Journal of Convex Analysis
Volume 17 (2010), No. 3&4, 1077–1088

On Gossez Type (D) Maximal Monotone Operators

M. Marques Alves*
IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil
maicon@impa.br

B. F. Svaiter†
IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil
benar@impa.br, tel: 55 (21) 25295112, fax: 55 (21) 25124115

Dedicated to Hedy Attouch on the occasion of his 60th birthday.

Received: April 7, 2009
Revised manuscript received: September 29, 2009

Gossez type (D) operators are defined in non-reflexive Banach spaces and share with the subdifferential operator a topological related property, characterized by bounded nets. In this work we present new properties and characterizations of these operators. The class (NI) was defined after Gossez defined the class (D) and seemed to generalize the class (D). One of our main results is the proof that these classes, type (D) and (NI), are identical.

Keywords: Maximal monotone operators, Gossez type (D) operators, non-reflexive Banach spaces, nets, Fitzpatrick functions

2000 Mathematics Subject Classification: 47H05, 46T99, 47N10

1. Introduction

Let X be a real Banach space with topological dual X^* and topological bidual X^{**}. Whenever necessary, we will identify X with its image under the canonical injection of X into X^{**}. We denote by $\langle \cdot, \cdot \rangle$ the duality product in both $X \times X^*$ and $X^* \times X^{**}$,

$$x^*(x) = \langle x, x^* \rangle, \qquad x^{**}(x^*) = \langle x^*, x^{**} \rangle.$$

A point-to-set operator $T : X \rightrightarrows X^*$ (respectively $T : X^{**} \rightrightarrows X^*$) is a relation on X to X^* (respectively on X^{**} to X^*):

$$T \subset X \times X^* \ \ \text{(respectively } T \subset X^{**} \times X^*\text{)},$$

and $r \in T(q)$ means $(q, r) \in T$. An operator $T : X \rightrightarrows X^*$, or $T : X^{**} \rightrightarrows X^*$, is *monotone* if

$$\langle q - q', r - r' \rangle \geq 0, \quad \forall (q, r), (q', r') \in T.$$

*Partially supported by Brazilian Ministry of Science and technology (MCT), scholarship PCI/DTI.
†Partially supported by CNPq grants 300755/2005-8, 475647/2006-8 and by PRONEX-Optimization.

ISSN 0944-6532 / $ 2.50

An operator $T : X \rightrightarrows X^*$ is *maximal monotone* (in $X \times X^*$) if it is monotone and maximal (whit respect to the inclusion) in the family of monotone operators of X into X^*. An operator $T : X^{**} \rightrightarrows X^*$ is maximal monotone (in $X^{**} \times X^*$) if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of X^{**} into X^*.

Since the canonical injection of X into X^{**} allows one to identity X with a subset of X^{**}, any maximal monotone operator $T : X \rightrightarrows X^*$ is also a monotone operator from X^{**} into X^*, and admits one (or more) maximal monotone extension in X^{**}. The problem of uniqueness of such *maximal monotone* extension to the bidual, has been previously addressed by Gossez [7, 8, 9, 10]. Gossez defined the class of maximal monotone operators of type (D), for which uniqueness of the extension to the bidual is guaranteed [10]. This class of maximal monotone operators has similar properties to the maximal monotone operators in *reflexive* Banach spaces. In the reflexive setting any maximal monotone operator is of Gossez type (D). Properties like surjectivity of perturbations by duality mappings, uniqueness of the extension to the bidual, etc., have been studied for this class of operators by Gossez himself and also by others authors [18, 2, 13, 11]. Attempting to generalize the Gossez type (D) class, Simons introduced and studied in [18] the class of maximal monotone operators of type (NI), and proved that condition (NI) guarantees uniqueness of the extension to the bidual and generalizes Gossez type (D) condition, but no example showing that these classes are distinct was provided. Equivalence between condition (D) and (NI) for *linear* (point-to-point) maximal monotone operators, bounded and unbouded was proved in [2] and [16] respectively.

