Cálculo IV- Prova 2-2012/01

Nome:

Cada resposta deve ser acompanhada de uma justificativa (e/ou cálculo) adequada, caso contrário não será considerada.

1. Encontre a solução geral da equação $x^2y''(x) + 3xy'(x) + y(x) = 0$, $x > 0$.
2. Desenvolver $f(z) = \frac{z}{(z^2+1)(z+2)}$ em uma série de Laurent válida para $|z+2| > \sqrt{5}$.
3. Calcule as seguintes integrais

   a. $\int_\gamma \frac{4\cos^2(z)}{(4z^2-\pi^2)(1+z^2)^2}dz$, γ é a fronteira do retângulo $[-\pi, \pi] \times [-2, 2]$, orientada no sentido anti-horário.

   b. $\int_\gamma \frac{1}{z^2 \text{sen}^2(z)}dz$, γ é a circunferência $|z| = 1$, orientada no sentido anti-horário.
4. Encontre uma transformação conforme entre: $D = \{z \in \mathbb{C},\ \pi/2 < Im(z) < \pi\}$ e $D^* = \{w \in \mathbb{C},\ |w-1| < 2\}$.

Fórmulas:

Equação característica para a equação de Euler: $r^2 + (\alpha - 1)r + \beta = 0$

Resíduo de f num polo z_0 de ordem 1: $Res(f, z_0) = \lim\limits_{z \to z_0} [(z-z_0)f(z)]$

Resíduo de f num polo z_0 de ordem m:

$$Res(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

Algumas séries de potências: $\text{sen}(z) = \sum\limits_{n=0}^{+\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$, $e^z = \sum\limits_{n=0}^{+\infty} \frac{z^n}{n!}$, $\cos(z) = \sum\limits_{n=0}^{+\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

Fórmula Integral de Cauchy: $\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$, $\quad n! \int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = 2\pi i f^{(n)}(z_0)$

Transformação de Möbius entre três pontos distintos:

$$(w_1 - w)(w_3 - w_2)(z_1 - z_2)(z_3 - z) = (z_1 - z)(z_3 - z_2)(w_1 - w_2)(w_3 - w)$$

Resolução P2 de Cálculo IV - 2012/01

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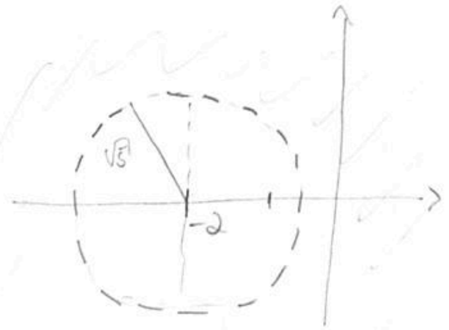
1) $x^2y''(x)+3xy'(x)+y(x)=0, \quad x>0$

Eq. Característica: $r^2+(3-1)r+1=0 \Rightarrow r^2+2r+1=0 \Rightarrow \boxed{r=-1}$

Sol. Geral: $y(x)=C_1x^{-1}+C_2x^{-1}\ln x$

2)



$$\frac{z}{(z^2+1)(z+2)}=\frac{z}{(z+i)(z-i)(z+2)}$$

$$\frac{z}{(z^2+1)(z+2)}=\frac{A}{z+i}+\frac{B}{z-i}+\frac{C}{z+2}$$

$$\frac{z}{(z^2+1)(z+2)}=\frac{A(z-i)(z+2)+B(z+i)(z+2)+C(z^2+1)}{(z^2+1)(z+2)}$$

$z=-2 \Rightarrow -2=C\cdot(4+1) \Rightarrow \boxed{C=-2/5}$

$z=i \Rightarrow i=2i(i+2)B \Rightarrow \boxed{B=\dfrac{1}{2(i+2)}}$

$z=-i \Rightarrow -i=-2i\,A(2-i) \Rightarrow \boxed{A=\dfrac{1}{2(2-i)}}$

$$\frac{1}{z+i}=\frac{1}{z+2-2+i}=\frac{1}{(z+2)\left[1-\dfrac{(2-i)}{z+2}\right]}=\frac{1}{z+2}\cdot\sum_{n=0}^{+\infty}\left(\frac{2-i}{z+2}\right)^n, \text{ pois}$$

