Prova I de B Cálculo II:

Nome:

Data:

1) Resolva as seguintes integrais, justificando rigorosamente sua resposta:

a) $\displaystyle\int \frac{dx}{1+2e^{x}-e^{-x}}$, $x \geq 0$

b) $\displaystyle\int \frac{dx}{\sqrt{4x^2-4x-3}}$, $x > 3/2$

c) $\displaystyle\int_0^{\pi/4} \operatorname{tg}^4\theta \sec^3\theta \, d\theta$

d) $\displaystyle\int \frac{dx}{3\,\operatorname{sen} x - 4\cos x}$, $x \in [0, x^*)$, onde $0 < x^* < \pi/2$ satisfaz $\operatorname{tg} x^* = 4/3$

e) $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$, $x \geq 3$ ou $x \leq -3$

f) $\displaystyle\int \frac{(2x^2-x+2)\,dx}{x^5+2x^3+x}$, $x > 0$

2) Escreva uma integral que represente o volume do sólido gerado pela rotação, em torno da reta $x=-4$, da região delimitada pelas parábolas $x = y - y^2$ e $x = y^2 - 3$. Esboce a região e o sólido.

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Prova I - B - Cálculo II - Resolução

1) Note que, $\forall x \geq 0$

a)

$$\frac{1}{1+2e^x-e^{-x}} = \frac{e^x}{e^x+2e^{2x}-1}$$

Definimos: $f:[1,+\infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{2x^2+x-1}$

$g:[0,+\infty) \to [1,+\infty)$, $g(x) = e^x$

Encontremos uma primitiva F para f: Do fato que

$$\frac{1}{2x^2+x-1} = \frac{A}{2x-1} + \frac{B}{x+1} \quad \text{com } A = 2/3 \text{ e } B = -1/3, \quad \forall x \neq -1,\ x \neq 1/2$$

temos

$$\int \frac{1}{2x^2+x-1}\,dx = \frac{2}{3}\int \frac{dx}{2x-1} - \frac{1}{3}\int \frac{dx}{x+1}$$

$$= \frac{1}{3}\ln(2x-1) - \frac{1}{3}\ln(x+1) + C, \quad \forall x \geq 1$$

Assim $F(x) = \frac{1}{3}\ln(2x-1) - \frac{1}{3}\ln(x+1)$. Além disso,

$$\frac{e^x}{e^x+2e^{2x}-1} = f(g(x)) \cdot g'(x), \text{ portanto}$$

$$\int \frac{e^x}{e^x+2e^{2x}-1}\,dx = \int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

$$= \frac{1}{3}\ln(2e^x-1) - \frac{1}{3}\ln(e^x+1) + C, \quad \forall x \geq 0$$

$$= \frac{1}{3}\ln\left(\frac{2e^x-1}{e^x+1}\right) + C, \quad \forall x \geq 0$$

2) Se $x > 3/2$, $4x^2-4x-3>0$ e podemos escrever

$$\frac{1}{\sqrt{4x^2-4x-3}} = \frac{1}{\sqrt{(2x-1)^2-4}}$$

Definimos $h:(3/2,+\infty)\to(2,+\infty)$, $h(x)=2x-1$

$$f:(2,+\infty)\to\mathbb{R}, \quad f(x)=\frac{1}{\sqrt{x^2-4}}$$

Então $\dfrac{1}{\sqrt{(2x-1)^2-4}} = \dfrac{1}{2}\cdot f(h(x))\,h'(x)$.

Basta encontrarmos uma primitiva para f em $(2,+\infty)$.

Seja $g(\theta)=2\sec\theta$, $g:(0,\pi/2)\to(2,+\infty)$

Então $g'(\theta)=2\sec\theta\,tg\,\theta$ e

$$\int f(g(\theta))\,g'(\theta)\,d\theta = \int \frac{1}{\sqrt{4\sec^2\theta-4}}\,2\sec\theta\,tg\,\theta\,d\theta$$

$$= \int \frac{\sec\theta\,tg\,\theta}{|tg\,\theta|}\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln(\sec\theta + tg\,\theta) + C$$

