Prova III - EDO - 2023/02

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Nome:

1. (2,5 pontos) Considere o sistema $\mathbf{x}'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}(t)$.

   a. *Usando o método dos autovalores e autovetores*, encontre um conjunto fundamental $\{\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\}$ de soluções para o sistema e escreva sua solução geral.

   b. Esboce o retrato de fase do sistema e classifique o ponto de equilíbrio $(0, 0)$.

2. (1,5 pontos) Determine a transformada de Laplace da função

$$f(t) = \int_0^t u \operatorname{sen}(u) e^{4(t-u)} du.$$

3. (1,5 pontos) Encontre a transformada inversa de Laplace da função

$$F(s) = \frac{e^{-\pi s}}{\sqrt[3]{8s-1}}.$$

4. (2,0) Use a transformada de Laplace para resolver o PVI

$$y''(t) + 6ty'(t) = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

5. (3,0 pontos) Considere o PVI:

$$y'' + 2y' + 4y = f(t), \quad y(0) = 0,\ y'(0) = 0,$$

sendo

$$f(t) = \begin{cases} 0, & t < \pi/2 \\ \operatorname{sen}(2t), & \pi/2 < t < \pi \\ e^{-2t}, & t > \pi \end{cases}.$$

   a. Escreva $f(t)$ em termos da função degrau unitário e encontre sua transformada de Laplace.

   b. Usando o item (a) e a transformada de Laplace, resolva o PVI.

Fórmulas: Soluções L.I. no caso de autovalores complexos $r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$:

$$\mathbf{u}(t) = e^{\lambda t}(\cos(\mu t)\mathbf{p} - \operatorname{sen}(\mu t)\mathbf{q})$$

$$\mathbf{v}(t) = e^{\lambda t}(\operatorname{sen}(\mu t)\mathbf{p} + \cos(\mu t)\mathbf{q}),$$

sendo $\mathbf{p} + \mathbf{q}i$ autovetor correspondente.

Soluções L.I. no caso de autovalor repetido λ com um único autovetor $\mathbf{v}$:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}, \quad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{u} + e^{\lambda t} t\mathbf{v},$$

sendo $(A - \lambda I)\mathbf{u} = \mathbf{v}$.

1. (2,5 pontos) Considere o sistema $\mathbf{x}'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}(t)$.

a. *Usando o método dos autovalores e autovetores*, encontre um conjunto fundamental $\{\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\}$ de soluções para o sistema e escreva sua solução geral.

b. Esboce o retrato de fase do sistema e classifique o ponto de equilíbrio $(0,0)$.

a) $|A-\lambda I| = 0 \iff \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0 \iff (2-\lambda)(-2-\lambda)+5=0$

$-4-2\lambda+2\lambda+\lambda^2+5=0 \Rightarrow \lambda^2+1=0 \to \lambda_1 = i, \ \lambda_2 = -i$

Autovetores: $\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = i \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow \begin{cases} 2V_1 - 5V_2 = iV_1 \\ V_1 - 2V_2 = iV_2 \end{cases}$

$(2-i)V_1 = 5V_2 \Rightarrow \boxed{V_2 = \left(\frac{2-i}{5}\right)V_1}$

$\Downarrow$

$V_1 = (2+i)V_2$

$(2-i)V_1 = (2-i)(2+i)V_2$

$\boxed{\frac{(2-i)V_1}{5} = V_2}$

Portanto: $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ \left(\frac{2-i}{5}\right)V_1 \end{bmatrix} = V_1 \begin{bmatrix} 1 \\ \frac{2-i}{5} \end{bmatrix}$ são os autovetores.

