Cálculo IV-MTM5186-2014.1

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Nome:

Responda todas as questões abaixo. Respostas sem justificativa e/ou cálculos não serão consideradas.

1. Encontre uma representação em série de Laurent para a função $f(z) = \frac{z+1}{(z-1)^2(z-2)}$ válida na região $1 < |z| < 2$.

2. Calcule as seguintes integrais:

   a. $\int_C \frac{4z^2-\pi^2}{\cos^2(z)} dz$, onde C é a circunferência $|z| = \pi$ orientada no sentido anti-horário.

   b. $\int_C \frac{(2z-1)}{z^2(z^3+1)} dz$, C é o retângulo formado pelas retas $x = -2$, $x = 1$, $y = -1/2$ e $y = 1/2$, orientado no sentido anti-horário.

3. Encontre uma transformação conforme entre as regiões $D_1 = \{z \in \mathbb{C}, |z| > 1\}$ e $D_2 = \{z \in \mathbb{C}, Re(z) < 0,\ Im(z) < 0\}$.

4. Denotando α_m como a m-ésima raiz da função de Bessel de ordem zero J_0, encontre os coeficientes c_m da expansão da função

$$f(x) = \begin{cases} \kappa_1 & se \quad 0 \le x < 1/3 \\ 0 & se \quad 1/3 \le x \le 2/3 \\ \kappa_2 & se \quad 2/3 < x < 1 \\ \kappa_3 & se \quad x = 1 \end{cases}$$

em termos de $\{J_0(\alpha_0 x), J_0(\alpha_1 x), J_0(\alpha_2 x), \ldots\}$. Analise a convergência da série

$$\sum_{m=0}^{+\infty} c_m J_0(\alpha_m x),$$

para cada $x \in [0, 1]$.

Resolução P2 - Cálculo IV - 2014.1:

(1) $$\frac{z+1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$z+1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

$z=1 \Rightarrow 2 = -B \Rightarrow \boxed{B=-2}$

$z=2 \Rightarrow 3 = C \Rightarrow \boxed{C=3}$

$z=-1 \Rightarrow 0 = (-2).(-3).A - 3B + 4C$

$0 = 6A + 6 + 12$

$\boxed{A=-3}$

$$\frac{-3}{z-1} = \frac{-3}{z(1-1/z)} = -\frac{3}{z}\cdot\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n = -3\sum_{n=0}^{+\infty}\frac{1}{z^{n+1}} \quad \text{pois } |z|>1$$

$$\frac{-2}{(z-1)^2} = 2\cdot\frac{d}{dz}\left(\frac{1}{z-1}\right) = 2\cdot\frac{d}{dz}\left(\sum_{n=0}^{+\infty}\frac{1}{z^{n+1}}\right) = 2\cdot\sum_{n=0}^{+\infty}\left[-(n+1)\cdot\left(\frac{1}{z}\right)^{n+2}\right]$$

$$= -2\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^{n+2}\cdot(n+1)$$

$$\frac{3}{z-2} = \frac{3}{2}\cdot\frac{1}{\frac{z}{2}-1} = -\frac{3}{2}\cdot\frac{1}{1-z/2} = -3/2\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n = -3\sum_{n=0}^{+\infty}\frac{z^n}{2^{n+1}} \quad \text{pois } |z|<2$$

Portanto:

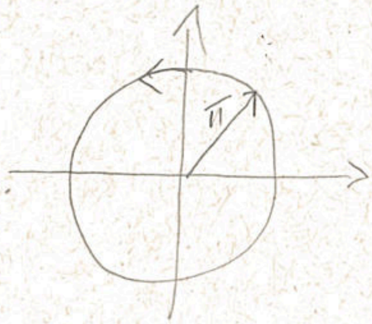
$$\frac{z+1}{(z-1)^2(z-2)} = -3\sum_{n=0}^{+\infty}\frac{1}{z^{n+1}} - 2\sum_{n=0}^{+\infty}(n+1)\cdot\left(\frac{1}{z}\right)^{n+2} - 3\sum_{n=0}^{+\infty}\frac{z^n}{2^{n+1}}$$