Como $\int f(g(\theta))\,g'(\theta)\,d\theta = F(g(\theta)) + C$, onde

F é a primitiva de f em $(2,+\infty)$, temos

que $F(g(\theta)) = \ln(\sec\theta + tg\,\theta)$, ou seja,

$$F(g(\theta)) = \ln\left(\frac{2\sec\theta}{2} + \frac{\sqrt{4\sec^2\theta - 4}}{2}\right)$$

$$= \ln\left(\frac{g(\theta)}{2} + \frac{\sqrt{(g(\theta))^2 - 4}}{2}\right)$$

Assim,
$$\int \frac{dx}{\sqrt{4x^2 - 4x - 3}} = \frac{1}{2}\int f(h(x))\, h'(x)\, dx$$

$$= \frac{1}{2} F(h(x)) + C$$

$$= \frac{1}{2} \ln\left(\frac{2x-1}{2} + \frac{\sqrt{(2x-1)^2 - 4}}{2}\right) + C$$

$$= \frac{1}{2} \ln\left(2x - 1 + \sqrt{4x^2 - 4x - 3}\right) + C$$

c) Antes resolvamos algumas integrais auxiliares:

$$\int \sec^3\theta\, d\theta = \int \sec\theta . \sec^2\theta\, d\theta = \int f(\theta)\, g'(\theta)\, d\theta, \quad 0 \leq \theta \leq \pi/4$$

Usando integração por partes: $f(\theta) = \sec\theta$, $g(\theta) = tg\,\theta$ definid para $[0, \pi/4]$,

$$\int \sec^3\theta\, d\theta = f(\theta)\, g(\theta) - \int f'(\theta)\, g(\theta)\, d\theta$$

$$= \sec\theta . tg\theta - \int \sec\theta . tg^2\theta\, d\theta$$

$$= \sec\theta . tg\theta - \int \sec\theta\,(-1 + \sec^2\theta)\, d\theta$$

$$= \sec\theta . tg\theta + \int \sec\theta - \int \sec^3\theta\, d\theta \implies$$

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\, tg\theta + \frac{1}{2}\int \sec\theta\, d\theta$$

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\, tg\theta + \frac{1}{2}\ln(\sec\theta + tg\theta) + C, \quad 0 \leq \theta \leq \pi/4$$

Por outro lado, usando novamente integração por partes,

$$\int \sec^5\theta\, d\theta = \int \sec^3\theta\, \sec^2\theta\, d\theta$$

$$= \sec^3\theta\, tg\theta - 3\int \sec^3\theta\, tg^2\theta\, d\theta$$

$$= \sec^3\theta\, tg\theta - 3\int \sec^3\theta(\sec^2\theta - 1)\, d\theta$$

$$= \sec^3\theta\, tg\theta - 3\int \sec^5\theta + 3\int \sec^3\theta\, d\theta$$

ou seja,

$$\int \sec^5\theta\, d\theta = \frac{1}{4}\sec^3\theta\, tg\theta + \frac{3}{4}\int \sec^3\theta\, d\theta, \quad \theta \in [0, \pi/4]$$

Também, $\int \sec^7\theta\, d\theta = \int \sec^5\theta \cdot \sec^2\theta\, d\theta = \sec^5\theta\, tg\theta - 5\int \sec^5\theta\, tg^2\theta\, d\theta$

$= \sec^5\theta\, tg\theta - 5\int \sec^5\theta(\sec^2\theta - 1)\, d\theta = \sec^5\theta\, tg\theta - 5\int \sec^7\theta\, d\theta + 5\int \sec^5\theta\, d\theta$

ou seja, $\int \sec^7\theta\, d\theta = \frac{1}{6}\sec^5\theta\, tg\theta + \frac{5}{6}\int \sec^5\theta\, d\theta, \quad \theta \in [0, \pi/4]$

