Lower Limits of Type (D) Monotone Operators in general Banach Spaces

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Abstract

We give, for general Banach spaces, a characterization of the sequential lower limit of maximal monotone operators of type (D) and prove its representability. As a consequence, using a recent extension of the Moreau-Yosida regularization for type (D) operators, we extend to general Banach spaces the definitions of the variational sum of monotone operators and the variational composition of monotone operators with continuous linear mappings, and we prove that both operators are representable.

Keywords: Banach spaces, Monotone operators, Variational Sum, Variational Composition, Type (D) operators, Moreau-Yosida regularization

1 Introduction

The sequential lower limit of maximal monotone operators was studied by García and Lassonde [7], for reflexive Banach spaces provided with a strictly convex norm and dual norm, and then applied to prove the representability of the *variational sum* (see [1, 18, 19]) and the *variational composition* (see [16]). The key tools for such work were Minty-Rockafellar surjectivity theorem (see Fact 4 or [20]) and the Moreau-Yosida regularization, adapted for reflexive Banach spaces by Brezis, Crandall and Pazy [3].

Unfortunately, such tools are not available in general for maximal monotone operators in non-reflexive Banach spaces. However, thanks to some recent work of Marques Alves and Svaiter [11, 12], the aforementioned results are available for maximal monotone operators *of type (D)*, introduced by Gossez [9]. Observe that type (D) operators have many nice properties that maximal monotone operators have in reflexive Banach spaces. Thus, for this kind of operators, we have a weak version of Minty-Rockafellar theorem (using J_ε instead of the duality mapping J, see Fact 3) and a general version of the Moreau-Yosida regularization (Fact 5).

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In this paper, we extend the characterization of the lower limit given in [7] to any Banach space (Proposition 4), and prove that the lower limit of a sequence of type (D) operators is representable (Theorem 5). We also extend the definition of variational sum and variational composition (section 4) to any Banach space when the operators involved are of type (D) and prove that the variational sum and the variational composition are representable (Theorems 8 and 6).

We must emphasize that our study is not a simple generalization from reflexive to non-reflexive, since even in the reflexive case we can drop the norm-regularity conditions asked in [7].

2 Preliminary definitions and notations

Let U, V be non-empty sets. For a set $S \subset U \times V$, we denote the projections over U and V as $\mathrm{Proj}_U(S) = \{u \in U \ : \ \exists\, v \in V,\ (u,v) \in S\}$ and $\mathrm{Proj}_V(S) = \{v \in V \ : \ \exists\, u \in U,\ (u,v) \in S\}$.

A *multivalued operator* $T : U \rightrightarrows V$ is a mapping $T : U \to \mathcal{P}(V)$, that is, for $u \in U$, $T(u) \subset V$. The *graph*, *domain* and *range* of T are defined, respectively, as

$$\mathrm{Gr}(T) = \big\{(u,v) \in U \times V \ : \ v \in T(u)\big\},$$
$$\mathrm{Dom}(T) = \big\{u \in U \ : \ T(u) \neq \emptyset\big\} = \mathrm{Proj}_U(\mathrm{Gr}(T)),$$
$$\mathrm{Ran}(T) = \bigcup_{u \in U} T(u) = \mathrm{Proj}_V(\mathrm{Gr}(T)).$$

In addition, $T^{-1} : V \rightrightarrows U$ is defined as $T^{-1}(v) = \{u \in U \ : \ v \in T(u)\}$. If V is a vector space, for multivalued operators $T, S : U \rightrightarrows V$, their *sum* $T + S : U \rightrightarrows V$ is the operator defined as $(T+S)(u) = T(u) + S(u) = \{v + w \ : \ v \in T(u),\ w \in S(u)\}$. From now on, we will identify multivalued operators with their graphs, so we will write $(u,v) \in T$ instead of $(u,v) \in \mathrm{Gr}(T)$.

Let X be a Banach space and X^*, X^{**} be its topological dual and bidual respectively. We will identify X with its image under the canonical injection into X^{**}, so in this way X is called *reflexive* if $X = X^{**}$ and *non-reflexive*, otherwise. The *duality product* $\pi : X \times X^* \to \mathbb{R}$ is defined as $\pi(x,x^*) = \langle x, x^*\rangle = x^*(x)$.

