI)^{-1}b} = \frac{\det(A-s_iI)}{c^TAdj(A-s_iI)b},$$

which is zero if s_i is an eigenvalue of A. So, we define

$$H\left((\lambda_1 ... \lambda_p)^T\right) = (\lambda_1 ... \lambda_p)^T$$

if $(\lambda_1 ... \lambda_p)^T$ is a p-uple of distinct eigenvalues. Therefore, any p-uple of distinct eigenvalues is a fixed-point of H.

Proposition 3 (DDPSE converges at least quadratically)**.** *Let λ_1, λ_2, ..., λ_p be p distinct eigenvalues of A. Then, there is a neighborhood V of $(\lambda_1 ... \lambda_p)^T$ such that, given any $S_0 \in V$, DDPSE converges at least quadratically to $(\lambda_1 ..., \lambda_p)^T$.*

Proof. Let $\lambda_1 = d_{k_1,k_1}$, ..., $\lambda_p = d_{k_p,k_p}$. Then,

$$\frac{\partial H}{\partial s_i}\left((d_{k_1,k_1} ... d_{k_p,k_p})\right) = e_i - \frac{1}{c^TPe_{k_i}e_{k_i}^TP^{-1}b}e_ie_i^T\left(Y(S)^TX(S)\right)^{-1}e.$$

From 7, $\left(Y(S)^TX(S)\right)^{-1} = \text{diag}\left(\left[c^TPe_{k_1}e_{k_1}^TP^{-1}b, ..., c^TPe_{k_p}e_{k_p}^TP^{-1}b\right]\right)$. So,

$$\frac{\partial H}{\partial s_i}\left((d_{k_1,k_1} ... d_{k_p,k_p})\right) = 0. \quad \square$$

Our test matrix J is sparse (density about 0.028%), of order $N = 13251$. The pencil $Jv = \lambda Ev$, where $E = diag(1, ..., 1, 0, ..., 0)$, corresponds to the problem $Ax = \lambda x$, where $A = J_1 - J_2J_4^{-1}J_3$ is of order $n = 1664$. This is the Jacobian matrix that corresponds to a planning model of the Brazilian Interconnected Power System and that had already used for tests in [8]. In the tests with DPSE and DDPSE, we have used data that can be obtained from a specific transfer function. From this, $D = 0$, and the input vector $B = \left(B_d^T\, B_a^T\right)$ and the output vector $C = \left(C_d^T\, C_a^T\right)$ are as follows: $B(524) = B(1442) = 1$, $B(1884) = B(1918) = -1$, and the others entries of B are null; $C(11558) = 26.5721$, $C(11559) = -13.1127$, $C(12502) = -29.2954$,

$C(12503) = 3.7609$ and these are all the non-zero elements of C. By using MATLAB, we verify that $C_a^T J_4^{-1} B_a = 0$, and then $d = 0$. Note that, from Equation 3, for any $s \notin \lambda(A)$, $C^T(J - sE)^{-1}B = c^T(A - sI)^{-1}b$, where $b = B_d - J_2 J_4^{-1} B_a$, $c = C_d - J_3^T J_4^{-T} C_a$.

Suppose we start with p complex values: $s^{(0)} = (s_1^{(0)} ... s_p^{(0)})^T \in \mathbb{C}^p$. Let $X = X^{(0)}$ and $Y = Y^{(0)}$ be two matrices $N \times p$, such that for $j = 1 : p$ $X(:, j) = (J - s_j^{(0)}E)^{-1}B / C^T(J - s_j^{(0)}E)^{-1}B$ and $Y(:, j) = (J^T - s_j^{(0)}E)^{-1}C / C^T(J - s_j^{(0)}E)^{-1}B$. Let $V = V^{(0)}$ and $W = W^{(0)}$ be two matrices $n \times p$ such that, for $j = 1 : p$,

$$V(:, j) = (A - s_j^{(0)}I)^{-1}b / c^T(A - s_j^{(0)}I)^{-1}b$$

and

$$W(:, j) = (A^T - s_j^{(0)}I)^{-1}c / c^T(A - s_j^{(0)}I)^{-1}b.$$

So, we have

$$W^T V = Y^T E X,$$

and for $j = 1 : p$

$$W^T A V e_j = W^T(A - s_j^{(0)}I + s_j^{(0)}I)(A - s_j^{(0)}I)^{-1}b / c^T(A - s_j^{(0)}I)^{-1}b =$$

