Cálculo IV- Prova 3

Nome:

Cada resposta deve ser acompanhada de uma justificativa (e/ou cálculo) adequada, caso contrário não será aceita.

1. Encontre todos os valores $z \in \mathbb{C}$ que satisfazem $\cosh(z) = 1/2$.

2. Calcule $Log(-1 - \sqrt{3}\,i)$.

3. Desenvolver $f(z) = \frac{3z-1}{(z^2-9)(z+1)}$ em uma série de Laurent válida para

   a. $1 < |z| < 3$

   b. $2 < |z+3| < 6$.

4. Calcule as seguintes integrais

   a. $\int_C \frac{1}{(1-z)^6} dz$, onde C é o arco da circunferência $z - 1 = e^{i\theta}$, com $0 \leq \theta \leq \pi/2$.

   b. $\int_C \frac{\text{sen}^4(z)}{z^4(1+z^2)} dz$, C é a circunferência de raio 2 centrado na origem, orientada positivamente.

   c. $\int_C z^{-3}\text{cossec}(\text{z}^2) dz$, C é o quadrado com lado igual a 1, centrado na origem, com lados paralelos aos eixos coordenados e orientado positivamente.

   d. $\int_C \frac{dz}{(z+4)z^3}$ ao longo da circunferência $|z| = 2$ (orientada positivamente).

Fórmulas:

Cosseno Hiperbólico: $\cosh(z) = \frac{e^z + e^{-z}}{2}$

Função Logarítmica: $Log(z) = ln|z| + i\, arg_0 z$

Resíduo de f num polo z_0 de ordem 1: $Res(f, z_0) = \lim_{z \to z_0} [(z - z_0) f(z)]$

Resíduo de f num polo z_0 de ordem m:

$$Res(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$$

Algumas séries de potências: $\text{sen(z)} = \sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$, $e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$

Fórmula Integral de Cauchy: $\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$, $\quad n! \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = 2\pi i f^{(n)}(z_0)$

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3) a) $\dfrac{3z-1}{(z^2-9)(z+1)} = \dfrac{A}{z+1} + \dfrac{B}{z-3} + \dfrac{C}{z+3}$

$$3z-1 = A(z^2-9) + B(z+1)(z+3) + C(z+1)(z-3)$$

$z=-1 \Rightarrow -3-1=-8A \Rightarrow \boxed{A=1/2}$

$z=3 \Rightarrow \quad 8 = 4B.6 \Rightarrow \boxed{B=1/3}$

$z=-3 \Rightarrow \quad -10 = C.(-2).(-6) \Rightarrow \quad C=\dfrac{-10}{12} \Rightarrow \boxed{C=-5/6}$

$1<|z|<3 \Rightarrow \quad \dfrac{|z|}{3}<1 \quad e \quad \dfrac{1}{|z|}<1.$

$$\frac{1}{z+1} = \frac{1}{z(1+1/z)} = \frac{1}{z}\cdot\sum_{n=0}^{+\infty}\left(1/z\right)^n$$

$$\frac{1}{z-3} = \frac{-1}{3(1-z/3)} = -1/3\sum_{n=0}^{+\infty}\left(z/3\right)^n = -1/3\sum_{n=0}^{+\infty} z^n/3^n$$

$$\frac{1}{z+3} = \frac{1}{3(1+z/3)} = 1/3\cdot\sum_{n=0}^{+\infty}\left(-z/3\right)^n = 1/3\sum_{n=0}^{+\infty}(-1)^n\frac{z^n}{3^n}$$