Escolhendo $V_1 = 1$, temos $\boxed{v = \begin{bmatrix} 1 \\ 2/5 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ -1/5 \end{bmatrix}}$

$\to p \qquad \to q$

As soluções LI são:

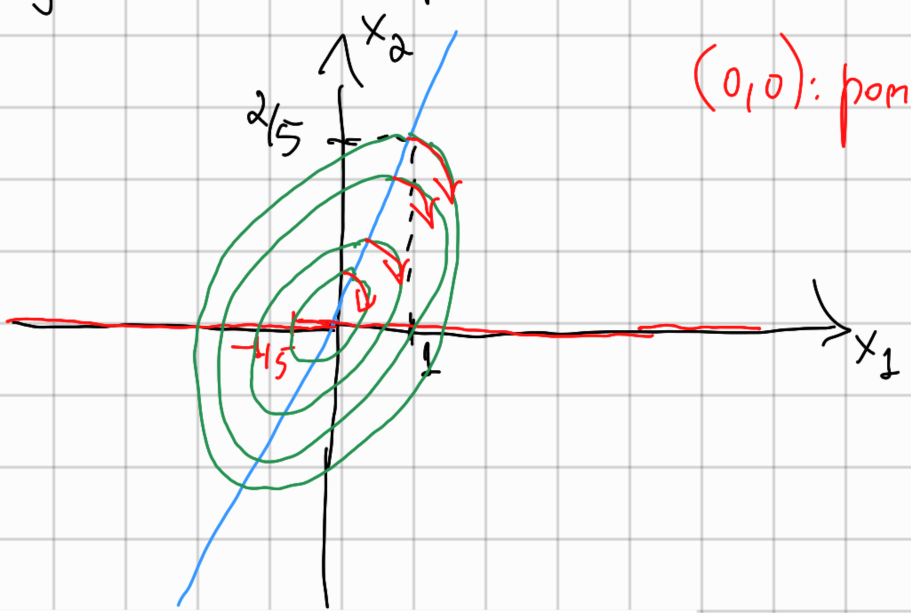
$u(t) = \cos(t)\begin{pmatrix} 1 \\ 2/5 \end{pmatrix} - \text{sen}(t)\begin{pmatrix} 0 \\ -1/5 \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \frac{2}{5}\cos(t) + \frac{1}{5}\text{sen}(t) \end{pmatrix}$

$v(t) = \text{sen}(t)\begin{pmatrix} 1 \\ 2/5 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ -1/5 \end{pmatrix} = \begin{pmatrix} \text{sen}(t) \\ \frac{2}{5}\text{sen}(t) - \frac{1}{5}\cos(t) \end{pmatrix}$

Solução do sistema:

$$X(t) = C_1 \cdot \begin{pmatrix} \cos(t) \\ \frac{2}{5}\cos(t) + \frac{1}{5}\operatorname{sen}(t) \end{pmatrix} + C_2 \begin{pmatrix} \operatorname{sen}(t) \\ \frac{2}{5}\operatorname{sen}(t) - \frac{1}{5}\cos(t) \end{pmatrix}$$

b) As trajetórias são periódicas:



(0,0): ponto periódico estável

2. (1,5 pontos) Determine a transformada de Laplace da função

$$f(t) = \int_0^t u \operatorname{sen}(u) e^{4(t-u)} du.$$

Note que $f(t) = (h * g)(t)$, sendo $h(t) = t\operatorname{sen}(t)$, $g(t) = e^{4t}$.

Logo, $F(s) = H(s) \cdot G(s)$, $H(s) = \mathcal{L}\{t\operatorname{sen}(t)\}(s)$, $G(s) = \mathcal{L}\{e^{4t}\}(s)$.

$$H(s) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{d}{ds}(s^2+1)^{-1} = (s^2+1)^{-2} \cdot 2s = \frac{2s}{(s^2+1)^2}, \ s>0$$

$$G(s) = \frac{1}{s-4}, \ s>4.$$ Portanto, $$F(s) = \frac{2s}{(s^2+1)^2}\left(\frac{1}{s-4}\right), \ s>4.$$

3. (1,5 pontos) Encontre a transformada inversa de Laplace da função

$$F(s) = \frac{e^{-\pi s}}{\sqrt[3]{8s-1}}.$$

$$\frac{1}{\sqrt[3]{8s-1}} = \frac{1}{\sqrt[3]{8(s-1/8)}} = \frac{1}{2} \cdot \frac{1}{(s-1/8)^{1/3}}$$