Studying convex representations of maximal monotone operators in non-reflexive spaces, the authors of the present paper have found several properties of the class (NI) and new characterizations in terms of the Fitzpatrick family [11, 12, 13, 14]. In this work, making use of some of these tools, we will prove that a maximal monotone operator is of type (NI) if and only if it is of Gossez type (D). Consequently, all properties studied recently in [11, 12, 13, 14] can be bridged to the class of operators of Gossez type (D). We summarize them in Theorem 4.4. Besides that, Lemma 4.1, and its consequence Theorem 4.2, were important tools for us, but they seem also to be relevant by themselves. In Theorem 4.5 (and its corollary) we give a partial answer to the question of equivalence between being of Gossez type (D) and having a unique extension to the bidual.

2. Basic definitions and classical results

We use the notation $\|\cdot\|$ for the norm in X, X^* and X^{**}. In the cartesian product of Banach spaces we use the max norm.

The *effective domain* and the *epigraph* of $f : X \to \bar{\mathbb{R}}$ are defined, respectively, as

$$\begin{aligned} \operatorname{dom}(f) &= \{x \in X \mid f(x) < \infty\}, \\ \operatorname{epi}(f) &= \{(x, \lambda) \in X \times \mathbb{R} \mid f(x) \le \lambda\}. \end{aligned}$$

The function f is *convex* if its epigraph is convex, and it is proper if $f > -\infty$ and $\operatorname{dom}(f) \neq \emptyset$. For $\varepsilon \ge 0$, the ε-subdifferential of f is $\partial_\varepsilon f : X \rightrightarrows X^*$,

$$\partial_\varepsilon f(x) = \{x^* \in X^* \mid f(y) \ge f(x) + \langle y - x, x^* \rangle - \varepsilon, \ \forall y \in X\}$$

and ∂f, the subdifferential of f, is the ε-subdifferential for $\varepsilon = 0$, that is $\partial f = \partial_0 f$.

The ε-subdifferential of a proper convex lower semicontinuous function is metrically close to the subdifferential of the function. This special property of the ε-subdifferential, proved in [4], and called *Brøndsted-Rockafellar property* was a fundamental result used by Rockafellar for proving the maximal monotonicity of the subdifferential. Brøndsted-Rockafellar property can be extended to maximal monotone operators as follows:

Definition 2.1 ([12]). A maximal monotone operator $T : X \rightrightarrows X^*$ satisfies the *strict Brøndsted-Rockafellar property* if, for any $\varepsilon > 0$ and every (x, x^*) such that

$$\inf_{(y,y^*)\in T} \langle x - y, x^* - y^* \rangle > -\varepsilon$$

it holds that for any $\lambda > 0$ there exists $(x_\lambda, x^*_\lambda) \in T$ such that

$$\|x - x_\lambda\| < \lambda, \qquad \|x^* - x^*_\lambda\| < \varepsilon/\lambda\ .$$

Fenchel-Legendre conjugate of $f : X \to \bar{\mathbb{R}}$ is $f^* : X^* \to \bar{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x\in X} \langle x, x^* \rangle - f(x).$$

Note that f^* is always convex and lower semicontinuous in the weak* (and hence strong) topology of X^*. Under suitable assumptions, f^{**} coincides with f in X.

Theorem 2.2 (Moreau). *If $f : X \to \bar{\mathbb{R}}$ is a proper, convex, lower semicontinuous function, then f^* is proper and $f^{**}(x) = f(x)$ for all $x \in X$.*

An elegant proof of the above theorem is provided in [3]. In the sequel we will need an auxiliary result, proved by Rockafellar inside the proof of [17, Proposition 1]. For the sake of completeness, a proof of this result is supplied in Appendix A.

Lemma 2.3. *Let $f : X \to \bar{\mathbb{R}}$ be a proper convex lower semicontinuous function. Then, for any $x^{**} \in X^{**}$ there exists $\alpha > 0$ such that $f + \delta_\alpha$ is proper, lower semicontinuous and*

$$f^{**}(x^{**}) = (f + \delta_\alpha)^{**}(x^{**}),$$

where δ_α denotes the indicator function of $B_X[0, \alpha]$ in X, $B_X[0, \alpha]$ being the closed ball centered at 0, with radius α.