$$= W^T b / c^T(A - s_j^{(0)}I)^{-1}b + s_j^{(0)} W^T V e_j = e / c^T(A - s_j^{(0)}I)^{-1}b + s_j^{(0)} W^T V e_j,$$

where $e = ones(p, 1)$. Therefore,

$$F = (W^T V)^{-1} W^T A V = (W^T V)^{-1} e v^T + S,$$

where $S = S^{(0)} = diag(s^{(0)})$, and

$$v^T = \left(v^{(0)}\right)^T = \left(1 / c^T(A - s_1^{(0)}I)^{-1}b \, ... \, 1 / c^T(A - s_p^{(0)}I)^{-1}b\right).$$

Note that, from Equation 4, we can use either DPSE or DDPSE to partially solve Equation 2, that is, we can carry out all the computation with J, B and C without explicitly computing A, b and c.

We have specified a relative error tolerance of 10^{-5} to both right and left vectors. Here we have used

$$\frac{||(J - s_j^{(k)}E)X^{(k-1)}e_j||}{||X^{(k-1)}e_j||} \text{ and } \frac{||(J^T - s_j^{(k)}E)Y^{(k-1)}e_j||}{||Y^{(k-1)}e_j||}$$

as the relative errors instead of using the equivalent formulae obtained from Equation 8, for we do not want to compute b explicitly. Convergence is achieved when both of these convergence criteria are satisfied. Suppose we have just obtained the first converged value $s_1^{(k)} = \lambda_1$. Then, we save their corresponding right and left generalized eigenvectors, $X^{(k-1)}e_1$ and $Y^{(k-1)}e_1$, to the respective columns of $X^{(r)}$ and $Y^{(r)}$ for $r \geq k$. In the case of a complex converged value, we add one more column to $X^{(r)}$ and $Y^{(r)}$, with the right and left generalized eigenvectors associated to its conjugate: $s_{p+1}^{(k)} = \overline{\lambda_1}$. Note that the respective columns of $V^{(r)}$ and $W^{(r)}$ are formed by the dynamical variables of $X^{(k-1)}e_1$ and $Y^{(k-1)}e_1$, respectively. Those converged eigenvalues can be deflated from the problem by this procedure. To see that, suppose $Ve_2 \approx a_1v_1+a_2v_2$, where $v_1 \approx V^{(k-1)}e_1$ and v_2 is an eigenvector corresponding to λ_2. Hence,

$$AVe_2 \approx a_1\lambda_1v_1 + a_2\lambda_2v_2 \approx a_1(\lambda_1 - \lambda_2)Ve_1 + \lambda_2Ve_2,$$

and so,

$$\left(W^TV\right)^{-1} W^TAVe_2 \approx a_1(\lambda_1 - \lambda_2)e_1 + \lambda_2e_2.$$

Thus,

$$F = \begin{pmatrix} \lambda_1 & \times & \times & \cdots & \times \\ 0 & \lambda_2 & \times & \cdots & \times \\ 0 & 0 & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \times & \cdots & \times \end{pmatrix}.$$

4. Numerical results

All the computed eigenvalues were actually complex conjugate pairs, but only their positive imaginary values are listed in the following tables for brevity. In Table 1 you see 7 eigenvalues calculated by DDPSE and DPSE from starting values $s_k = k.(-1 + i)$, $k = 1 : 7$. There you can observe that the number of LU factorizations required by DDPSE is about 13% smaller than that required by DPSE. Also, DDPSE found six poles that have the largest measures of relative dominance, while DPSE found four. However, for $k > 7$, e.g., $k = 10$, DPSE had a better performance regarding the number of LU factorizations than DDPSE. Table 2 compares the eigenvalues calculated respectively by DDPSE, DPSE and SADPA from starting values