Agora, $\int tg^4\theta\, \sec^3\theta\, d\theta = \int (\sec^2\theta - 1)^2 \sec^3\theta\, d\theta = \int (\sec^4\theta - 2\sec^2\theta + 1)\sec^3\theta\, d\theta$

$$= \int \sec^7\theta\, d\theta - 2\int \sec^5\theta\, d\theta + \int \sec^3\theta\, d\theta$$

$$= \frac{1}{6}\sec^5\theta\, tg\theta + \frac{5}{6}\int \sec^5\theta\, d\theta - 2\int \sec^5\theta\, d\theta + \int \sec^3\theta\, d\theta$$

$$= \frac{1}{6}\sec^5\theta\, tg\theta - \frac{7}{6}\int \sec^5\theta\, d\theta + \int \sec^3\theta\, d\theta$$

$$= \frac{1}{6}\sec^5\theta\, tg\theta - \frac{7}{24}\sec^3\theta\, tg\theta - \frac{7}{8}\int \sec^3\theta\, d\theta + \int \sec^3\theta\, d\theta$$

$$= \frac{1}{6}\sec^5\theta\, tg\theta - \frac{7}{24}\sec^3\theta\, tg\theta + \frac{1}{8}\int \sec^3\theta\, d\theta.$$

Assim,

$$\int tg^4\theta \sec^3\theta\, d\theta = \frac{1}{6}\sec^5\theta\, tg\theta - \frac{7}{24}\sec^3\theta\, tg\theta + \frac{1}{16}\sec\theta\, tg\theta$$

$$+ \frac{1}{16}\ln(\sec\theta + tg\theta) + C, \quad \theta \in [0, \pi/4], \quad e$$

$$\int_0^{\pi/4} tg^4\theta \sec^3\theta\, d\theta = \frac{1}{6}\cdot\left(\frac{2}{\sqrt{2}}\right)^5 - \frac{7}{24}\left(\frac{2}{\sqrt{2}}\right)^3 + \frac{1}{16}\cdot\frac{2}{\sqrt{2}} +$$

$$\frac{1}{16}\ln\left(\frac{2}{\sqrt{2}}+1\right) = \frac{1}{6}\cdot\frac{32}{(2^{1/2})^4\cdot\sqrt{2}} - \frac{7}{24}\cdot\frac{8}{2\sqrt{2}} + \frac{1}{16}\frac{2}{\sqrt{2}} + \frac{1}{16}\ln\left(\frac{2+\sqrt{2}}{\sqrt{2}}\right)$$

$$= \frac{4}{3\sqrt{2}} - \frac{7}{6\sqrt{2}} + \frac{1}{8\sqrt{2}} + \frac{1}{16}\ln\left(\frac{2+\sqrt{2}}{\sqrt{2}}\right)$$

$$= \frac{7}{24\sqrt{2}} + \frac{1}{16}\ln\left(\frac{2+\sqrt{2}}{\sqrt{2}}\right)$$

d) Seja $g: [0, x^*) \to [0, tg(\frac{x^*}{2}))$, $g(x) = tg(x/2)$

Por simplicidade denotamos $z = g(x)$. Então $z' = g'(x) = \sec^2(\frac{x}{2})\cdot\frac{1}{2} = \left(1 + tg^2(\frac{x}{2})\right)\frac{1}{2} = \frac{1+z^2}{2}$. Conforme visto em sala de aula:

$$\cos x = \frac{1-z^2}{1+z^2} \quad e \quad \operatorname{sen} x = \frac{2z}{1+z^2}.$$

Então: $$\frac{1}{3\operatorname{sen}x - 4\cos x} = \frac{1}{\frac{6z}{1+z^2} - 4\left(\frac{1-z^2}{1+z^2}\right)} = \frac{1+z^2}{6z - 4 + 4z^2}$$

$$= \frac{1+z^2}{2}\cdot\frac{1}{2z^2+3z-2} = g'(x)\cdot\frac{1}{2(g(x))^2 + 3g(x) - 2}$$

Ou seja, $\int \frac{dx}{3\operatorname{sen} x - 4\cos x} = \int f(g(x))\, g'(x)\, dx$,

onde $f(x) = \frac{1}{2x^2+3x-2}$, $f: [0, tg(\frac{x^*}{2})) \longrightarrow \mathbb{R}$

Basta encontrarmos $F: [0, tg(\frac{x^*}{2})) \rightarrow \mathbb{R}$ primitiva de f. Para isso, note que

$$2x^2 + 3x - 2 = (2x-1)(x+2) \quad \text{e assim}$$

$$\frac{1}{2x^2+3x-2} = \frac{2}{5} \cdot \frac{1}{2x-1} - \frac{1}{5} \cdot \frac{1}{x+2}, \quad x \neq 1/2,\ x \neq -2$$

Note que $tg\left(\frac{x}{2}\right) = 1/2 \Rightarrow \cos x = \frac{1 - 1/4}{1 + 1/4} = 3/5$ e

$\operatorname{sen} x = \frac{1}{1 + 1/4} = 4/5$, ou seja, $tg\, x = 4/3$. Logo

se $x \in (0, \pi/2)$, temos $x = x^*$, já que a função tangente é injetora.