We will say that $(x,x^*), (y,y^*) \in X \times X^*$ are *monotonically related* if

$$\langle x - y, x^* - y^*\rangle \geq 0,$$

and this will be denoted by $(x,x^*) \sim (y,y^*)$. Analogously, (x,x^*) is *monotonically related* to an operator T if it is monotonically related to every point in T, and this will be denoted by $(x,x^*) \sim T$. A multivalued operator $T : X \rightrightarrows X^*$ is called *monotone* if every $(x,x^*), (y,y^*) \in T$ are monotonically related, and it is called *maximal monotone* if it is monotone and $T \subset S$ and S monotone implies $T = S$. In the same way, an operator $T : X^* \to X$ is (maximal) monotone if T^{-1} is so too. We can also consider the *monotone polar* of T defined as

$$T^\mu = \{(x,x^*) \in X \times X^* \ : \ (x,x^*) \sim T\}.$$

Given a function $f : X \to \mathbb{R}\cup\{+\infty\}$, we denote its *effective domain* by $\mathrm{dom}(f) = \{x \in X \ : \ f(x) < \infty\}$ and its *epigraph* as $\mathrm{epi}(f) = \{(x, \lambda) \in X \times \mathbb{R} \ : \ f(x) \le \lambda\}$. We respectively call f *proper*, *convex* and *lower semicontinuous*, whenever $\mathrm{epi}(f)$ is a non-empty, convex and closed set in $X \times \mathbb{R}$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, proper, convex function. The *Fenchel conjugate* $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined as $f^*(x^*) = \sup_{x\in X}\{\langle x, x^*\rangle - f(x)\}$ and is also lower semicontinuous, proper and convex, whenever f is. The *subdifferential* of f is the multivalued operator defined as

$$\partial f = \big\{(x, x^*) \ : \ f(y) \ge f(x) + \langle y - x, x^*\rangle, \, \forall\, y \in X\big\}.$$

In the same way, the ε*-subdifferential* of f is defined as

$$\partial_\varepsilon f = \big\{(x, x^*) \ : \ f(y) \ge f(x) + \langle y - x, x^*\rangle - \varepsilon, \, \forall\, y \in X\big\}.$$

The subdifferential can be related to the Fenchel conjugate, recall the Fenchel-Young inequality,

$$f(x) + f^*(x^*) \ge \langle x, x^*\rangle,$$

where the equality holds if, and only if, $(x, x^*) \in \partial f$. In the same way, $(x, x^*) \in \partial_\varepsilon f$ if, and only if,

$$\langle x, x^*\rangle \le f(x) + f^*(x^*) \le \langle x, x^*\rangle + \varepsilon.$$

For instance, if $j : X \to \mathbb{R}$, $j(x) = \frac{1}{2}\|x\|^2$, we have $j^* : X^* \to \mathbb{R}$, $j^*(x) = \frac{1}{2}\|x^*\|^2$ and we will denote $J = \partial\frac{1}{2}\|\cdot\|^2$ and $J_\varepsilon = \partial_\varepsilon\frac{1}{2}\|\cdot\|^2$. We must emphasize that most of these definitions were introduced by Moreau [15].

Let $T : X \rightrightarrows X^*$ be a maximal monotone operator, denote $\widehat{T}, \widetilde{T} : X^{**} \rightrightarrows X^*$ as

$$\widehat{T} = T, \qquad \text{and} \qquad \widetilde{T} = \{(x^{**}, x^*) \in X^{**} \times X^* \ : \ (x^{**}, x^*) \sim \widehat{T}\}.$$

The following definition was due to Gossez.

Definition 1 ([9])**.** We say that T is of *type (D)*, if for every $(x^{**}, x^*) \in \widetilde{T}$, there exists a bounded net $(x_\alpha, x^*_\alpha)_\alpha \subset T$ such that $(x_\alpha, x^*_\alpha) \to (x^{**}, x^*)$ in the weak*×strong topology of $X^{**} \times X^*$.

For instance, subdifferentials of convex, proper and lower semicontinuous functions are of type (D).

Another equivalent class of maximal monotone operators was given by Simons.

Definition 2 ([21])**.** We say that T is of *type (NI)* if for all $(x^{**}, x^*) \in X^{**} \times X^*$,

$$\inf_{(y,y^*)\in T}\langle x^{**} - y, x^* - y^*\rangle \le 0.$$

The equivalency between these two classes was proven by Simons [21], and Marques Alves and Svaiter [10].

When X is non-reflexive, if T is a type (D) operator, then $\widehat{T}$ has a unique maximal monotone extension on $X^{**} \times X^*$ and, in this case, such maximal extension is $\widetilde{T}$. Note that every maximal monotone operator in a reflexive space is trivially of type (D).