$$= -3\sum_{n=1}^{+\infty}\frac{1}{z^n} - 2\sum_{n=2}^{+\infty}(n-1)\left(\frac{1}{z}\right)^n - 3\sum_{n=0}^{+\infty}\frac{z^n}{2^{n+1}}$$

$$= -3\cdot\frac{1}{z} + \sum_{n=2}^{+\infty}(-3-2n+2)\left(\frac{1}{z^n}\right) - 3\sum_{n=0}^{+\infty}\frac{z^n}{2^{n+1}}$$

$$= -\frac{3}{z} + \sum_{n=2}^{+\infty}(-1-2n)\frac{1}{z^n} - 3\sum_{n=0}^{+\infty}\frac{z^n}{2^{n+1}} \quad \text{para } 1<|z|<2$$

2) a)



$\cos^2(z)=0 \Rightarrow z_1=\pi/2$ e $z_2=-\pi/2$

$$f(z)=\frac{4z^2-\pi^2}{\cos^2(z)}=\frac{4\left[z-\pi/2\right]\left[z+\pi/2\right]}{\cos^2(z)}$$

$z_1=\pi/2$ e $z_2=-\pi/2$ são polos de ordem 1 de $f(z)$ pois são zeros de ordem 2 para $\cos^2(z)$.

Logo, $\operatorname{Res}(f,\pi/2)=\lim\limits_{z\to\pi/2}(z-\pi/2)\cdot\dfrac{4(z-\pi/2)(z+\pi/2)}{\cos^2(z)}$

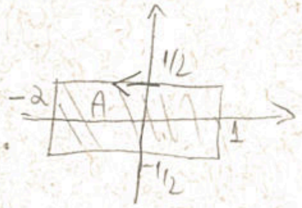
$$=\lim_{z\to\pi/2}\frac{4(z-\pi/2)^2(z+\pi/2)}{\operatorname{sen}^2(z-\pi/2)}=4\cdot\left(\pi/2+\pi/2\right)\cdot\lim_{z\to\pi/2}\left[\frac{z-\pi/2}{\operatorname{sen}(z-\pi/2)}\right]^2$$

$$=4\cdot\pi\cdot1=4\pi$$

$$\operatorname{Res}(f,-\pi/2)=\lim_{z\to-\pi/2}4\cdot(z-\pi/2)\cdot\frac{(z+\pi/2)^2}{\operatorname{sen}^2(z+\pi/2)}=-4\pi$$

Portanto $\displaystyle\int_C\frac{4z^2-\pi^2}{\cos^2(z)}dz=2\pi i\left[4\pi-4\pi\right]=0$

b)



$z^3+1=0\Rightarrow z^3=-1=e^{i\left(\frac{\pi+2k\pi}{3}\right)}$, $k=0,1,2$

$k=0$, $z_1=e^{i\pi/3}=\cos\pi/3+i\operatorname{sen}\pi/3=\frac{1}{2}+\frac{\sqrt{3}}{2}i$ que não pertence à região A

$k=2$, $z_2=e^{i\frac{\pi+4\pi}{3}}=e^{\frac{5\pi}{3}i}=\cos\left(\frac{5\pi}{3}\right)+i\operatorname{sen}\left(\frac{5\pi}{3}\right)=\frac{1}{2}-i\frac{\sqrt{3}}{2}$

que não pertence à região A

Logo f tem duas singularidades isoladas na região A:

$z_3=0$ e $z_4=-1$

$\rightsquigarrow z_3=0$ é polo duplo e $z_4=-1$ é polo simples.