Assim $F: [0, 1/2) \rightarrow \mathbb{R}$ e $f: [0, 1/2) \longrightarrow \mathbb{R}$. e

$$\int \frac{dx}{2x^2+3x-2} = 2/5 \ln|2x-1| - \frac{1}{5}\ln(x+2) + C$$

$$0 \leq x < x^*$$

Ou seja, $F(x) = 2/5 \ln|2x-1| - \frac{1}{5} \ln(x+2)$, e

$$\int \frac{dx}{3\operatorname{sen} x - 4\cos x} = F(g(x)) + C$$

$$= \frac{2}{5} \ln\left|2 tg\left(\frac{x}{2}\right) - 1\right| - \frac{1}{5} \ln\left(tg\left(\frac{x}{2}\right) + 2\right) + C$$

$$0 \leq x < x^*$$

e) Seja $g(\theta) = 3\sec(\theta)$, $g: [0, \pi/2) \to [3, +\infty)$. Então

$g'(\theta) = 3\sec\theta \, tg\theta$.

Defina $f: [3, +\infty) \to \mathbb{R}$, $f(x) = \frac{\sqrt{x^2 - 9}}{x^3}$. Então

$$f(g(\theta)) = \frac{\sqrt{9\sec^2\theta - 9}}{27\sec^3\theta} = \frac{3\,|tg\theta|}{27\sec^3\theta} = \frac{1}{9}\frac{tg\theta}{\sec^3\theta}$$

Vamos calcular $\int f(g(\theta))\, g'(\theta)\, d\theta = F(g(\theta)) + C$,

onde $F: [3, +\infty) \to \mathbb{R}$ é uma primitiva de f.

Para isso, note que $\int f(g(\theta))\, g'(\theta)\, d\theta = \frac{1}{3}\int \frac{tg\theta}{\sec^3\theta} \cdot \sec\theta \, tg\theta \, d\theta$

$$= \frac{1}{3}\int \frac{tg^2\theta}{\sec^2\theta}\, d\theta = \frac{1}{3}\int \left(\frac{\sec^2\theta - 1}{\sec^2\theta}\right) d\theta = \frac{1}{3}\int \left(1 - \cos^2\theta\right) d\theta$$

$$= \frac{1}{3}\int sen^2\theta\, d\theta = \frac{1}{3}\cdot\frac{1}{2}\int (1 - \cos(2\theta))\, d\theta = \frac{1}{6}\theta - \frac{1}{12} sen(2\theta$$

$$= \frac{1}{6} arc\,sec\left(\frac{1}{3} \cdot 3\sec\theta\right) - \frac{1}{12} \cdot 2\, sen\theta \cos\theta + C$$

$$= \frac{1}{6} arc\,sec\left(\frac{1}{3} g(\theta)\right) - \frac{1}{6}\sqrt{1 - \cos^2\theta}\, \cos\theta + C$$

$$= \frac{1}{6} arc\,sec\left(\frac{1}{3} g(\theta)\right) - \frac{1}{6}\sqrt{1 - \frac{9}{9\sec^2\theta}} \cdot \frac{1}{\sec\theta} + C$$

$$= \frac{1}{6} arc\,sec\left(\frac{1}{3} g(\theta)\right) - \frac{1}{6}\frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta} \cdot \frac{3}{3\sec\theta} + C$$

$= \frac{1}{6} \operatorname{arcsec}\left(\frac{1}{3} g(\theta)\right) - \frac{1}{2} \frac{\sqrt{(g(\theta))^2 - 9}}{[g(\theta)]^2} + C$

Ou seja, (*) $\boxed{\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \frac{1}{6}\operatorname{arcsec}\left(\frac{1}{3}x\right) - \frac{1}{2}\frac{\sqrt{x^2-9}}{x^2} + C}$

$x \geq 3$

Para $x \leq -3$, defina $g : [\pi, 3\pi/2) \to (-\infty, -3]$,

$g(\theta) = 3\sec\theta$ e $f : (-\infty, -3] \to \mathbb{R}$, $f(x) = \frac{\sqrt{x^2-9}}{x^3}$