A convex, proper and lower semicontinuous function $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ is a *representative* of a monotone operator $T : X \rightrightarrows X^*$ if $h(x, x^*) \geq \langle x, x^* \rangle$, for all $(x, x^*) \in X \times X^*$, and $T = \{(x, x^*) \in X \times X^* \ : \ h(x, x^*) = \langle x, x^* \rangle\}$. If T has a representative, then it is called a *representable* operator. The theory of convex representations began with the seminal work of Fitzpatrick [5], which was independently rediscovered by Burachik and Svaiter [4], and Martinez-Legaz and Théra [14]. Fitzpatrick defined, for a (non-empty) monotone operator $T : X \rightrightarrows X^*$, its *Fitzpatrick function* $\varphi_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$,

$$\varphi_T(x, x^*) = \langle x, x^* \rangle - \inf_{(y,y^*)\in T} \langle y - x, y^* - x^* \rangle.$$

Fact 1 ([5, Theorem 3.4]). *Let $T : X \rightrightarrows X^*$ be a monotone operator and $(x, x^*) \in T$. Then $\varphi_T(x, x^*) = \langle x, x^* \rangle$, $(x^*, x) \in \partial\varphi_T(x, x^*)$, and $\varphi_T^*(x^*, x) = \langle x, x^* \rangle$.*

Fitzpatrick also proved that maximal monotone operators are representable by φ_T. Representable operators are closed by arbitrary intersections, in the following sense.

Fact 2 ([13, Corollary 10]). *Let $\{R_\lambda : X \rightrightarrows X^*\}_{\lambda\in\Lambda}$ be a family of representable operators, then*

$$R = \bigcap_{\lambda\in\Lambda} R_\lambda,$$

is also representable.

The reader can find more information about convex representations in [17, 13, 22], and the references therein.

A representative function $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ of a monotone operator T is called a *strong representative* of T if $h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle$, for all $(x^*, x^{**}) \in X^* \times X^{**}$. The following fact was proven by Marques Alves and Svaiter.

Fact 3 ([12, Theorem 3.6] and [10, Theorem 4.4]). *For a maximal monotone operator $T : X \rightrightarrows X^*$, the following conditions are equivalent:*

1. *T is of type (D);*
2. *T has a strong representative;*
3. $\mathrm{Ran}(T + J_\varepsilon(\cdot - x_0)) = X^*$, *for all $x_0 \in X$ and $\varepsilon > 0$.*

The following result due to Rockafellar is well known.

Fact 4 ([20]). *Let X be reflexive and $T : X \rightrightarrows X^*$ monotone. Then T is maximal if, and only if,* $\mathrm{Ran}(T + J) = X^*$.

As a consequence of this result, when X is reflexive with a norm which is strictly convex as well as its dual norm, given a maximal monotone operator $T : X \rightrightarrows X^*$ and $\lambda > 0$, the inclusion

$$0 \in \lambda T(\cdot) + J(\cdot - x)$$

has a solution $R_\lambda(x)$, which is also unique. Thus, the *resolvent* of T is the operator $R_\lambda : X \to X$ and the *Moreau-Yosida regularization* of T is the operator $T_\lambda : X \to$

X^*, defined as $T_\lambda(x) = \dfrac{1}{\lambda} J(x - R_\lambda(x))$. These operators were studied by Moreau and Yosida in Hilbert spaces, and generalized to reflexive spaces by Brezis *et al.* [3]. Recently, Marques Alves and Svaiter [11] extended these definitions to general Banach spaces, for maximal monotone operators of type (D). For X any Banach space and given $T : X \rightrightarrows X^*$ a type (D) operator and $\lambda > 0$, we consider the solutions (x^*, z^{**}) of the system

$$x^* \in \widetilde{T}(z^{**}), \qquad 0 \in \lambda x^* + \widetilde{J}(z^{**} - x). \tag{1}$$

Thus, the new versions of R_λ and T_λ are

$$R_\lambda : X \rightrightarrows X^{**}, \quad \operatorname{Gr}(R_\lambda) = \left\{ (x, z^{**}) \ : \ \begin{array}{l} \exists\, x^* \in X^* \text{ such that} \\ (x^*, z^{**}) \text{ is solution of (1)} \end{array} \right\}$$

$$T_\lambda : X \rightrightarrows X^{*}, \quad \operatorname{Gr}(T_\lambda) = \left\{ (x, x^{*}) \ : \ \begin{array}{l} \exists\, z^{**} \in X^{**} \text{ such that} \\ (x^*, z^{**}) \text{ is solution of (1)} \end{array} \right\}$$

Some properties of T_λ are the following.