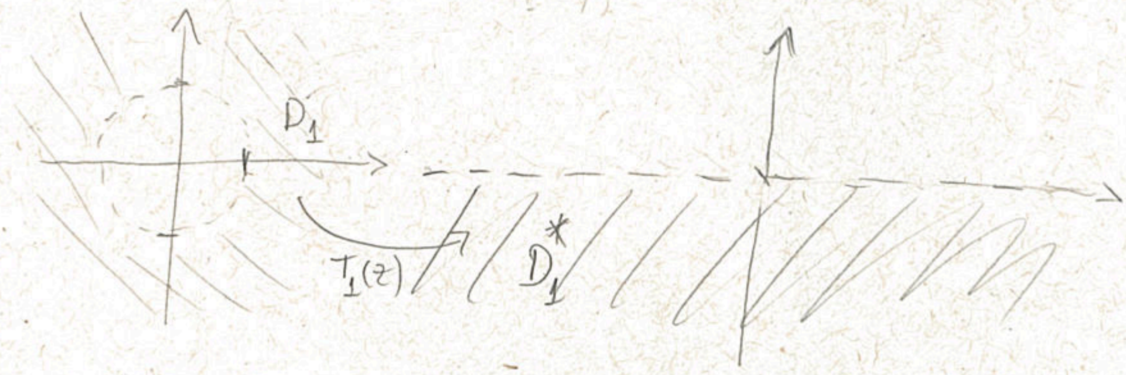
$$\mathrm{Res}(f_1, z_3) = \mathrm{Res}(f_1, 0) = \lim_{z \to 0} \frac{d}{dz}\left[z^2 \cdot \frac{2z-1}{z^2(z^3+1)} \right] = \lim_{z\to 0} \left(\frac{2z-1}{z^3+1} \right)'$$

$$= \lim_{z\to 0} \frac{2(z^3+1) - 3z^2 \cdot (2z-1)}{(z^3+1)^2} = 2$$

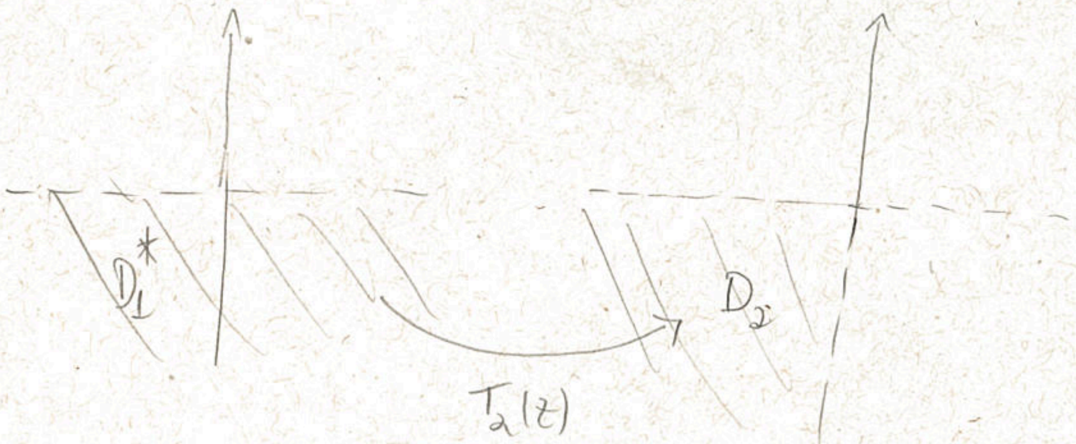
$$\mathrm{Res}(f_1, z_4) = \lim_{z\to -1} (z+1) \frac{(2z-1)}{z^2(z^3+1)} = \lim_{z \to -1} \frac{2z-1}{z^2(z^2-z+1)} = \frac{-2-1}{1.(1+1+1)} = -1$$

$$\int_C \frac{2z-1}{z^2(z^3+1)}\, dz = 2\pi i \left[2 - 1 \right] = 2\pi i$$

3



1º passo



2º passo

Resultado final: $T(z) = T_2(T_1(z))$

Note que se $z \in D_1^*$, $z = |z| \cdot e^{i\theta}$ com $-\pi < \theta < 0$.