$$\left|\frac{2-i}{z+2}\right|=\frac{|2-i|}{|z+2|}=\frac{\sqrt{5}}{|z+2|}<\frac{\sqrt{5}}{\sqrt{5}}=1, \text{ já que } |z+2|>\sqrt{5}.$$

$$\frac{1}{z-i} = \frac{1}{z+2-2-i} = \frac{1}{(z+2)\left[1-\frac{2+i}{z+2}\right]} = \frac{1}{z+2}\cdot\sum_{n=0}^{+\infty}\left(\frac{2+i}{z+2}\right)^n, \text{ pois}$$ (2)

$|z+2| > \sqrt{5} = |2+i|$

Assim:

$$\frac{z}{(z^2+1)(z+2)} = -\frac{2}{5}\cdot\frac{1}{(z+2)} + \frac{1}{2(2-i)}\cdot\frac{1}{(z+2)}\sum_{n=0}^{+\infty}\frac{(2-i)^n}{(z+2)^n} + \frac{1}{2(2+i)}\cdot\frac{1}{(z+2)}\sum_{n=0}^{+\infty}\frac{(2+i)^n}{(z+2)^n}$$

$$= -\frac{2}{5}\cdot\frac{1}{(z+2)} + \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(2-i)^{n-1}}{(z+2)^{n+1}} + \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(2+i)^{n-1}}{(z+2)^{n+1}}$$

$$\boxed{\frac{z}{(z^2+1)(z+2)} = -\frac{2}{5}\cdot\frac{1}{(z+2)} + \frac{1}{2}\sum_{n=0}^{+\infty}\left[\frac{(2-i)^{n-1}+(2+i)^{n-1}}{(z+2)^{n+1}}\right]}$$

3) a) $4z^2-\pi^2=0 \Rightarrow (2z-\pi)(2z+\pi)=0$ → $z_1=\pi/2$, $z_2=-\pi/2$

$1+z^2=0$ → $z_3=i$, $z_4=-i$

$$\int_\gamma \frac{4\cos^2(z)}{(4z^2-\pi^2)(1+z^2)^2}\,dz = 2\pi i\left[\operatorname{Res}(f,\pi/2)+\operatorname{Res}(f,-\pi/2)+\operatorname{Res}(f,i)+\operatorname{Res}(f,-i)\right]$$

→ $\operatorname{Res}(f,\pi/2)=\operatorname{Res}(f,-\pi/2)=0$ pois z_1 e z_2 são singularidades removíveis, já que $h(z)=\cos^2 z$ tem zeros de ordem 2 nesses pontos.

→ z_3 e z_4 são pólos de ordem 2 pois $\cos^2(i)\neq 0$ e $\cos^2(-i)\neq 0$ e esses pontos são zeros de ordem 2 para $g(z)=(4z^2-\pi^2)(1+z^2)^2$.

$$\operatorname{Res}(f, i) = \lim_{z \to i} \frac{d}{dz}\left\{(z-i)^2 \cdot \frac{4\cos^2 z}{(z-i)^2(z+i)^2(4z^2-\pi^2)}\right\} = \lim_{z \to i} \frac{d}{dz}\left\{\frac{4\cos^2 z}{(z+i)^2(4z^2-\pi^2)}\right\}$$

