# Prova II - Cálculo IV - 2015/01

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Nome:

3,5 1. Encontre a solução do problema de valor inicial

$$y''(x) - xy'(x) - y(x) = 0, \quad y(0) = 1, \; y'(0) = 0,$$

na forma de série de potências (**Somente serão aceitas soluções que apresentem a forma genérica dos coeficientes**).

2,0 2. Encontre a série de Taylor de $f(x) = \cos(x)$ em torno de $x_0 = \pi$.

2,5 3. Estime o erro ao utilizarmos a aproximação

$$\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{x^2}{2}$$

quando $|x| \leq 2/5$.

2,0 4. Encontre todos os $z \in \mathbb{C}$ tais que $z^6 - 1 + i = 0$. Represente as soluções graficamente.

## Fórmulas:

*Série de Taylor:* $\displaystyle\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

*Desigualdade de Taylor:* $|r_n(x)| \leq \frac{M|x-x_0|^{n+1}}{(n+1)!}$, *se* $|f^{(n+1)}(y)| \leq M$ *quando* $y \in [x_0, x_0+h]$ *(ou* $y \in [x_0 - h, x_0]$*)*

*Série Binomial:*

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots + \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}x^n + \ldots, \qquad |x| < 1$

*Erro série alternada:* $|s - s_N| \leq b_{N+1}$

*Raízes complexas:* $z_k = r^{1/n} e^{\frac{\theta + 2\pi k}{n} i}$, $k = 0, 1, \ldots, n-1$.

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1) $y(x) = c_0 + \sum_{n\geq 1} c_n x^n \Rightarrow y(0)=1 \Rightarrow \boxed{c_0=1}$

$y'(x) = c_1 + \sum_{n\geq 2} n c_n x^{n-1} \Rightarrow y'(0)=0 \Rightarrow \boxed{c_1=0}$

$y''(x) = 2c_2 + \sum_{n\geq 3} n c_n (n-1) x^{n-2}$

$$2c_2 + \sum_{n\geq 3} n(n-1)c_n x^{n-2} - c_1 x - \sum_{n\geq 2} n c_n x^n - c_0 - \sum_{n\geq 1} c_n x^n = 0$$

$$2c_2 + \sum_{n\geq 3} n(n-1)c_n x^{n-2} - \sum_{n\geq 2} n c_n x^n - 1 - \sum_{n\geq 1} c_n x^n = 0$$

$$2c_2 + \sum_{n\geq 1} (n+2)(n+1)c_{n+2} x^n - \sum_{n\geq 2} n c_n x^n - 1 - \sum_{n\geq 1} c_n x^n = 0$$

$$2c_2 + 3\cdot 2\cdot c_3 \cdot x - c_1 x - 1 + \sum_{n\geq 2} [(n+2)(n+1)c_{n+2} - nc_n - c_n] x^n$$

$\boxed{c_2 = 1/2}$ $\quad 6c_3 - c_1 = 0 \Rightarrow \boxed{c_3 = 0}$

$n \geq 2 \Rightarrow (n+2)(n+1)c_{n+2} - (n+1)c_n = 0$

$$c_{n+2} = \frac{c_n}{(n+2)}, \quad n \geq 2$$

$n=2 \Rightarrow c_4 = \frac{c_2}{4} = \frac{1}{2}\cdot\frac{1}{4}$

$n=3 \Rightarrow c_5 = \frac{c_3}{5} = 0$

$n=4 \Rightarrow c_6 = \frac{c_4}{6} = \frac{1}{2}\cdot\frac{1}{4}\cdot$

$n=5 \Rightarrow c_7 = 0$

$n=6 \Rightarrow c_8 = \frac{c_6}{8} = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}\cdot$

$C_1 = C_3 = C_5 = \ldots = C_{2m-1} = 0, \quad m \geq 1$

$$C_{2m} = \frac{1}{2} \cdot \frac{1}{4} \cdots \frac{1}{2m} = \frac{1}{2^m} \frac{1}{1.2 \ldots m} = \frac{1}{2^m} \cdot \frac{1}{m!}$$

$$\boxed{y(x) = 1 + \sum_{m=1}^{+\infty} \frac{1}{2^m} \cdot \frac{1}{m!} x^{2m}}$$

(2) $\cos(x - \pi) = \cos x . \cos \pi + \operatorname{sen} x . \operatorname{sen} \pi = -\cos x$

Logo $\cos(x - \pi) = -\sum_{m=0}^{+\infty} \frac{(-1)^m (x-\pi)^{2m}}{(2m)!} = \sum_{m=0}^{+\infty} \frac{(-1)^{m+1} (x-\pi)^{2m}}{(2m)!}$