$s_k = (k - 1/2)(-1 + i)$, $k = 1 : 10$. The Subspace Accelerated Dominant Pole Algorithm (SADPA) was the algorithm chosen for comparison with DPSE and DDPSE due to excellent results formerly obtained with our test matrix, which was reported for instance in [7]. To this end, we have used the script test_dpa_family.m as starting point, with options 3 (our sparse matrix of order 13251) and 2 (SADPA), which is available at `https://sites.google.com/site/rommes/software`. The default (maximum) number of eigenvalues to be calculated by SADPA is ten. All the measures of relative dominance of the eigenvalues obtained by SADPA were computed by DPSE. In order to compare the results from the three methods we have used the same $(p =)10$ initial shifts given in the left complex semi-plane: $s_r = r*(-1+i)/2$, $r = 1 : 2 : 20$. You see that with those 10 initial shifts DDPSE did not converge to $-0.0335 \pm 1.0787\,i$, which are the most dominant poles of the system. On the other hand, our eigenvalue problem is so ill-conditioned that if $s_k = (k - 1/2)(-1 + 81i/82)$, $k = 1 : 10$, are used instead of $s_k = (k-1/2)(-1+i)$, $k = 1 : 10$, our results will change a lot, as you see in Table 3. Note that the eigenvalues of A are very clustered around zero, according to Figure 2. To improve DDPSE's performance in respect to the number of LU factorizations, we propose to carry out some steps of DPSE before applying DDPSE. You see in Table 4 that after executing only one step of DPSE the number of LU factorizations required by DDPSE lowered from 93 (Table 2) to 68. This initial procedure resulted in better initial estimates to be used by DDPSE, which has only local convergence. Our experiments with DPSE have shown that it has large basins of attraction. The tests have been performed in the MATLAB R2011b 64 bits at a HP Compaq 6000 Pro, with processor Intel Core 2 Duo E8400 3.00 GHz.

5. References

[1] L. H. Bezerra and C. Tomei, *Spectral transformation algorithms for computing unstable modes*, Comp. Appl. Math, vol. 18, no. 1, pp. 1–15, 1999.

[2] L. H. Bezerra, *An eigenvalue method for calculating dominant poles of a transfer function*, Appl. Math. Lett. 21(3), pp. 244–247, 2008.

[3] Z. Du, W. Liu, and W. Fang, *Calculation of rightmost eigenvalues in power systems using the Jacobi-Davidson method*, IEEE Trans. Power Syst., vol. 21, no. 1, pp. 234–239, 2006.

DDPSE		DPSE	
Eigenvalues	Dominance	Eigenvalues	Dominance
$-0.0335 + 1.0787i$	760.11	$-0.0335 + 1.0787i$	760.11
$-0.5567 + 3.6097i$	14.87	$-0.6120 + 0.3587i$	12.40
$-0.6120 + 0.3587i$	12.40	$-2.9445 + 4.8214i$	6.85
$-0.4548 + 4.7054i$	5.78	$-1.2936 + 1.4028i$	5.59
$-1.2936 + 1.4028i$	5.59	$-1.0829 + 0.8747i$	3.78
$-0.9931 + 0.1082i$	2.00	$-1.4463 + 1.4565i$	3.58
$-7.5416 + 6.2292i$	1.07	$-4.0233 + 4.2124i$	2.60
Number of LU factorizations: 60		Number of LU factorizations: 68	

Table 1: Eigenvalues calculated respectively by DDPSE and DPSE from starting values $s_k = k.(-1 + i)$, $k = 1 : 7$, and their measure of relative dominance

DDPSE		DPSE		SADPA	
Eigenvalues	Dominance	Eigenvalues	Dominance	Eigenvalues	Dominance
$-0.6120 + 0.3587i$	12.40	$-0.0335 + 1.0787i$	760.11	$-0.0335 + 1.0787i$	760.11
$-2.9445 + 4.8214i$	6.85	$-0.6120 + 0.3587i$	12.40	$-0.5567 + 3.6097i$	14.87
$-1.8415 + 6.9859i$	5.11	$-2.9445 + 4.8214i$	6.85	$-0.1151 + 0.2397i$	2.20
$-1.4463 + 1.4565i$	3.58	$-1.4463 + 1.4565i$	3.58	$-0.5208 + 2.8814i$	0.79
$-4.0233 + 4.2124i$	2.60	$-4.0233 + 4.2124i$	2.60	$-0.0356 + 0.0000i$	0.21
$-5.7475 + 6.7761i$	1.43	$-5.7475 + 6.7761i$	1.43	$-0.1139 + 0.0000i$	0.10
$-5.8148 + 4.8704i$	1.36	$-5.8148 + 4.8704i$	1.36	$-0.1291 + 0.0000i$	0.04
$-6.9172 + 3.2292i$	0.62	$-5.5632 + 7.7510i$	1.22	$-0.1440 + 0.0000i$	0.04
$-3.1087 + 0.1476i$	0.49	$-7.5416 + 6.2292i$	1.07	$-0.1276 + 0.0000i$	0.02
$-8.0955 + 9.8277i$	0.16	$-6.9657 + 11.0840i$	0.15		
Number of LU factorizations: 93		Number of LU factorizations: 61		Number of LU factorizations: 73	