(3)

$$= \lim_{z \to i} 4 \left\{\frac{2\cos z(-\operatorname{sen} z)(z+i)^2(4z^2-\pi^2) - \cos^2 z\left[8z(z+i)^2 - 2(z+i)(4z^2-\pi^2)\right]}{(z+i)^4(4z^2-\pi^2)^2}\right\}$$

$$= 4\left\{\frac{-2\cos(i)\operatorname{sen}(i)(2i)^2(-4-\pi^2) - \cos^2(i)\left[(8i)(2i)^2 - 2(2i)(-4-\pi^2)\right]}{(2i)^4(-4-\pi^2)^2}\right\}$$

$$= 4\left\{\frac{-4\operatorname{sen}(2i)(4+\pi^2) - \cos^2(i)\left[-32i + 4i(4+\pi^2)\right]}{16(4+\pi^2)^2}\right\} = \frac{1}{4} \cdot \frac{1}{(4+\pi^2)^2}\left\{-4(4+\pi^2)\operatorname{sen}(2i)\right.$$

$$\left.-\cos^2(i)\left(-32i + 4i(4+\pi^2)\right)\right\} = \frac{-1}{4(4+\pi^2)^2}\left\{4(4+\pi^2)\operatorname{sen}(2i) + \cos^2(i)\left(-16+4\pi^2\right)i\right\}$$

$$\operatorname{Res}(f, -i) = \lim_{z \to -i} \frac{d}{dz}\left\{(z+i)^2 \cdot \frac{4\cos^2 z}{(z+i)^2(z-i)^2(4z^2-\pi^2)}\right\} = \lim_{z \to -i} \frac{d}{dz}\left\{\frac{4\cos^2 z}{(z-i)^2(4z^2-\pi^2)}\right\}$$

$$= \lim_{z \to -i} 4 \left\{\frac{2\cos z(-\operatorname{sen} z)(z-i)^2(4z^2-\pi^2) - \cos^2 z\left(2(z-i)(4z^2-\pi^2) + 8z(z-i)^2\right)}{(z-i)^4(4z^2-\pi^2)^2}\right\}$$

$$= 4\left\{\frac{-\operatorname{sen}(2i)(-2i)^2(-4-\pi^2) - \cos^2(i)\left(2(-2i)(-4-\pi^2) - 8i(-2i)^2\right)}{(-2i)^4(-4-\pi^2)^2}\right\}$$

$$= 4\left\{\frac{-4(4+\pi^2)\operatorname{sen}(2i) - \cos^2(i)\left(16i + 4\pi^2 i + 32i\right)}{16(4+\pi^2)^2}\right\}$$

$$= \frac{1}{4} \cdot \frac{1}{(4+\pi^2)^2}\left\{-4\operatorname{sen}(2i)^{(4+\pi^2)} - \cos^2(i)\left(48+4\pi^2\right)i\right\}$$

Logo $\qquad$ (4)

$$\int_\gamma \frac{4\cos^2 z}{(4z^2-\pi^2)(1+z^2)^2}\,dz = \frac{2\pi i}{4(4+\pi^2)^2}\left\{-8(4+\pi^2)\,\text{sen}(2i) - \cos^2(i)\left(-16i+4\pi^2 i+48i+4\pi^2 i\right)\right\}$$

$$= \frac{\pi i}{2(4+\pi^2)^2}\left\{-8(4+\pi^2)\,\text{sen}(2i) - \cos^2(i)\,(32+4\pi^2)\,i\right\} //$$

) A única singularidade de $f(z) = \dfrac{1}{z^2\,\text{sen}^2(z)}$ ocorre em $z_0=0$

e é um polo de ordem 4. Então

$$\int_\gamma \frac{1}{z^2\,\text{sen}^2 z}\,dz = \text{Res}(f,0)\,.\,2\pi i.$$

Mas $\text{sen}^2 z = \frac{1}{2}\left(1-\cos(2z)\right) = \frac{1}{2}\left(1-\sum_{n=0}^{+\infty}\frac{(-1)^n(2z)^{2n}}{(2n)!}\right)$

$$= \frac{1}{2}\left(-\sum_{n=1}^{+\infty}\frac{(-1)^n(2z)^{2n}}{(2n)!}\right)$$

$$\cancel{z^2\,\text{sen}^2 z = -\frac{1}{2}\sum_{n=1}^{+\infty}\frac{(-1)^n 2^{2n} z^{2n+2}}{(2n)!} = -\frac{1}{2}\left[-z^3+\frac{1}{6}z^4-\frac{1}{90}z^5+\dots\right]}$$