Fact 5 ([11, Theorem 4.6]). *Let X be a Banach space, $\lambda > 0$ and let $T : X \rightrightarrows X$ be a maximal monotone operator of type (D). Then the following holds:*

1. *T_λ is maximal monotone of type (D),*
2. $\operatorname{Dom}(T_\lambda) = X$*,*
3. *T_λ maps bounded sets into bounded sets .*

3 Sequential lower limits in general Banach spaces

Let X be a Banach space and let $\{T_n : X \rightrightarrows X^*\}_{n\in\mathbb{N}}$ be a sequence of maximal monotone operators. The *sequential lower limit* (with respect to the strong×strong topology) of $\{T_n\}_n$ is

$$T = \liminf T_n = \big\{ (x, x^*) \ : \ \exists\, (y_n, y_n^*) \in T_n, \text{ s.t. } \lim_n (y_n, y_n^*) = (x, x^*) \big\},$$

where the convergence is taken in the strong×strong topology of $X \times X^*$. García and Lassonde [7], in reflexive Banach spaces, proved the following characterization of T: $(x, x^*) \in T$ if, and only if, $x = \lim x_n$, where x_n is the unique solution of

$$x^* \in T_n(x_n) + J(x_n - x).$$

Note that Fact 4 is central for $(x_n)_n$ to be well defined.

Our aim is to recover this result for a sequence of type (D) operators in any Banach space, so from now on, X will be any real Banach space.

Let $\{T_n : X \rightrightarrows X^*\}_{n\in\mathbb{N}}$ be a sequence of maximal monotone operators of type (D) and let $T : X \rightrightarrows X^*$ be its lower limit. For every T_n, consider $\widetilde{T}_n : X^{**} \rightrightarrows X^*$ be its unique extension to the bidual, which exists as T_n is of type (D). Thus, the sequence $\{\widetilde{T}_n\}_n$ also has its lower limit, say $S : X^{**} \rightrightarrows X^*$. Note that $T_n \subset \widetilde{T}_n$, since T_n is monotone, and this implies $T \subset S$, considering $X \subset X^{**}$.

Consider $(\varepsilon_n)_n$ to be any sequence of positive numbers converging to 0. Given $(x, x^*) \in X \times X^*$ and $n \in \mathbb{N}$, by Fact 3, the inclusion

$$x^* \in T_n(x_n) + J_{\varepsilon_n}(x_n - x), \tag{2}$$

has a solution $x_n \in X$. Thus, there exist $x_n^* \in T_n(x_n)$ and $w_n^* \in J_{\varepsilon_n}(x_n - x)$ such that $x^* = x_n^* + w_n^*$.

Lemma 3. *For any $(x, x^*) \in X \times X^*$, let $(x_n)_n$ be the sequence of solutions of the inclusion* (2) *and let $(x_n^*)_n, (w_n^*)_n \subset X^*$ such that $x^* = x_n^* + w_n^*$, $x_n^* \in T_n(x_n)$ and $w_n^* \in J_{\varepsilon_n}(x_n - x)$. Then $(x_n)_n$, $(x_n^*)_n$ and $(w_n^*)_n$ are bounded. Moreover, given any weak*-limit points $\bar{x}^{**}$ and $\bar{w}^*$ of $(x_n)_n$ and $(w_n^*)_n$, respectively, there exist subnets $(x_{n_\alpha})_\alpha$ and $(w_{n_\alpha}^*)_\alpha$ of $(x_n)_n$ and $(w_n^*)_n$, respectively, such that*

$$\langle u^{**} - \bar{x}^{**}, u^* - (x^* - \bar{w}^*)\rangle \geq \frac{1}{2}\lim_{\alpha} \|x_{n_\alpha} - x\|^2 + \frac{1}{2}\lim_{\alpha}\|w_{n_\alpha}^*\|^2 + \langle x - \bar{x}^{**}, \bar{w}^*\rangle, \tag{3}$$

for all $(u^{**}, u^*) \in S$. In particular, $(\bar{x}^{**}, x^* - \bar{w}^*) \sim S$.

