Segunda Avaliação de Cálculo IV (MTM 5178) - 2006/01

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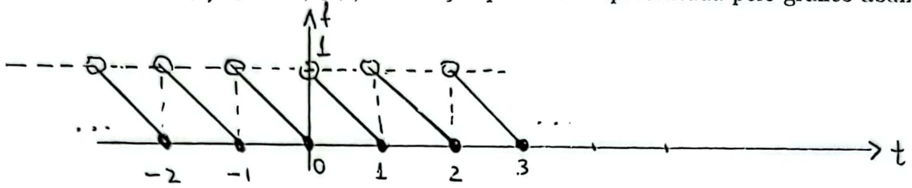
Nome: Data:

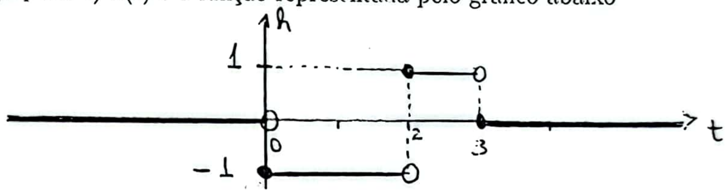
1. Calcule a transformada de Laplace da função dada

(a) (0,5 pontos) $g(t) = \int_0^t e^{-(t-\tau)} sen(\tau) d\tau$;

(b) (1,0 pontos) $f(t)$ é a função periódica representada pelo gráfico abaixo



(c) (1,0 pontos) $h(t)$ é a função representada pelo gráfico abaixo



2. (2.0 pontos) Ache a solução do problema de valor inicial

$$y''(t) + 2y'(t) + 3y(t) = \delta(t - \pi) + sen(t)$$
$$y(0) = 0, \; y'(0) = 1$$

3. (2,5 pontos) Resolva o sistema abaixo

$$\begin{aligned} &2x_1'(t) + x_2(t) = e^t u_2(t) \\ &x_2'(t) + 2x_1(t) = 0 \\ &x_1(0) = x_2(0) = 0 \end{aligned}$$

4. (2.0 pontos) Use a transformada de Laplace para encontrar a solução do problema

$$y''(t) + 2ty'(t) - 4y(t) = 1, \; y(0) = 0, y'(0) = 0.$$

5. (1,0 pontos) Encontre uma expressão para $\phi(t)$ sabendo que esta função é solução da equação integral

$$\phi(t) + \int_0^t (t - \xi)\phi(\xi) d\xi = f(t),$$

onde f é uma função conhecida que possui Transformada de Laplace. (Dica: Aplique a Transformada de Laplace em ambos os membros da equação).

Resolução da 2ª Prova: Cálculo IV (MTM 5178) - Prof. Luciano Bedin:

① a) $g(t) = \int_0^t e^{-(t-\tau)} \operatorname{sen}(\tau)\, d\tau = e^{-t} * \operatorname{sen} t$, $\mathcal{L}\{g(t)\} = \mathcal{L}\{e^{-t}\} \cdot \mathcal{L}\{\operatorname{sen} t\}$

$$= \frac{1}{s+1} \cdot \frac{1}{1+s^2}$$

$$= \frac{1}{(s+1)(s^2+1)}$$

b) $f(t) = \begin{cases} 1-t & \text{se } 0 < t \leq 1 \\ 0 & \text{se } t = 0 \end{cases}$, $f(t+1) = f(t)$, f periódica de período $T = 1$.

$$\mathcal{L}\{f(t)\} = \frac{\int_0^1 (1-t)e^{-st}\,dt}{1-e^{-s}} = \frac{1}{(1-e^{-s})}\left[\int_0^1 e^{-st}\,dt - \int_0^1 t e^{-st}\,dt\right]$$

$$= \frac{1}{(1-e^{-s})} \cdot \left[\left[-\frac{1}{s} e^{-st}\right]_0^1 - \left(-\frac{1}{s} t e^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt\right)\right]$$

$$= \frac{1}{1-e^{-s}}\left[-\frac{1}{s}(e^{-s}-1) - \left(-\frac{1}{s}e^{-s} - \frac{1}{s^2} e^{-st}\Big|_0^1\right)\right] = \frac{1}{1-e^{-s}}\left[\frac{1}{s} + \frac{1}{s^2}(e^{-s}-1)\right]$$

$$= \frac{1}{s(1-e^{-s})} - \frac{1}{s^2} = \frac{s-(1-e^{-s})}{s^2(1-e^{-s})} = \frac{s+e^{-s}-1}{s^2(1-e^{-s})}$$

c) $h(t) = -1\,[u_0(t) - u_2(t)] + 1[u_2(t) - u_3(t)] = -u_0(t) + 2u_2(t) - u_3(t)$

$$\mathcal{L}\{h(t)\} = -\frac{1}{s} + \frac{2e^{-2s}}{s} - \frac{e^{-3s}}{s}$$

② $\mathcal{L}\{y''\}+2\mathcal{L}\{y'\}+3\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\pi)\}+\mathcal{L}\{\operatorname{sen} t\}$