3. Background materials

In [7] Gossez introduced a class of maximal monotone operators in *non-reflexive* Banach spaces which share many properties with maximal monotone operators defined in reflexive spaces. This class, coined by Gossez as of type (D) operators, is the main concern of this article.

Definition 3.1 ([7]). Gossez's monotone closure (with respect to $X^{**} \times X^*$) of a maximal monotone operator $T : X \rightrightarrows X^*$, is $\overline{T}^g : X^{**} \rightrightarrows X^*$,

$$\overline{T}^g = \{(x^{**}, x^*) \in X^{**} \times X^* \,|\, \langle x^* - y^*, x^{**} - y \rangle \geq 0,\ \forall (y, y^*) \in T\}. \tag{1}$$

A maximal monotone operator $T : X \rightrightarrows X^*$, is of Gossez type (D) if for any $(x^{**}, x^*) \in \overline{T}^g$, there exists a *bounded* net $\{(x_i, x^*_i)\}_{i\in I}$ in T which converges to (x^{**}, x^*) in the $\sigma(X^{**}, X^*)\times$strong topology of $X^{**} \times X^*$.

In [18] Simons introduced a what seemed to be a new class of maximal monotone operators, called (NI), proved that this class encompasses Gossez type (D) class, and also generalized to this class some previous results of Gossez for type (D) operators.

Definition 3.2 ([18]). A maximal monotone operator $T : X \rightrightarrows X^*$ is of Simons type (NI) if

$$\inf_{(y,y^*)\in T} \langle y^* - x^*, y - x^{**}\rangle \le 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

As we shall see, the classes Simons type (NI) and Gossez type (D) are identical. We will also present new characterizations of the Gossez type (D) operators. Our main tool will be the Fitzpatrick functions which is the next topic.

The Fitzpatrick function and the Fitzpatrick family of a maximal monotone operator $T : X \rightrightarrows X^*$ are defined, respectively, as [6]:

$$\varphi_T : X \times X^* \to \bar{\mathbb{R}}, \qquad \varphi_T(x, x^*) = \sup_{(y,y^*)\in T} \langle y, x^*\rangle + \langle y^*, x\rangle - \langle y, y^*\rangle \tag{2}$$

and

$$\mathcal{F}_T = \left\{ h \in \bar{\mathbb{R}}^{X\times X^*} \;\middle|\; \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ \langle x, x^*\rangle \le h(x, x^*),\ \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^*\rangle \end{array} \right\}. \tag{3}$$

Next we summarize Fitzpatrick's results:

Theorem 3.3 ([6, Theorem 3.10]). *Let $T : X \rightrightarrows X^*$ be maximal monotone. Then for any $h \in \mathcal{F}_T$ and $(x, x^*) \in X \times X^*$*

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^*\rangle,$$

and φ_T is the smallest element of the family $\mathcal{F}_T$.

Burachik and Svaiter defined and studied in [5] the largest element of $\mathcal{F}_T$, the $\mathcal{S}$-function:

$$\mathcal{S}_T = \sup_{h\in\mathcal{F}_T} h. \tag{4}$$

Moreover, they also characterized it by

$$\mathcal{S}_T = \operatorname{cl}\operatorname{conv}(p + \delta_T), \tag{5}$$

where δ_T denotes the indicator function of T, cl conv is the convex lower semicontinuous closure in the strong topology of $X \times X^*$ and $p : X \times X^* \to \mathbb{R}$ is the duality product,

$$p(x, x^*) = \langle x, x^*\rangle.$$

Burachik and Svaiter also proved that the family $\mathcal{F}_T$ is invariant under a generalized conjugation, which we describe next. Recall that $(X \times X^*)^* = X^* \times X^{**}$. For $f : X \times X^* \to \bar{\mathbb{R}}$ $\big(f : X^* \times X^{**} \to \bar{\mathbb{R}}\big)$, define $\mathcal{J}f : X \times X^* \to \bar{\mathbb{R}}$ $\big(\mathcal{J}_* f : X^* \times X^{**} \to \bar{\mathbb{R}}\big)$ by

$$\mathcal{J}f(x, x^*) = f^*(x^*, x) \quad \big(\mathcal{J}_* f(x^*, x^{**}) = f^*(x^{**}, x^*)\big).$$