Série de Laurent:

$$\frac{1}{2}\sum_{n=0}^{+\infty}\frac{1}{z^{n+1}} - \frac{1}{9}\sum_{n=0}^{+\infty} z^n/3^n - \frac{5}{18}\sum_{n=0}^{+\infty}(-1)^n\, z^n/3^n$$

$$= 1/2\sum_{n=0}^{+\infty} 1/z^{n+1} - \sum_{n=0}^{+\infty} z^n\left(1/3^{n+2} + \frac{5(-1)^n}{18}/3^n\right), \quad \forall\, 1<|z|<3$$

b) Se $|z| > 3$, $\frac{1}{|z|} < \frac{1}{3} \Rightarrow \frac{3}{|z|} < 1$

$$\frac{1}{z+1} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z} \cdot \sum_{n=0}^{+\infty} \frac{1}{z^n}$$

$$\frac{1}{z+3} = \frac{1}{z\left(1+\frac{3}{z}\right)} = \frac{1}{z} \sum_{n=0}^{+\infty} \left(-\frac{3}{z}\right)^n = \frac{1}{z} \sum_{n=0}^{+\infty} (-1)^n \frac{3^n}{z^n}$$

$$\frac{1}{z-3} = \frac{1}{z\left(1-\frac{3}{z}\right)} = \frac{1}{z} \sum_{n=0}^{+\infty} \left(\frac{3}{z}\right)^n = \frac{1}{z} \sum_{n=0}^{+\infty} \frac{3^n}{z^n}$$

Série de Laurent:

$$\frac{1}{2} \sum_{n=0}^{+\infty} \frac{1}{z^{n+1}} + \frac{1}{3} \sum_{n=0}^{+\infty} \frac{3^n}{z^{n+1}} - \frac{5}{6} \sum_{n=0}^{+\infty} (-1)^n \frac{3^n}{z^{n+1}}$$

$$= \sum_{n=0}^{+\infty} \left(\frac{1}{2} + 3^{n-1} - \frac{5}{2} (-1)^n 3^{n-1}\right) z^{-n-1} \quad //$$

c) $0 < |z-1| < 2 \Rightarrow \frac{|z-1|}{2} < 1$

$$\frac{1}{z+1} = \frac{1}{z+2-1} = \frac{1}{2+(z-1)} = \frac{1}{2\left(1+\frac{z-1}{2}\right)} = \frac{1}{2} \sum_{n=0}^{+\infty} (-1)^n \frac{(z-1)^n}{2^n}$$

$$\frac{1}{z-3} = \frac{1}{z-2-1} = \frac{1}{-2+(z-1)} = \frac{-1}{2-(z-1)} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z-1}{2}}$$

$$= -\frac{1}{2} \sum_{n=0}^{+\infty} \left(\frac{z-1}{2}\right)^n$$

d) Singularidade de f: $z_0 = 0$

$$f(z) = \frac{1/z+4}{z^3} = \frac{g(z)}{h(z)}$$

Como $g(0) \neq 0$ e h tem um zero de ordem 3 em $z_0 = 0$, f tem um pólo de ordem 3 em $z_0 = 0$.

Assim $\text{Res}(f, 0) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left(z^3 \cdot \frac{1}{z^3(z+4)} \right)$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d}{dz} \left[-(z+4)^{-2} \right]$$

$$= \frac{1}{2!} \lim_{z \to 0} 2(z+4)^{-3} = \frac{1}{2!} 2 \cdot 4^{-3} = \frac{1}{64}$$

Logo

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \cdot 1/64 = \pi i/32$$

(1) $\cosh(z) = \dfrac{e^z + e^{-z}}{2} = \frac{1}{2} \Rightarrow e^z + e^{-z} = 1$

Se $z = x + iy$ temos $e^x \cdot e^{iy} + e^{-x} \cdot e^{-iy} = 1$

$$e^x [\cos(y) + i\,\text{sen}(y)] + e^{-x} [\cos(-y) + i\,\text{sen}(-y)] = 1$$

Logo: $e^x \cos y + e^{-x} \cos(-y) = 1 \Rightarrow \boxed{\cos(y)\,(e^x + e^{-x}) = 1}$

$e^x\,\text{sen}\,y - e^{-x}\,\text{sen}\,y = 0 \Rightarrow \boxed{\text{sen}\,y\,(e^x - e^{-x}) = 0}$