Proof. Let $(u^{**}, u^*) \in S$ be fixed and let $(u_n^{**}, u_n^*) \in \widetilde{T}_n$ such that (u_n^{**}, u_n^*) converges strongly to (u^{**}, u^*). Using the monotonicity of $\widetilde{T}_n$ and the facts that $T_n \subset \widetilde{T}_n$ and $(x_n, x_n^*) \in T_n$, we have

$$\langle u_n^{**} - x_n, u_n^* - x_n^*\rangle \geq 0,$$

and this, together with $x^* = x_n^* + w_n^*$, implies

$$\langle u_n^{**} - x_n, u_n^* - x^*\rangle \geq \langle x_n, w_n^*\rangle - \langle u_n^{**}, w_n^*\rangle. \tag{4}$$

Moreover, since $w_n^* \in J_{\varepsilon_n}(x_n - x)$,

$$\frac{1}{2}\|x_n - x\|^2 + \frac{1}{2}\|w_n^*\|^2 \leq \langle x_n, w_n^*\rangle - \langle x, w_n^*\rangle + \varepsilon_n, \tag{5}$$

so we put (5) into (4) to obtain

$$\begin{aligned}\langle u_n^{**} - x_n, u_n^* - x^*\rangle &\geq \langle x_n, w_n^*\rangle - \langle u_n^{**}, w_n^*\rangle \\ &\geq \frac{1}{2}\|x_n - x\|^2 + \frac{1}{2}\|w_n^*\|^2 + \langle x - u_n^{**}, w_n^*\rangle - \varepsilon_n. \qquad (6)\end{aligned}$$

On the other hand,

$$\langle u_n^{**} - x, w_n^*\rangle \leq \frac{1}{2}\|u_n^{**} - x\|^2 + \frac{1}{2}\|w_n^*\|^2,$$

together with (6) shows that

$$\begin{aligned}\langle u_n^{**} - x_n, u_n^* - x^*\rangle &\geq \frac{1}{2}\|x_n - x\|^2 + \frac{1}{2}\|w_n^*\|^2 + \langle x - u_n^{**}, w_n^*\rangle - \varepsilon_n \qquad (7)\\ &\geq \frac{1}{2}\|x_n - x\|^2 - \frac{1}{2}\|u_n^{**} - x\|^2 - \varepsilon_n. \qquad (8)\end{aligned}$$

The right side in equation (8) has a quadratic term depending on $(x_n)_n$, which is bounded by a linear one on the left side of (7). This, along with the fact that $(u_n^{**})_n$ and $(u_n^*)_n$ are bounded, implies that $(x_n)_n$ is bounded. Similarly, rearranging (7), we obtain that $(w_n^*)_n$ is also bounded, and so is $(x_n^*)_n$.

Take any weak*-limit points of $(x_n)_n$ and $(w_n^*)_n$, say $\bar{x}^{**}$ and $\bar{w}^*$, which exist by the Banach-Alaoglu Theorem. Let $(x_{n_\alpha})_\alpha$ and $(w_{n_\alpha}^*)_\alpha$ be subnets such that $x_{n_\alpha} \to \bar{x}^{**}$ and $w_{n_\alpha}^* \to \bar{w}^*$, in the weak* topology of X^{**} and X^*, respectively. Without loss of generality, we also assume that both $(\|x_{n_\alpha} - x\|)_\alpha$ and $(\|w_{n_\alpha}^*\|)_\alpha$ are convergent and, since the norm is lower semi-continuous in the weak* topology,

$$\lim_\alpha \|x_{n_\alpha} - x\| \geq \|\bar{x}^{**} - x\|, \quad \text{and} \quad \lim_\alpha \|w_{n_\alpha}^*\| \geq \|\bar{w}^*\|. \tag{9}$$

We replace the previously chosen subnets in (7) and take limits, so we obtain

$$\langle u^{**} - \bar{x}^{**}, u^* - x^* \rangle \geq \frac{1}{2} \lim_\alpha \|x_{n_\alpha} - x\|^2 + \frac{1}{2} \lim_\alpha \|w_{n_\alpha}^*\|^2 + \langle x - u^{**}, \bar{w}^* \rangle,$$

which is exactly (3). Now, combining (3) and (9), for all $(u^{**}, u^*) \in S$,

$$\begin{aligned}\langle u^{**} - \bar{x}^{**}, u^* - (x^* - \bar{w}^*) \rangle &\geq \frac{1}{2} \lim_\alpha \|x_{n_\alpha} - x\|^2 + \frac{1}{2} \lim_\alpha \|w_{n_\alpha}^*\|^2 + \langle x - \bar{x}^{**}, \bar{w}^* \rangle \\ &\geq \frac{1}{2}\|\bar{x}^{**} - x\|^2 + \frac{1}{2}\|\bar{w}^*\|^2 - \langle \bar{x}^{**} - x, \bar{w}^* \rangle \geq 0,\end{aligned}$$

which implies $(\bar{x}^{**}, x^* - \bar{w}^*) \sim S$. □

So we have the following proposition.