$$s^2Y(s)-sy(0)-y'(0)+2sY(s)-2y(0)+3Y(s)=e^{-\pi s}+\frac{1}{1+s^2}$$

$$(s^2+2s+3)Y(s)-1=e^{-\pi s}+1/_{1+s^2} \Rightarrow Y(s)=\frac{1}{s^2+2s+3}+\frac{e^{-\pi s}}{s^2+2s+3}+\frac{1}{(1+s^2)(s^2+2s+3)}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+2s+3}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(1+s^2)(s^2+2s+3)}\right\}$$

$$\frac{1}{(1+s^2)(s^2+2s+3)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2s+3}=\frac{(As+B)(s^2+2s+3)+(Cs+D)(s^2+1)}{(s^2+1)(s^2+2s+3)}$$

$$1=As^3+2As^2+3As+Bs^2+2Bs+3B+Cs^3+Cs+Ds^2+D$$

$A+C=0 \Rightarrow C=-A$

$2A+B+D=0$

$3A+2B+C=0$

$3B+D=1 \Rightarrow D=1-3B$

$$\Rightarrow \begin{cases} 2A+B+1-3B=0 \\ 3A+2B-A=0 \end{cases} \Rightarrow \begin{cases} 2A-2B=-1 \\ 2A+2B=0 \Rightarrow A=-B \Rightarrow \boxed{B=1/4} \end{cases}$$

$4A=-1 \Rightarrow \boxed{A=-1/4}$

$\boxed{C=1/4}$

$\boxed{D=1-3/4=1/4}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(1+s^2)(s^2+2s+3)}\right\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{-s+1}{s^2+1}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+3}\right\}$$

$$=-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{1+s^2}\right\}+1/4\,\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2}\right\}$$

$$=-\frac{1}{4}\cos t+\frac{1}{4}\operatorname{sen} t+\frac{1}{4}\cdot\frac{\cancel{1}}{\cancel{\sqrt{2}}}\left[\cos(\sqrt{2}t)\right]e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\}=\frac{1}{\sqrt{2}}e^{-t}\,\text{sen}\,\sqrt{2}\,t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+2s+3}\right\}=\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+2}\right\}=\frac{1}{\sqrt{2}}u_\pi(t)\,e^{-(t-\pi)}\,\text{sen}\,\sqrt{2}(t-\pi)$$

$$y(t)=-\frac{1}{4}\cos t+\frac{1}{4}\text{sen}\,t+\frac{1}{4}e^{-t}\cos(\sqrt{2}t)+\frac{1}{\sqrt{2}}e^{-t}\text{sen}(\sqrt{2}t)+\frac{1}{\sqrt{2}}u_\pi(t)\,e^{-(t-\pi)}\,\text{sen}\left(\sqrt{2}(t-\pi)\right)$$

3 $\begin{cases} 2sX_1(s)+X_2(s)=\mathcal{L}\{e^{t-2}\cdot e^2\cdot u_2(t)\}=e^2\mathcal{L}\{e^{t-2}u_2(t)\}=e^2\cdot\dfrac{e^{-2s}}{s-1} \\ sX_2(s)+2X_1(s)=0 \end{cases}$

$$\begin{cases} 2sX_1(s)+X_2(s)=e^2\cdot\dfrac{e^{-2s}}{s-1} \\ -2sX_1(s)-s^2X_2(s)=0 \end{cases} \Rightarrow \quad (1-s^2)X_2(s)=\frac{e^2\cdot e^{-2s}}{s-1}$$

$$X_2(s)=\frac{e^2\cdot e^{-2s}}{(1-s^2)(s-1)}$$

$$X_2(s)=\frac{e^2\cdot e^{-2s}}{(1-s)(1+s)(s-1)}=\frac{-e^2\cdot e^{-2s}}{(1-s)^2(1+s)}$$

$$\frac{1}{(1-s)^2(1+s)}=\frac{A}{(1-s)^2}+\frac{B}{(1-s)}+\frac{C}{(1+s)}\Rightarrow A(1+s)+B(1-s)(1+s)+C(1-s)^2=1$$

$s=1\Rightarrow 2A=1\Rightarrow \boxed{A=1/2}$

$s=-1\Rightarrow 4C=1\Rightarrow \boxed{C=1/4}$

$s=2\Rightarrow \frac{1}{2}\cdot 3+B\cdot(-1)\cdot 3+C=1\Rightarrow \frac{3}{2}-3B+1/4=1$