$$\cancel{= \frac{1}{2}z^3 - \frac{1}{12}z^4 + \frac{1}{180}z^5 + \dots}$$

$$\cancel{\frac{z^3}{2} - \frac{z^4}{12} + \frac{1}{180}z^5}$$

$$\cancel{\frac{2}{z^3} + \frac{1}{3}\cdot\frac{1}{z^2} + 2\left(\frac{1}{36}-\frac{1}{90}\right)\cdot\frac{1}{z}+\dots}$$

$$\cancel{1+\frac{1}{6}z-\frac{1}{90}z^2+\dots}$$

$$\cancel{-\frac{1}{6}z+\frac{1}{36}z^2+\dots}$$

$$\cancel{\text{Logo} \int_\gamma z^{-2}\,\text{sen}^{-2} z\,dz = 2\pi i\,.\,2\left(\frac{1}{36}-\frac{1}{90}\right)}$$

$$\cancel{= 2\pi i\cdot\left(\frac{1}{18}-\frac{1}{45}\right) = \frac{2\pi i}{30} = \frac{\pi i}{15}}$$

$$\operatorname{sen}^2 z = -\tfrac{1}{2}\sum_{m=1}^{+\infty} \frac{(-1)^m 2^{2m} z^{2m}}{(2m)!} = -\tfrac{1}{2}\left[-\frac{4}{2}z^2 + \frac{2^4}{4!}z^4 - \frac{2^6}{6!}z^6 + \cdots\right]$$

$$= \tfrac{1}{2}\left[2z^2 - \tfrac{2}{3}z^4 + \frac{4}{45}z^6 + \cdots\right]$$

$$= z^2\left[1 - \frac{1}{3}z^2 + \frac{2z^4}{45} + \cdots\right]$$

$$\frac{1}{z^2\operatorname{sen}^2 z} = \frac{1}{z^4}\cdot\frac{1}{1 - \tfrac{1}{3}z^2 + \tfrac{2}{45}z^4 + \cdots}$$