Proposition 4. *Let X be a real Banach space, $\{T_n : X \rightrightarrows X^*\}_n$ be a sequence of maximal monotone operators of type (D) and let $(\varepsilon_n)_n$ be any sequence of positive numbers converging to zero. If $T = \liminf T_n$, then $(x, x^*) \in T$ if, and only if, $x = \lim x_n$, where x_n is a solution of*

$$x^* \in T_n(x_n) + J_{\varepsilon_n}(x_n - x).$$

Proof. In the proof of Lemma 3, consider the particular case when $(x, x^*) \in T$. Choose $(u^{**}, u^*) = (x, x^*)$ so we have $u_n^{**} \to u^{**} = x$ and $u_n^* \to u^* = x^*$, strongly. Thus, taking the limits in (7), since all the sequences involved are bounded, we have $x_n \to x$ and $w_n^* \to 0$, also strongly. This proves the "only if" part. Conversely, if $x_n \to x$, equation (5) implies that $w_n^* \to 0$. Hence $x_n^* \to x^*$. Since $(x_n, x_n^*) \in T_n$, $(x, x^*) \in T$ and the proposition follows. □

We end this section by proving that the lower limit T is a representable operator.

Theorem 5. *Let X be a real Banach space and $\{T_n : X \rightrightarrows X^*\}_{n \in \mathbb{N}}$ be a sequence of maximal monotone type (D) operators. Then, the lower limit $T = \liminf T_n$ is representable.*

Proof. Recall that S^{-1} is monotone, since S is. Also consider the definition of the Fitzpatrick function of S^{-1}:

$$\varphi_{S^{-1}}(z^*, z^{**}) = \langle z^*, z^{**}\rangle - \inf_{(u^{**},u^*)\in S}\langle u^* - z^*, u^{**} - z^{**}\rangle.$$

Let $(x, x^*) \in X \times X^*$ be arbitrary and consider sequences $(x_n)_n$ and $(w_n^*)_n$ as in Lemma 3. Let $\bar{x}^{**}$ and $\bar{w}^*$ be any weak*-limit points of $(x_n)_n$ and $(w_n^*)_n$, respectively. Also by Lemma 3, equation (3) holds for certain subnets $(x_{n_\alpha})_\alpha$ and $(w^*_{n_\alpha})_\alpha$ of $(x_n)_n$ and $(w_n^*)_n$, respectively.

Taking the infimum over every $(u^*, u^{**}) \in S^{-1}$ in (3), and using the definition of $\varphi_{S^{-1}}(x^* - \bar{w}^*, \bar{x}^{**})$, we obtain

$$\varphi_{S^{-1}}(x^* - \bar{w}^*, \bar{x}^{**}) \leq \langle \bar{x}^{**} - x, \bar{w}^*\rangle + \langle x^* - \bar{w}^*, \bar{x}^{**}\rangle \\ - \frac{1}{2}\lim_\alpha \|x_{n_\alpha} - x\|^2 - \frac{1}{2}\lim_\alpha \|w^*_{n_\alpha}\|^2 \quad (10)$$

Now, by the Fenchel-Young inequality

$$\varphi^*_{S^{-1}}(x, x^*) \geq \langle x^* - \bar{w}^*, x\rangle + \langle \bar{x}^{**}, x^*\rangle - \varphi_{S^{-1}}(x^* - \bar{w}^*, \bar{x}^{**}). \quad (11)$$

Using (10) in (11) we obtain

$$\varphi^*_{S^{-1}}(x, x^*) \geq \langle x^* - \bar{w}^*, x\rangle + \langle \bar{x}^{**}, x^*\rangle - \langle \bar{x}^{**} - x, \bar{w}^*\rangle \\ - \langle x^* - \bar{w}^*, \bar{x}^{**}\rangle + \frac{1}{2}\lim_\alpha \|x_{n_\alpha} - x\|^2 + \frac{1}{2}\lim_\alpha \|w^*_{n_\alpha}\|^2.$$