$\Rightarrow y = m\pi,\ m \in \mathbb{Z}$ ou $x = 0$

Se $x = 0$, temos $\cos(y) \cdot (e^x + e^{-x}) = 1 \Rightarrow \cos(y) \cdot 2 = 1$

$\cos y = \frac{1}{2} \Rightarrow y = \frac{\pi}{3} + k \cdot 2\pi$ ou $y = -\frac{\pi}{3} + k \cdot 2\pi,\ k \in \mathbb{Z}$

Se $y = m\pi,\ m \in \mathbb{Z}$, $\cos(m\pi)\,(e^x + e^{-x}) = 1$. Dois casos:

$\rightarrow m$ ímpar $\Rightarrow (-1)(e^x + e^{-x}) = 1 \Rightarrow e^x + e^{-x} = -1$ que não tem solução

$\rightarrow m$ par: $e^x + e^{-x} = 1 \Rightarrow$ não tem solução

De fato, se $x \geq 0$, $e^x \geq e^0 = 1$ e $e^x + e^{-x} > 1$ já que $e^{-x} > 0$

Se $x < 0$, $-x > 0$ e $e^{-x} > e^0 = 1 \Rightarrow e^x + e^{-x} > 1$.

Solução final: $\left(\pm\frac{1}{3} + 2k\right)\pi i$, $k \in \mathbb{Z}$.

$$\frac{1}{z+3} = \frac{1}{z-1+4} = \frac{1}{4} \cdot \frac{1}{1+\frac{z-1}{4}} = \frac{1}{4} \sum_{m=0}^{+\infty} (-1)^m \left(\frac{z-1}{4}\right)^m$$

Série de Laurent:

$$\frac{1}{4} \sum_{m=0}^{+\infty} (-1)^m \frac{(z-1)^m}{2^m} - \frac{1}{3} \cdot \frac{1}{2} \sum_{m=0}^{+\infty} \frac{(z-1)^m}{2^m} - \frac{5}{24} \sum_{m=0}^{+\infty} (-1)^m \frac{(z-1)^m}{4^m}$$

$$\sum_{m=0}^{+\infty} \left(\frac{(-1)^m}{2^{m+2}} - \frac{1}{3} \cdot \frac{1}{2^{m+1}} - \frac{5}{6} \cdot \frac{1}{4^{m+1}} (-1)^m \right) (z-1)^m$$

d) $2<|z+3|<6 \Rightarrow \left(\frac{|z+3|}{6} < 1 \; , \; \frac{2}{|z+3|} < 1 \right)$

$$\frac{1}{z+1} = \frac{1}{z+3-2} = \frac{-1}{2-(z+3)} = \frac{-1}{(z+3)\left[\frac{2}{z+3} - 1\right]} = +\frac{1}{z+3} \cdot \frac{1}{1-\frac{2}{z+3}}$$

$$= \frac{1}{z+3} \cdot \sum_{m=0}^{+\infty} \left(\frac{2}{z+3}\right)^m$$

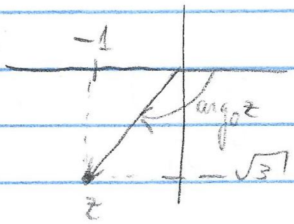
$$\frac{1}{z-3} = \frac{1}{z-6+3} = \frac{1}{-6+z+3} = \frac{-1}{6-(z+3)} = \frac{-1}{6} \cdot \frac{1}{1-\frac{z+3}{6}}$$

$$= -\frac{1}{6} \sum_{m=0}^{+\infty} \left[\frac{z+3}{6}\right]^m$$

Série: $\frac{1}{2} \sum_{m=0}^{+\infty} \frac{2^m}{(z+3)^{m+1}} - \frac{1}{6} \cdot \frac{1}{3} \sum_{m=0}^{+\infty} \frac{(z+3)^m}{6^m} - \frac{5}{6} \cdot \frac{1}{z+3}$

$$\frac{1}{6} \cdot \frac{1}{z+3} + \sum_{m=1}^{+\infty} \frac{2^{m-1}}{(z+3)^{m+1}} - \frac{1}{3} \sum_{m=0}^{+\infty} \frac{(z+3)^m}{6^{m+1}}$$