$$1 - \tfrac{1}{3}z^2 + \frac{2}{45}z^4 + \cdots$$

$$1 + \tfrac{1}{3}z^2 + \left(\frac{1}{9} - \frac{2}{45}\right)z^4 + \cdots$$

$$1$$

$$1 + \tfrac{1}{3}z^2 - \tfrac{2}{45}z^4$$

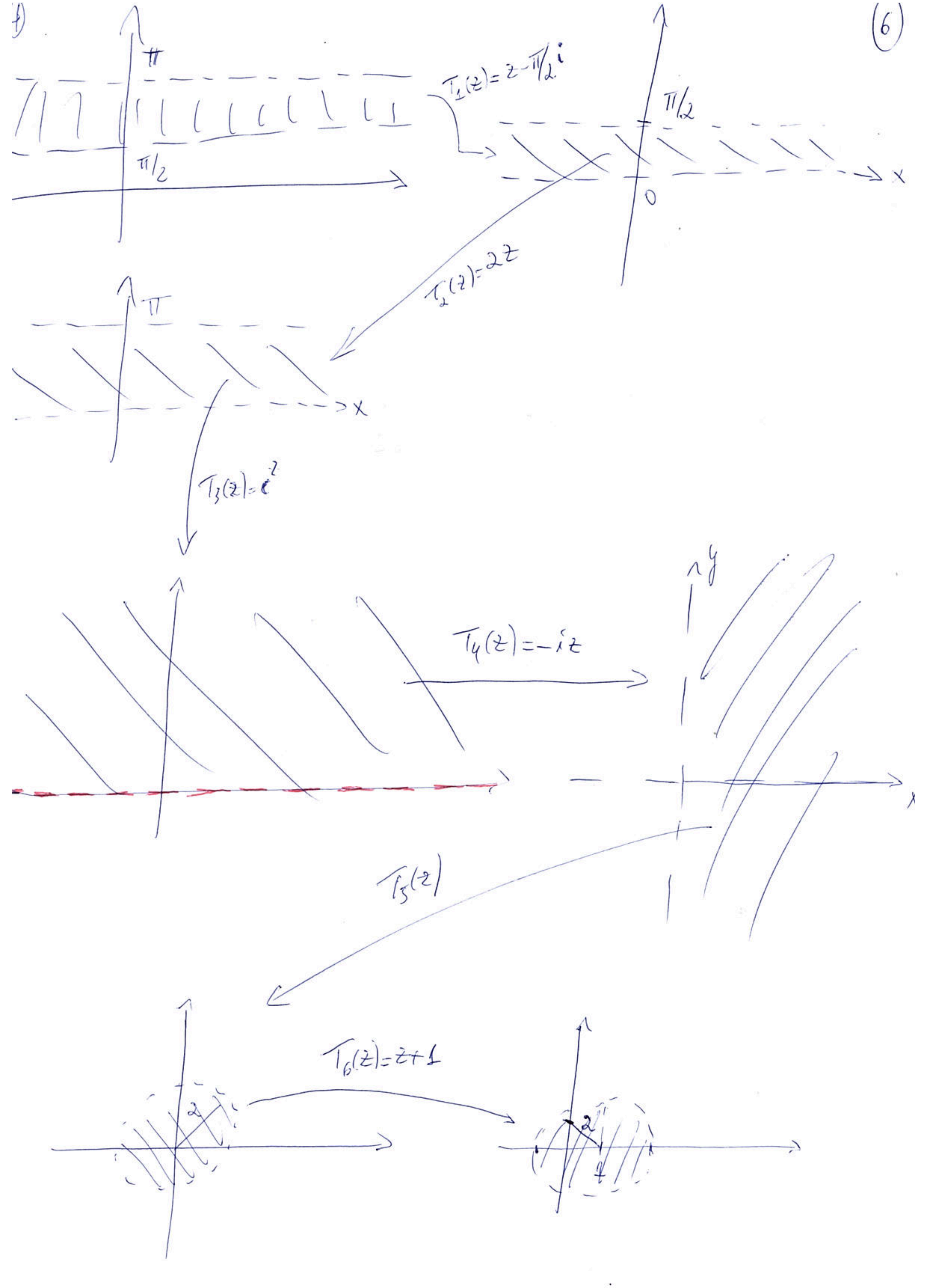
$$-\tfrac{1}{3}z^2 + \frac{1}{9}z^4$$

$$\frac{1}{z^2\operatorname{sen}^2 z} = \tfrac{1}{z^4}\cdot\left\{1 + \tfrac{1}{3}z^2 + \left(\frac{1}{9} - \frac{2}{45}\right)z^4 + \cdots\right\}$$

Logo $\operatorname{Res}\left(\frac{1}{z^2\operatorname{sen}^2 z}, 0\right) = 0$ $\qquad f$

$$\int_\gamma f\, dz = 2\pi i \cdot 0 = 0.$$

f)



Para encontrar $T_5(z)$:

Considere a Transformação: $i \to 1$, $0 \to i$, $-i \to -1$

$$(1-w)(-1-i)(-i-z)\cdot i = (i-z)(-i)(1-i)(-1-w) \Rightarrow w = T^*(z) = -i\left(\frac{z-1}{z+1}\right)$$

Como $T^*(1) = 0$, T^* aplica  em 

Então $T_5(z) = 2T^*(z) = -2i\left(\frac{z-1}{z+1}\right)$ e

$$T(z) = (T_6 \circ T_5 \circ T_4 \circ T_3 \circ T_2 \circ T_1)(z) =$$

$$= T_6 \circ T_5 \circ T_4 \circ \left(e^{2z-\pi i}\right)$$

$$= T_6 \circ T_5 \circ \left[-i e^{2z-\pi i}\right]$$

$$= T_6 \circ \left[-2i \cdot \left(\frac{-ie^{2z-\pi i} - 1}{-ie^{2z-\pi i} + 1}\right)\right]$$

$$T(z) = -2i \cdot \left(\frac{-ie^{2z-\pi i} - 1}{-ie^{2z-\pi i} + 1}\right) + 1$$