(3) $f(x) = (1 - x^2)^{-1/2}$

Para $K = -1/2$ e $|x| < 1$,

$$(1-x^2)^{-1/2} = [1 + (-x^2)]^{-1/2} = 1 + \left(\frac{-1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)}{2!} . (-x^2)^2 + \ldots = 1 + \frac{x^2}{2} + \ldots$$

$$f'(x) = -\frac{1}{2}(1 - x^2)^{-1/2 - 1} . (-2x) = x(1-x^2)^{-3/2}$$

$$f''(x) = (1-x^2)^{-3/2} + x . \left(-\frac{3}{2}\right)(1-x^2)^{-3/2 - 1} . (-2x)$$

$$f''(x) = (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2} = (1-x^2)^{-5/2}[1 - x^2 + 3x^2]$$

$$= (1-x^2)^{-5/2}(1 + 2x^2)$$

Como $f'(0)=0$ e $f''(0)=1$,

$$(1-x^2)^{-1/2} \approx f(0)+f'(0).x+f''(0).\frac{x^2}{2} = 1+\frac{x^2}{2}$$

é uma aproximação de Taylor de ordem 2.

Uma estimativa para o erro é então

$$|R_2(x)| \le M\,\frac{|x|^3}{3!}.$$

Para encontrar $M$, primeiro calculamos $f'''(x)$:

$$f'''(x) = -\frac{5}{2}(1-x^2)^{-7/2}\cdot(-2x)\cdot(1+2x^2)+4x\cdot(1-x^2)^{-5/2}$$

$$f'''(x) = 5x(1-x^2)^{-7/2}(1+2x^2)+4x(1-x^2)^{-5/2}$$

$$f'''(x) = (1-x^2)^{-7/2}\left[5x(1+2x^2)+4x(1-x^2)\right]$$

$$f'''(x) = (1-x^2)^{-7/2}\left[5x+10x^3+4x-4x^3\right]$$

$$f'''(x) = (1-x^2)^{-7/2}\left[6x^3+9x\right] = 3(1-x^2)^{-7/2}(2x^3+3x)$$

$$|x| \le 2/5 \Leftrightarrow -\frac{2}{5} \le x \le 2/5 \Leftrightarrow -\frac{8}{125} \le x^3 \le \frac{8}{125}$$

$$-\frac{8}{125}.2-\frac{6}{5} \le 2x^3+3x \le 2.\frac{8}{125}+3.\frac{2}{5} = \frac{16}{125}+\frac{6}{5}$$

$$-\frac{166}{125} \le 2x^3+3x \le \frac{16+150}{125} = \frac{166}{125}$$

$$0 \le x^2 \le \frac{4}{25} \Rightarrow -\frac{4}{25} \le -x^2 \le 0 \Rightarrow \frac{21}{25} \le 1-x^2 \le 1$$

$$\left[\frac{21}{25}\right]^{7/2} \leq (1-x^2)^{7/2} \leq 1 \Rightarrow \quad (1-x^2)^{-7/2} \leq \left(\frac{25}{21}\right)^{7/2}$$

Logo $$|f'''(x)| \leq 3.\left(\frac{25}{21}\right)^{7/2} \cdot \frac{166}{125} = \frac{3.5^7}{(21)^{7/2}} \cdot \frac{166}{5^3} = \frac{3.5^4.166}{(21)^{7/2}} = M$$

e

$$|R_2(x)| \leq \frac{3.5^4.166}{(21)^{7/2}} \cdot \frac{|x|^3}{3!} \leq \frac{3.5^4.166}{(21)^{7/2}} \cdot \left(\frac{2}{5}\right)^3 \cdot \frac{1}{3!}$$

$$|R_2(x)| \leq \frac{3.5^4.166}{3^{7/2}.7^{7/2}} \cdot \frac{2^3}{5^3} \cdot \frac{1}{3.2} = \frac{5.166.2^2}{3^{7/2}.7^{7/2}} \quad \text{para } |x| \leq 2/5$$

4) $z^6 - 1 + i = 0 \Leftrightarrow z = (1-i)^{1/6}$

$$1 - i = \sqrt{2}\, e^{i.\left(\frac{-\pi}{4}\right)}$$

$$(1-i)^{1/6} = \boxed{\sqrt{2}}^{1/6} \cdot e^{\left(\frac{-\pi/4 + K.2\pi}{6}\right) i} = \underbrace{2^{\frac{1}{12}} \cdot e^{\boxed{\frac{-\pi}{24} + \frac{K.\pi}{3}} i}}_{z_K}$$

$K = 0, 1, \ldots, 5$