$3B=3/2+1/4-1$

$\boxed{B=1/4}$

$$x_2(t) = \mathcal{L}^{-1}\left\{\frac{-e^2 e^{-2s}}{2(1-s)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-e^2 e^{-2s}}{4(1-s)}\right\} + \mathcal{L}^{-1}\left\{\frac{-e^2 e^{-2s}}{4(1+s)}\right\}$$

$$x_2(t) = -\frac{e^2}{2} \cdot u_2(t) \cdot (t-2)\, e^{t-2} + \frac{e^2}{4} u_2(t)\, e^{t-2} - \frac{e^2}{4} u_2(t)\, e^{-(t-2)}$$

$$X_1(s) = -\frac{s}{2} X_2(s) = -\frac{s}{2}\left[\frac{-e^2 \cdot e^{-2s}}{(1-s)^2(1+s)}\right\} = \frac{e^2}{2}\left\{\frac{s}{(1-s)^2(1+s)} e^{-2s}\right\}$$

$$\frac{s}{(1-s)^2(1+s)} = \frac{A}{(1-s)^2} + \frac{B}{(1-s)} + \frac{C}{1+s} \Rightarrow s = A(1+s) + B(1-s)(1+s) + C(1-s)^2$$

$s=-1 \Rightarrow -1 = 4C \Rightarrow \boxed{C = -1/4}$

$s=1 \Rightarrow 1 = 2A \Rightarrow \boxed{A = 1/2}$

$s=0 \Rightarrow 0 = A + B + C$

$0 = 1/2 + B - 1/4 \Rightarrow B = 1/4 - 1/2$

$\boxed{B = -1/4}$

$$x_1(t) = \frac{e^2}{2} \cdot \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(1-s)^2}\right\} + \frac{e^2}{2} \cdot \left(-\frac{1}{4}\right) \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{1-s}\right\} + \frac{e^2}{2}\left(\frac{-1}{4}\right) \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{1+s}\right\}$$

$$x_1(t) = \frac{e^2}{4} \cdot u_2(t)(t-2)\, e^{t-2} + \frac{e^2}{8} u_2(t)\, e^{t-2} - \frac{e^2}{8} u_2(t)\, e^{-(t-2)}$$

④ $y''(t) + 2ty'(t) - 4y(t) = 1 \qquad y(0) = y'(0) = 0$

$$s^2 Y(s) - s\,y(0) - y'(0) - 2\frac{d}{ds}\{s\,Y(s) - y(0)\} - 4Y(s) = 1/s$$

$$s^2 Y(s) - 2s\,Y'(s) - 2Y(s) - 4Y(s) = 1/s \Rightarrow \quad -2s\,Y'(s) + (s^2 - 6)\,Y(s) = 1/s$$

$$Y'(s) - \left(\frac{s^2}{2s} - \frac{6}{2s}\right) Y(s) = -\frac{1}{2s^2}$$

$$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right) Y(s) = -\frac{1}{2} \cdot \frac{1}{s^2}$$

Fator Integrante: $e^{\int\left(\frac{3}{s} - \frac{s}{2}\right)} = e^{3\ln s} \cdot e^{-s^2/4} = s^3 \cdot e^{-s^2/4}$

$$\frac{d}{ds}\left[Y(s) \cdot s^3 e^{-s^2/4}\right] = -\frac{1}{2} \cdot s^3 \cdot e^{-s^2/4} \cdot \frac{1}{s^2}$$

$$s^3 e^{-s^2/4}\, Y(s) = \int -\frac{1}{2}\, s \cdot e^{-s^2/4}\, ds \Rightarrow u = -s^2/4 \qquad du = -\frac{2s}{4} ds = -s/2$$

$$s^3 e^{-s^2/4}\, Y(s) = e^{-s^2/4} + C \Rightarrow Y(s) = \frac{1}{s^3} + \frac{C}{s^3} e^{s^2/4}$$

Como $Y(s) \to 0$ quando $s \to +\infty$, $Y(s) = 1/s^3 \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = t^2/2$

⑤ $\mathcal{L}\{\phi(t)\} + \mathcal{L}\left\{\int_0^t (t-\xi)\,\phi(\xi)\,d\xi\right\} = \mathcal{L}\{f(t)\} \Rightarrow \quad \Phi(s) + \mathcal{L}\{t * \phi\} = F(s)$

$$\Phi(s) + \frac{1}{s^2} \cdot \Phi(s) = F(s) \Rightarrow \left(\frac{s^2+1}{s^2}\right)\Phi(s) = F(s)$$

$$\Phi(s) = \frac{s^2}{1+s^2} F(s) \Rightarrow \left(1 - \frac{1}{s^2+1}\right) F(s)$$

$$\Phi(s) = \left(1 - \frac{1}{s^2+1}\right) F(s) \Rightarrow \phi(t) = \mathcal{L}^{-1}\{F(s)\} - \mathcal{L}^{-1}\{F(s) \cdot \tfrac{1}{s^2+1}\}$$

$$\begin{array}{r|l} s^2 & s^2+1 \\ \hline -s^2-1 & 1 \\ -1 & \end{array}$$

$$\phi(t) = f(t) - f(t) * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{\phi(t) = f(t) - f(t) * \operatorname{sen} t = f(t) - \int_0^t \operatorname{sen}(t-\xi)\, f(\xi)\, d\xi}$$

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