$\to n+1 = 1/3$

$n = 1/3 - 1 = -2/3$

$$\mathcal{L}\{t^{-2/3}\}(s) = \frac{\Gamma(-2/3+1)}{s^{-2/3+1}}$$

$$\boxed{\mathcal{L}\{t^{-2/3}\}(s) = \frac{\Gamma(1/3)}{s^{1/3}}}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^{1/3}}\right\} = \frac{1}{\Gamma(1/3)} \cdot t^{-2/3}}$$

Logo, $\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{(s-1/8)^{1/3}}\right\}(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1/8)^{1/3}}\right\}(t)$

$$= \underbrace{\frac{1}{2} \cdot \frac{1}{\Gamma(1/3)}\, t^{-2/3} \cdot e^{1/8\, t}}_{\to f(t)}$$

e assim

$$F(s) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{\sqrt[3]{8s-1}}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{2} \cdot \frac{1}{(s-1/8)^{1/3}} \cdot e^{-\pi s}\right\}(t)$$

$$= f(t-\pi) \cdot \mathcal{U}_\pi(t) = \boxed{\frac{1}{2} \cdot \frac{1}{\Gamma(1/3)} (t-\pi)^{-2/3} \cdot e^{1/8(t-\pi)} \cdot \mathcal{U}_\pi(t)}$$

4. (2,0) Use a transformada de Laplace para resolver o PVI

$$y''(t) + 6ty'(t) = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - s$$

$$\mathcal{L}\{t y'(t)\} = -\frac{d}{ds}\mathcal{L}\{y'(t)\} = -\frac{d}{ds}\left[sY(s) - y(0)\right] = -\left[sY'(s) + Y(s)\right]$$

Logo: $s^2 Y(s) - s - 6sY'(s) - 6Y(s) = 0$

$$-6sY'(s) + (s^2 - 6)Y(s) = s \Rightarrow Y'(s) + \frac{(s^2-6)}{-6s}Y(s) = \frac{s}{-6s}$$

$$\Rightarrow Y'(s) + \left(\frac{1}{s} - \frac{s}{6}\right)Y(s) = -\frac{1}{6}$$

$$I(s) = e^{\int\left(\frac{1}{s} - \frac{s}{6}\right)ds} = e^{\ln(s) - \frac{s^2}{12}} = s \cdot e^{-s^2/12}$$

$$Y(s) = \frac{C}{I(s)} + \frac{1}{I(s)}\int s \cdot e^{-s^2/12} \cdot \left(-\frac{1}{6}\right) ds$$

$u = -s^2/12$, $du = -\frac{2s}{12}ds = -\frac{s}{6}ds$

$$Y(s) = \frac{C}{I(s)} + \frac{1}{I(s)} \cdot \int e^u du = \frac{C}{I(s)} + \frac{1}{I(s)} \cdot e^{-s^2/12}$$

$$C = 0 \Rightarrow Y(s) = \frac{1}{s e^{-s^2/12}} \cdot e^{-s^2/12} = \frac{1}{s} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\boxed{y(t) = 1}$$

5. (3,0 pontos) Considere o PVI:

$$y'' + 2y' + 4y = f(t), \quad y(0) = 0, \; y'(0) = 0,$$

sendo

$$f(t) = \begin{cases} 0, & t < \pi/2 \\ \operatorname{sen}(2t), & \pi/2 < t < \pi \\ e^{-2t}, & t > \pi \end{cases} .$$

a. Escreva $f(t)$ em termos da função degrau unitário e encontre sua transformada de Laplace.

b. Usando o item (a) e a transformada de Laplace, resolva o PVI.