Repetindo o argumento anterior e observando que $\operatorname{sen}\theta \leq 0$ e $\operatorname{tg}\theta \geq 0$ em $[\pi, 3\pi/2)$

$$\int f(g(\theta))\, g'(\theta)\, d\theta = \frac{1}{6}\operatorname{arcsec}\left(\frac{1}{3} g(\theta)\right) + \frac{1}{6}\sqrt{1 - \frac{9}{9\sec^2\theta}} \cdot \frac{3}{3\sec\theta} + C$$

Como $\sec\theta < 0$ em $[\pi, 3\pi/2)$ temos

$$\sqrt{1 - \frac{9}{9\sec^2\theta}} \cdot \frac{3}{3\sec\theta} = \frac{\sqrt{9\sec^2\theta - 9}}{|3\sec\theta|} \cdot \frac{3}{3\sec\theta}$$

$$= -\frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta} \cdot \frac{3}{3\sec\theta} \quad \text{e}$$

a expressão (*) é válida também para o caso $x \leq -3$

1) Observe que $x^5+2x^3+x = x(x^4+2x^2+1) = x(x^2+1)^2$

Assim, temos de encontrar A, B, C, D e E satisfazendo

$$\frac{2x^2-x+2}{x^5+2x^3+x} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{x^2+1}, \quad \forall x \neq 0$$

Ou seja, A, B, C, D e E têm de satisfazer:

$$2x^2-x+2 = A(x^2+1)^2 + (Bx+C)x + (Dx+E)x(x^2+1)$$

Colocando $x=0$, temos $\boxed{2=A}$. Além disso,

$$2x^2-x+2 = A(x^4+2x^2+1) + Bx^2+Cx + (Dx^2+Ex)(x^2+1)$$

$$= A(x^4+2x^2+1) + Bx^2+Cx + Dx^4+Dx^2+Ex^3+Ex$$

$$= x^4(A+D) + Ex^3 + x^2(2A+D+B) + x(C+E) + A$$

Temos o sistema:

$$\begin{cases} A+D=0 \\ \boxed{E=0} & \Rightarrow \boxed{D=-2} \\ 2A+D+B=2 & \Rightarrow 4-2+B=2 \Rightarrow \boxed{B=0} \\ C+E=-1 & \Rightarrow \boxed{C=-1} \\ A=2 \end{cases}$$

Assim

$$\int \frac{2x^2-x+2}{x^5+2x^3+x}\,dx = 2\int\frac{dx}{x} - \int\frac{dx}{(x^2+1)^2} - 2\int\frac{x\,dx}{x^2+1}$$

$$= 2\ln x - \int\frac{dx}{(x^2+1)^2} - \ln(x^2+1), \quad x>0$$

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Resta resolvermos $\int \frac{dx}{(x^2+1)^2}$. Para isso, defina

$g:(0,\pi/2) \to (0,+\infty)$, $g(\theta)=tg\,\theta$. Então $g'(\theta)=\sec^2\theta\, d\theta$

Definimos também $f:(0,+\infty)\to\mathbb{R}$, $f(x)=\frac{1}{(x^2+1)^2}$

Então $f(g(\theta))=\frac{1}{(tg^2\theta+1)^2}=\frac{1}{(\sec^2\theta)^2}=\cos^4\theta$ e

$$\int f(g(\theta))\,g'(\theta)\,d\theta=\int\cos^2\theta\,d\theta=\frac{1}{2}\int(1+\cos(2\theta))\,d\theta$$

$$=\frac{1}{2}\theta+\frac{1}{4}\operatorname{sen}(2\theta)+C=\frac{1}{2}\operatorname{arctg}(tg\,\theta)+\frac{1}{4}\cdot 2\operatorname{sen}(\theta)\cos\theta$$

$$=\frac{1}{2}\operatorname{arctg}(tg\,\theta)+\frac{1}{2}tg\,\theta\cos^2\theta=\frac{1}{2}\operatorname{arctg}(g(\theta))+\frac{1}{2}g(\theta)\frac{1}{\sec^2\theta}$$

$$=\frac{1}{2}\operatorname{arctg}(g(\theta))+\frac{1}{2}g(\theta)\cdot\frac{1}{(g(\theta))^2+1}+C$$