(2) $z=-1-\sqrt{3}\,i \Rightarrow |z|=\sqrt{1+3}=2$



$$\arg_0 z = -\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = -\left(\frac{3\pi+\pi}{6}\right) = -\frac{2\pi}{3}$$

$$\text{Log}(z) = \ln 2 - \frac{2\pi}{3}\,i$$

(4) a) $$\int_C \frac{dz}{(1-z)^6} = \int_0^{\pi/2} \frac{ie^{i\theta}}{(e^{i\theta})^6}\,d\theta = \int_0^{\pi/2} ie^{-5i\theta}\,d\theta = -\frac{1}{5}e^{-5i\theta}\Big|_0^{\pi/2} = -\frac{1}{5}(-i-1) = \frac{i+1}{5}$$

Pois

$z-1=e^{i\theta} \Rightarrow z=1+e^{i\theta} \Rightarrow r(\theta)=1+e^{i\theta} \Rightarrow r'(\theta)=ie^{i\theta}$

b) Singularidades da função: $z_0=0$, $z_1=i$ e $z_2=-i$

Como $f(z)=\dfrac{\text{sen}^4 z}{z^4(1+z^2)}$, podemos escrever $f(z)=\dfrac{g(z)}{h(z)}$

onde $g(z)=\text{sen}^4 z$ e $h(z)=z^4(1+z^2)$

g tem um zero de ordem 4 em $z_0=0$,

h tem um zero de ordem 4 em $z_0=0$,

logo $z_0=0$ é singularidade removível.

Como consequência, $\text{Res}(f,0)=0$.

Por outro lado $g(i)\neq 0$ e $g(-i)\neq 0$. Como $h(z)=z^4(z+i)(z-i)$, z_1 e z_2 são zeros de ordem 1 para h. Portanto $z_1 = i$ e $z_2 = -i$ são pólos simples para f.

$$Res(f, i) = \lim_{z \to i} \cancel{(z-i)} \frac{sen^4 z}{z^4 (z+i)\cancel{(z-i)}} = \frac{sen^4(i)}{i^4 \, 2i} = \frac{sen^4(i)}{2i}$$

$$Res(f, -i) = \lim_{z \to -i} (z+i) \frac{sen^4 z}{z^4 (z+i)(z-i)} = \frac{sen^4(-i)}{(-i)^4 \cdot -2i} = \frac{sen^4(i)}{-2i}$$

$$\int_C \frac{sen^4 z}{z^4(1+z^2)} dz = 2\pi i \left[\frac{sen^4(i)}{2i} + \frac{sen^4(i)}{-2i} + 0 \right] = 0$$

c) Para $z \neq 0$, $z^{-3} \operatorname{cossec}(z^2) = \frac{1}{z^3} \cdot \frac{1}{sen(z^2)} = \frac{1}{z^3} \cdot \frac{1}{\sum_{n=0}^{+\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!}}$

$$= \frac{1}{\sum_{n=0}^{+\infty} (-1)^n \frac{z^{4n+5}}{(2n+1)!}} = \frac{1}{z^5 - \frac{z^9}{3!} + \frac{z^{13}}{5!} + \cdots}$$

Dividindo temos

$$\begin{array}{l|l} 1 & z^5 - z^9/3! + z^{13}/5! + \cdots \\ \hline -1 + \frac{z^4}{3!} - \frac{z^8}{5!} + \cdots & 1/z^5 + 1/3! \cdot 1/z + \cdots \\ \hline z^4/3! - z^8/5! + \cdots & \\ -1/3!\, z^4 + \left(1/3!\right)^2 z^8 + \cdots & \end{array}$$

Logo $Res(f, 0) = 1/3! = 1/6$

e

$$\int_C z^{-3} \operatorname{cossec}(z^2)\, dz = 2\pi i/6 = \pi i/3$$