This implies,

$$\varphi^*_{S^{-1}}(x, x^*) \geq \langle x^*, x\rangle + \frac{1}{2}\lim_\alpha \|x_{n_\alpha} - x\|^2 + \frac{1}{2}\lim_\alpha \|w^*_{n_\alpha}\|^2 \geq \langle x, x^*\rangle. \quad (12)$$

Now we prove that T is representable. Define $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ as

$$h(x, x^*) = \varphi^*_{S^{-1}}(x, x^*) = \sup_{(a^*,a^{**})\in X^*\times X^{**}} \Big\{\langle a^*, x\rangle + \langle a^{**}, x^*\rangle - \varphi_{S^{-1}}(a^*, a^{**})\Big\}.$$

Then h is convex and strongly lower semicontinuous and, in view of (12),

$$h(x, x^*) \geq \langle x, x^*\rangle, \qquad \forall\, (x, x^*) \in X \times X^*.$$

In addition, if $(x, x^*) \in T$ then $(x^*, x) \in S^{-1}$, so using Fact 1,

$$h(x, x^*) = \varphi^*_{S^{-1}}(x, x^*) = \langle x, x^*\rangle.$$

Conversely, if $h(x, x^*) = \langle x, x^*\rangle$ then equation (12) implies that $x_{n_\alpha} \to x$ and $w^*_{n_\alpha} \to 0$, strongly. In view of equation (9) in the proof of Lemma 3, we have $\bar{x}^{**} = x$ and $\bar{w}^* = 0$, so these weak*-limit points are in fact strong limit points. Since these were chosen arbitrarily, the bounded sequences $(x_n)_n$ and $(w_n^*)_n$ possesses a unique strong limit point. Hence $(x_n)_n$ is strongly convergent to x and, by Proposition 4, $(x, x^*) \in T$. □

4 The variational sum and composition

The *variational sum* of maximal monotone operators was defined by Attouch *et al.* [1] in Hilbert Spaces and then generalized to reflexive Banach spaces by Revalski and Théra [18, 19]. Recent results about this kind of sum were given by García *et al.* [6, 7, 8]. Similarly, the *variational composition* was introduced by Pennanen *et al.* [16], also for reflexive Banach spaces. We now extend such notions to general Banach spaces, for maximal monotone operators of type (D).

Given a Banach space X, and two maximal monotone type (D) operators $T_1, T_2 : X \rightrightarrows X^*$, their *variational sum* is defined as follows

$$T_1 \underset{v}{+} T_2 = \bigcap_{\mathcal{I}} \liminf_n (T_{1,\lambda_n} + T_{2,\mu_n}),$$

where

$$\mathcal{I} = \{(\lambda_n, \mu_n)_n \subset \mathbb{R}^2 \, : \, \lambda_n, \mu_n \geq 0, \, \lambda_n + \mu_n > 0, \, \lambda_n, \mu_n \to 0\} \tag{13}$$

and T_λ denotes the Moreau-Yosida regularization of T, for $\lambda \geq 0$ (T_0 simply denotes the operator T). In the same way, for a type (D) operator $T : X \rightrightarrows X^*$ and $A : Y \to X$ linear and continuous, the *variational composition* $(A^*TA)_v$ is defined as

$$(A^*TA)_v = \bigcap_{\mathcal{J}} \liminf_n A^* T_{\lambda_n} A,$$

where

$$\mathcal{J} = \{(\lambda_n)_n \subset \mathbb{R} \, : \, \lambda_n > 0, \, \lambda_n \to 0^+\}. \tag{14}$$

García and Lassonde [7], proved that both the variational sum and composition are representable, when the underlying space was reflexive, strictly convex with strictly convex dual. To recover this result in the general case, we need the following result due to Marques Alves and Svaiter.

Fact 6 ([12, Lemma 3.5]). *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators of type (D), and let $h_1, h_2 : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be representatives of T_1 and T_2, respectively. If*

$$\bigcup_{\lambda > 0} \lambda \Big(\mathrm{Proj}_X(\mathrm{dom}(h_1)) - \mathrm{Proj}_X(\mathrm{dom}(h_2)) \Big)$$

is a closed subspace, then $T_1 + T_2$ is a maximal monotone operator of type (D).

Observe that this fact implies, in particular, that $T_1 + T_2$ is of type (D) whenever T_1 (or T_2) is everywhere defined.