Table 2: Eigenvalues calculated respectively by DDPSE, DPSE and SADPA from starting values $s_k = (k - 1/2)(-1 + i)$, $k = 1 : 10$, and their measure of relative dominance

DDPSE		DPSE	
Eigenvalues	Dominance	Eigenvalues	Dominance
$-0.0335 + 1.0787i$	760.11	$-0.6120 + 0.3587i$	12.40
$-0.6120 + 0.3587i$	12.40	$-2.9445 + 4.8214i$	6.85
$-2.9445 + 4.8214i$	6.85	$-1.0829 + 0.8747i$	3.78
$-1.4463 + 1.4565i$	3.58	$-1.4463 + 1.4565i$	3.58
$-5.8148 + 4.8704i$	1.36	$-4.0233 + 4.2124i$	2.60
$-5.5632 + 7.7510i$	1.22	$-5.7475 + 6.7761i$	1.43
$-7.5416 + 6.2292i$	1.07	$-5.8148 + 4.8704i$	1.36
$-6.4231 + 8.6949i$	0.41	$-5.5632 + 7.7510i$	1.22
$-2.7460 + 0.0000i$	0.31	$-7.5416 + 6.2292i$	1.07
$-7.8864 + 10.3395i$	0.06	$-6.9657 + 11.0840i$	0.15
Number of LU factorizations: 88		Number of LU factorizations: 70	

Table 3: Eigenvalues calculated respectively by DDPSE and DPSE from starting values $s_k = (k - 1/2)(-1 + 81i/82)$, $k = 1 : 10$

Eigenvalue		Relative Dominance
Real	Imag	m
-0.6120	0.3587	12.40
-2.9445	4.8214	6.85
-1.0829	0.8747	3.78
-1.4463	1.4565	3.58
-4.0233	4.2124	2.61
-5.7475	6.7761	1.43
-5.8148	4.8704	1.36
-5.5632	7.7510	1.22
-7.5416	6.2292	1.07
-7.8864	10.3395	0.06
Number of LU factorizations: 68		

Table 4: Eigenvalues calculated by DDPSE after only one step of DPSE from starting values $s_k = (k - 1/2)(-1 + i)$, $k = 1 : 10$

[4] L. T. G. Lima, L. H. Bezerra, C. Tomei, and N. Martins, *New methods for fast small-signal stability assessment of large-scale power systems*, IEEE Trans. Power Syst., vol. 10, no. 4, pp. 1979–1985, Nov. 1995.

[5] N. Martins, L. T. G. Lima, and H. J. P. Pinto, *Computing Dominant Poles of Power System Transfer Functions*, IEEE Trans. Power Syst. vol. 11, no. 1, pp. 162–170, 1996.

[6] N. Martins, *The Dominant Pole Spectrum Eigensolver*, IEEE Trans. Power Syst. vol. 12, no. 1, pp. 245–254, 1997.

[7] J. Rommes and N. Martins, *Efficient computation of transfer function dominant poles using subspace acceleration* IEEE Trans. Power Syst. vol. 21, no. 3, pp. 1218–1226, 2006.

[8] J. Rommes, *Arnoldi and Jacobi-Davidson methods for generalized eigenvalue problems $Ax = \lambda Bx$ with singular B*, Math. Comput., vol. 77, no. 262, pp. 9951015, Apr. 2008

[9] J. Rommes and G. L. G. Sleijpen, *Convergence of the dominant pole algorithm and Rayleigh quotient iteration*, SIAM J. Matrix Anal. Appl., vol. 30, no. 1, pp. 346–363, 2008.

[10] J. Rommes, N. Martins and F. Freitas, *Computing Rightmost Eigenvalues for Small-Signal Stability Assessment of Large-Scale Power Systems*, IEEE Trans. Power Syst. vol. 25, no. 2, pp. 929–938, 2010.