Logo $\int\frac{dx}{(x^2+1)^2}=\frac{1}{2}\operatorname{arctg}(x)+\frac{1}{2}\frac{x}{x^2+1}+C$

Assim

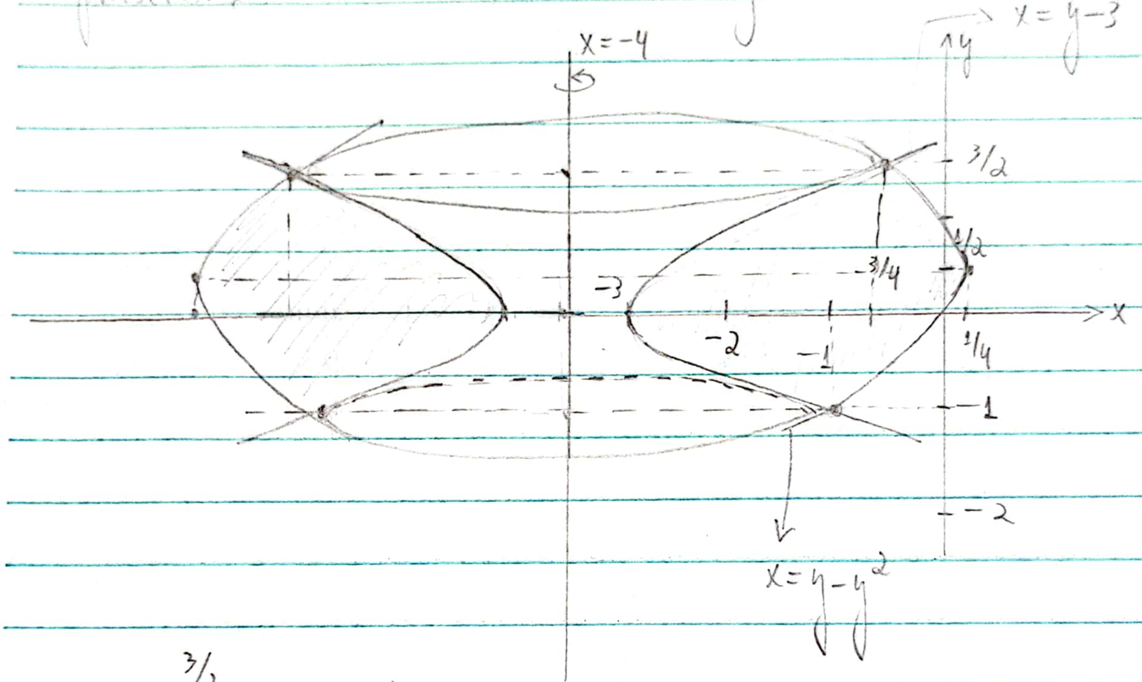
$$\int\frac{2x^2-x+2}{x^5+2x^3+x}\,dx=\ln\left(\frac{x^2}{x^2+1}\right)-\frac{1}{2}\operatorname{arctg}(x)-\frac{1}{2}\frac{x}{x^2+1}+C,\quad x>0$$

(2) Pontos de intersecção: $y - y^2 = y^2 - 3 \Rightarrow 2y^2 - y - 3 = 0$

$$y = \frac{1 \pm \sqrt{1+24}}{4} \begin{cases} \rightarrow \frac{1+5}{4} = 3/2 \rightsquigarrow x = 9/4 - 3 = -3/4 \\ \rightarrow \frac{1-5}{4} = -1 \rightarrow x = 1 - 3 = -2 \end{cases}$$

$P_1(-2,-1)$ e $P_2(-3/4, 3/2)$

Representação Gráfica da Região e do sólido



$$V = \pi \int_{-1}^{3/2} (R^2 - r^2)\, dy, \quad \begin{matrix} R \text{ raio maior do anel} \\ r \text{ " menor do anel} \end{matrix}$$

$R(y) = 4 + f(y), \quad f(y) = y - y^2$

$r(y) = 4 + g(y), \quad g(y) = y^2 - 3$

$$V = \pi \int_{-1}^{3/2} \left[(4 + y - y^2)^2 - (4 + y^2 - 3)^2 \right] dy$$