We have the following theorem.

Theorem 6. *Let X be a real Banach space and let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators of type (D). Then their variational sum $T_1 \underset{v}{+} T_2$ is representable.*

Proof. Take any sequence $(\lambda_n, \mu_n)_n \in \mathcal{I}$ (see (13)). From Fact 5, for any $n \in \mathbb{N}$, the Moreau-Yosida regularizations T_{1,λ_n} and T_{2,μ_n} of T_1 and T_2, respectively are also of type (D) and everywhere defined and, by Fact 6, $T_{1,\lambda_n} + T_{2,\mu_n}$ is maximal monotone of type (D). Thus, $(T_{1,\lambda_n} + T_{2,\mu_n})_n$ is a sequence of type (D) operators, so its lower limit is representable and, by Fact 2, arbitrary intersection of representable operators is also representable, hence so is the variational sum. □

As done in [7, §5], we can express the variational composition in terms of a variational sum. Let X, Y be real Banach spaces, $T : X \rightrightarrows X^*$ be maximal monotone and $A : Y \to X$ be linear continuous. Define $T^{\#}, N_A : Y \times X \rightrightarrows Y^* \times X^*$, respectively as $T^{\#}(y,x) = \{0\} \times T(x)$ and

$$N_A(y,x) = \partial\delta_{\mathrm{Gr}(A)} = \begin{cases} \{(A^*x^*, -x^*) \,:\, x^* \in X^*\}, & \text{if } (y,x) \in A, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\delta_{\mathrm{Gr}(A)}$ is the indicator function of the graph of A. Then

$$y^* \in A^*TA(y) \quad \iff \quad (y^*, 0) \in (T^{\#} + N_A)(y, Ay).$$

This allows us to obtain the following lemma, which was already proved by Voisei and Zălinescu [23] (see also [2]). For the sake of completeness, we present its proof here.

Lemma 7. *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator of type (D) and $A : Y \to X$ be linear continuous. If $\mathrm{Dom}(T) = X$ then A^*TA is maximal monotone of type (D).*

Proof. Note that both $T^{\#}$ and N_A are maximal monotone of type (D) in $Y \times X$, as T is of type (D) and N_A is a subdifferential. Moreover, $T^{\#}$ is everywhere defined since T is. Hence, by Fact 6, $T^{\#} + N_A$ is of type (D).

Observe that $\mathrm{Dom}(T^{\#} + N_A) = \mathrm{Dom}(N_A) = \mathrm{Gr}(A)$, therefore

$$T^{\#} + N_A = \left\{ (y, Ay, A^*x^*, w^* - x^*) \in Y \times X \times Y^* \times X^* \;:\; \begin{matrix} y \in Y,\, x^* \in X^* \\ w^* \in T(A(y)) \end{matrix} \right\}.$$

The maximal monotonicity of A^*TA is straightforward, so it remains to prove that A^*TA is maximal monotone of type (D). In view of the equivalency between the type (D) and type (NI) classes, we will prove that A^*TA is of type (NI). Given any $(v^{**}, v^*) \in Y^{**} \times Y^*$, taking $(u^{**}, u^*) = (A^{**}v^{**}, 0) \in X^{**} \times X^*$ and using that $T^{\#} + N_A$ is of type (D), we have

$$\inf_{\substack{y \in Y \\ w^* \in T(A(y)) \\ x^* \in X^*}} \langle v^{**} - y, v^* - A^*x^* \rangle + \langle A^{**}v^{**} - Ay, x^* - w^* \rangle \le 0$$

Rearranging the latter inequality, we obtain

$$\inf_{\substack{y \in Y \\ w^* \in T(A(y))}} \langle v^{**} - y, v^* - A^*w^* \rangle \le 0,$$

and the lemma follows. □

Finally, we have the following theorem.

Theorem 8. *Let X, Y be real Banach spaces, let $T : X \rightrightarrows X^*$ be a maximal monotone operator of type (D) and let $A : Y \to X$ be a linear continuous map. Then the variational composition $(A^*TA)_v$ is representable.*

Proof. From Fact 5 and Lemma 7, for any sequence $(\lambda_n)_n \in \mathcal{J}$ (see (14)) and $n \in \mathbb{N}$, $A^*T_{\lambda_n}A$ is maximal monotone of type (D). Thus, $(A^*T_{\lambda_n}A)_n$ is a sequence of type (D) operators, so its lower limit is representable and, so is the variational composition. $\square$

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