a) $f(t) = u_{\pi/2}(t)\,\operatorname{sen}(2t) - u_{\pi}(t)\,\operatorname{sen}(2t) + e^{-2t} \cdot u_{\pi}(t)$

$$\operatorname{sen}(2t) = \operatorname{sen}\left(2\left(t - \pi/2 + \pi/2\right)\right) = \operatorname{sen}\left(2\left(t-\pi/2\right)\right) \cdot \overbrace{\cos\left(2 \cdot \pi/2\right)}^{-1}$$
$$+ \cos\left(2\left(t - \pi/2\right)\right) \cdot \overbrace{\operatorname{sen}\left(2 \cdot \pi/2\right)}^{0}$$

$$\operatorname{sen}(2t) = -\operatorname{sen}\left(2\left(t - \pi/2\right)\right)$$

Logo $\mathcal{L}\left\{u_{\pi/2}(t)\,\operatorname{sen}(2t)\right\}(s) = -\mathcal{L}\left\{u_{\pi/2}(t)\,\operatorname{sen}\left(2\left(t-\pi/2\right)\right)\right\}(s)$

$$= -e^{-\pi/2\, s} \cdot \frac{2}{s^2+4}$$

Também:

$$\operatorname{sen}(2t) = \operatorname{sen}\left(2\left(t - \pi + \pi\right)\right) = \operatorname{sen}\left(2(t-\pi)\right) \cdot \overbrace{\cos(2\pi)}^{1} +$$
$$+ \underbrace{\operatorname{sen}(2\pi)}_{0} \cdot \cos\left(2(t-\pi)\right)$$

$\operatorname{sen}(2t) = \operatorname{sen}\left(2(t-\pi)\right)$, logo:

$$\mathcal{L}\left\{u_{\pi}(t)\,\operatorname{sen}(2t)\right\}(s) = e^{-\pi s} \cdot \frac{2}{s^2+4}$$

$e^{-2t} = e^{-2(t-\pi+\pi)} = e^{-2\pi} \cdot e^{-2(t-\pi)}$, logo

$$\mathcal{L}\{u_\pi(t)\, e^{-2t}\}(s) = e^{-2\pi} \cdot e^{-\pi s} \cdot \frac{1}{s+2}$$

Assim:

$$\boxed{F(s) = -e^{-\pi/2 s} \cdot \frac{2}{s^2+4} - e^{-\pi s} \cdot \frac{2}{s^2+4} + e^{-2\pi} \cdot e^{-\pi s} \cdot \frac{1}{s+2}}$$

b) $\mathcal{L}\{y''\}(s) + 2\mathcal{L}\{y'\}(s) + 4\mathcal{L}\{y\}(s) = F(s)$

$$s^2 Y(s) + 2sY(s) + 4Y(s) = F(s)$$

$$Y(s) = \frac{F(s)}{s^2+2s+4}$$

$$Y(s) = -e^{-\pi/2 s} \cdot \left(\frac{2}{s^2+4}\right)\left(\frac{1}{s^2+2s+4}\right) - e^{-\pi s} \frac{2}{(s^2+4)(s^2+2s+4)} + \frac{e^{-2\pi} \cdot e^{-\pi s}}{(s+2)(s^2+2s+4)}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\}(t)}$$

$$\rightsquigarrow \mathcal{L}^{-1}\left\{\left(\frac{2}{s^2+4}\right)\left(\frac{1}{s^2+2s+4}\right)\right\}$$

$$\frac{2}{(s^2+4)(s^2+2s+4)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+2s+4}$$

$$= \frac{(As+B)(s^2+2s+4) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+2s+4)}$$

$$(As+B)(s^2+2s+4)+(Cs+D)(s^2+4)=2 \longrightarrow A=-1/4 \quad , \quad C=1/4$$
$$\longrightarrow B=0 \quad , \quad D=1/2$$

Logo:

$$\mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)(s^2+2s+4)}\right\}(t)=-\frac{1}{4}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}}_{\cos(2t)}(t)+\frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{s+2}{\underbrace{s^2+2s+4}_{(s+1)^2+3}}\right\}(t)$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2s+4}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3}\right\}+\frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{1\cdot\sqrt{3}}{(s+1)^2+3}\right\}$$

$$=e^{-t}\cdot\cos(\sqrt{3}\,t)+\frac{1}{\sqrt{3}}e^{-t}\,\mathrm{sen}(\sqrt{3}\,t)$$