Então $z^{1/2} = |z|^{1/2} e^{i\theta/2}$ e $-i z^{1/2} = e^{-\pi/2 i} \cdot |z|^{1/2} \cdot e^{i\theta/2}$

$$= |z|^{1/2} \cdot e^{\frac{i(\theta-\pi)}{2}}$$

sendo que $-2\pi < \theta - \pi < -\pi$ e $-\pi < \frac{\theta - \pi}{2} < -\frac{\pi}{2}$

Logo $T_2(z) = -i z^{1/2}$

Para encontrarmos $T_1(z)$ procuramos uma transformação de Möbius que transforma:

$z_1 = 1 \rightsquigarrow 0 = w_1$

$z_2 = i \rightsquigarrow 1 = w_2$

$z_3 = -i \rightsquigarrow -1 = w_3$

$$(0-w).(-1-1).(1-i).(-i-z) = (1-z).(-i-i)(0-1).(-1-w)$$

$$(+w).(-2)(1-i)(i+z) = (z-1)(2i)(1+w)$$

$$w\left[-2(1-i)(i+z) - 2i(z-1)\right] = 2i(z-1)$$

$$w.\left[(-2+2i)(i+z) - 2iz + 2i\right] = 2i(z-1)$$

$$w = \frac{2i(z-1)}{-2i-2z-2+\cancel{2iz}-\cancel{2iz}+\cancel{2i}} = \frac{2i(z-1)}{-2z-2}$$

$$\boxed{w = \frac{-i(z-1)}{z+1}}$$

Testando $z=2 \Rightarrow w = \frac{-i.(2-1)}{2+1} = -i/3$ que está localizada em D_1^*.

Portanto $T_1(z) = \frac{-i(z-1)}{z+1}$

$$T(z) = T_2(T_1(z)) = -i.\left[T_1(z)\right]^{1/2} = -i\left[\frac{-i(z-1)}{z+1}\right]^{1/2}$$

4

$$f(x) = \sum_{m=0}^{+\infty} c_m J_0(\alpha_m x), \quad \text{sendo}$$

$$C_m = \frac{\int_0^1 x\, J_0(\alpha_m x)\, f(x)\, dx}{\boxed{\int_0^1 x \cdot J_0^2(\alpha_m x)\, dx}}$$

$$\hookrightarrow = \frac{1}{2} J_1^2(\alpha_m)$$

$$\int_0^1 x\, J_0(\alpha_m x)\, f(x)\, dx = K_1 \int_0^{1/3} x\, J_0(\alpha_m x)\, dx + K_2 \int_{2/3}^1 x\, J_0(\alpha_m x)\, dx$$

Mas $(\alpha_m x) J_0(\alpha_m x) = \left[J_1(\alpha_m x) \cdot (\alpha_m x)\right]' \Rightarrow x\, J_0(\alpha_m x) = \left[x \cdot J_1(\alpha_m x)\right]'$, logo

$$\int_0^1 x\, J_0(\alpha_m x)\, f(x)\, dx = K_1 \int_0^{1/3} \left[x\, J_1(\alpha_m x)\right]' dx + K_2 \int_{2/3}^1 \left[x\, J_1(\alpha_m x)\right]' dx$$

$$= K_1 \cdot \tfrac{1}{3} J_1\left(\frac{\alpha_m}{3}\right) + K_2 J_1(\alpha_m) - \frac{2}{3} K_2 J_1\left(\frac{2\alpha_m}{3}\right)$$

$$C_m = \frac{\left(K_1/3\right) J_1(\alpha_m/3) + K_2 J_1(\alpha_m) - \left(\tfrac{2}{3} K_2\right) J_1\left(\tfrac{2}{3}\alpha_m\right)}{\left(1/2\right) J_1^2(\alpha_m)}$$

$$\sum_{m=0}^{+\infty} c_m J_0(\alpha_m x) = \begin{cases} K_1 & \text{se} \quad 0 < x < 1/3 \\ 0 & \text{se} \quad 1/3 < x < 2/3 \\ K_2 & \text{se} \quad 2/3 < x < 1 \\ \dfrac{K_1 + K_2}{2} & \text{se} \quad x = 0 \text{ ou } x = 1 \\ K_1/2 & \text{se} \quad x = 1/3 \\ K_2/2 & \text{se} \quad x = 2/3 \end{cases}$$