Logo $$\mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)(s^2+2s+4)}\right\}(t)=-\frac{1}{4}\cos(2t)+\frac{1}{4}e^{-t}\cos(\sqrt{3}\,t)$$
$$+\frac{1}{4\cdot\sqrt{3}}e^{-t}\,\mathrm{sen}(\sqrt{3}\,t)=g(t)$$

$$\mathcal{L}^{-1}\left\{e^{-\pi/2\,s}\cdot\frac{2}{(s^2+4)(s^2+2s+4)}\right\}(t)=\mathcal{U}_{\pi/2}(t)\cdot g(t-\pi/2)$$

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{2}{(s^2+4)(s^2+2s+4)}\right\}(t)=\mathcal{U}_{\pi}(t)\,g(t-\pi)$$

$$\rightsquigarrow \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+2s+4)}\right\}(t):$$

$$\frac{1}{(s+2)(s^2+2s+4)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+2s+4}=\frac{A(s^2+2s+4)+(Bs+C)(s+2)}{(s+2)(s^2+2s+4)}$$

$$A(s^2+2s+4)+(Bs+C)(s+2)=1 \Longrightarrow A=\tfrac{1}{4} \quad C=0$$
$$B=-\tfrac{1}{4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+2s+4)}\right\}(t)=\tfrac{1}{4}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t)}_{\Downarrow\; e^{-2t}}-\frac{1}{4}\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+4}\right\}}_{\Downarrow\; \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+3}\right\}}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+3}\right\}=\underbrace{\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3}\right\}}_{\Downarrow\; e^{-t}\cos(\sqrt{3}t)}-\frac{1}{\sqrt{3}}\underbrace{\mathcal{L}^{-1}\left\{\frac{1\cdot\sqrt{3}}{(s+1)^2+3}\right\}}_{\Downarrow\; -\frac{1}{\sqrt{3}}\cdot e^{-t}\operatorname{sen}(\sqrt{3}t)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+2s+4)}\right\}(t)=\frac{1}{4}e^{-2t}-\frac{1}{4}e^{-2t}\cos(\sqrt{3}t)+\frac{1}{4\sqrt{3}}e^{-t}\operatorname{sen}(\sqrt{3}t)=h(t)$$

$$\mathcal{L}^{-1}\left\{e^{-2\pi}\cdot e^{-\pi s}\cdot\frac{1}{(s+2)(s^2+2s+4)}\right\}(t)=e^{-2\pi}u_\pi(t)\,h(t-\pi).$$

Solução: $y(t)=-u_{\pi/2}(t)\,g(t-\pi/2)-u_\pi(t)\,g(t-\pi)+e^{-2\pi}u_\pi(t)\,h(t-\pi)$

Sendo:

$$g(t-\pi/2)=-\frac{1}{4}\cos\left(2(t-\pi/2)\right)+\frac{1}{4}e^{-(t-\pi/2)}\cos\left(\sqrt{3}(t-\pi/2)\right)+\frac{1}{4\sqrt{3}}e^{-(t-\pi/2)}\operatorname{sen}\left(\sqrt{3}(t-\pi/2)\right)$$

$$g(t-\pi)=-\frac{1}{4}\cos\left(2(t-\pi)\right)+\frac{1}{4}e^{-(t-\pi)}\cos\left(\sqrt{3}(t-\pi)\right)+\frac{1}{4\sqrt{3}}e^{-(t-\pi)}\operatorname{sen}\left(\sqrt{3}(t-\pi)\right)$$

$$h(t-\pi)=\tfrac{1}{4}e^{-2(t-\pi)}-\frac{1}{4}e^{-(t-\pi)}\cos\left(\sqrt{3}(t-\pi)\right)+\frac{1}{4\sqrt{3}}e^{-(t-\pi)}\cdot\operatorname{sen}\left(\sqrt{3}(t-\pi)\right)$$