To simplify the notation, define

$$\Lambda : X^{**} \times X^* \to X^* \times X^{**}, \qquad \Lambda(x^{**}, x^*) = (x^*, x^{**}). \tag{6}$$

Note that $\Lambda(X \times X^*) = X^* \times X$.

Theorem 3.4 ([5, Theorem 5.3]). *Let $T : X \rightrightarrows X^*$ $\big(T : X^{**} \rightrightarrows X^*\big)$ be maximal monotone. Then*

$$\mathcal{J}(\mathcal{F}_T) \subset \mathcal{F}_T \quad \big(\mathcal{J}_*(\mathcal{F}_{\Lambda T}) \subset \mathcal{F}_{\Lambda T}\big)$$

In the sequel we will need a variant of [12, Theorem 3.4], which we state and prove next.

Theorem 3.5. *Let $h : X \times X^* \to \bar{\mathbb{R}}$ be a proper convex lower semicontinuous function. Suppose that*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*,$$
$$h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

If $h(z, z^*) - \langle z, z^* \rangle \leq \varepsilon < \infty$ then, for any $\eta, \mu > 0$ such that $\eta\,\mu > 1$, there exists $(y, y^*) \in X \times X^*$ such that

$$h(y, y^*) = \langle y, y^* \rangle, \qquad \|y - z\| \leq \eta\sqrt{\varepsilon}\,, \qquad \|y^* - z^*\| \leq \mu\sqrt{\varepsilon}$$

with strict inequalities if $\varepsilon > 0$.

Proof. If $\varepsilon = 0$ this means that $h(z, z^*) = \langle z, z^* \rangle$ and the pair $(y, y^*) = (z, z^*)$ has the desired properties.

Suppose that $0 < \varepsilon < \infty$. In this case, defining $\tilde{\varepsilon} = \eta\mu\varepsilon$ we have

$$h(z, z^*) - \langle z, z^* \rangle < \tilde{\varepsilon}.$$

Using now the above equation and Theorem 3.4 of [12], we conclude that for any $\lambda > 0$ there exists (y_λ, y_λ^*) such that

$$h(y_\lambda, y_\lambda^*) = \langle y_\lambda, y_\lambda^* \rangle, \qquad \|y - z\| < \lambda, \qquad \|y^* - z^*\| < \tilde{\varepsilon}/\lambda.$$

Now, the conclusion follows taking $\lambda = \eta\sqrt{\varepsilon}$. □

4. Main results

The next two results are the main tools of our analysis.

Lemma 4.1. *Take $f : X \times X^* \to \bar{\mathbb{R}}$ convex proper lower semicontinuous (in the strong topology) and define $\tilde{f} : X^{**} \times X^* \to \bar{\mathbb{R}}$,*

$$\tilde{f}(x^{**}, x^*) = \begin{cases} f(x^{**}, x^*), & x^{**} \in X, \\ \infty & \textit{otherwise.} \end{cases}$$

Then,

$$\operatorname{cl\,conv}_{w-*\times s} \tilde{f}(x^{**}, x^*) = f^{**}(x^{**}, x^*), \quad \forall (x^{**}, x^*) \in X^{**} \times X^*$$

where $\operatorname{cl}\operatorname{conv}_{w-*\times s}$ *stands for the convex lower semicontinuous closure in the* $\sigma(X^{**},X^*)$ $\times$*strong topology of* $X^{**}\times X^*$. *Equivalently,*

$$f^{**}(x^{**},x^*)=\liminf_{(y,y^*)\to(x^{**},x^*)} f(y,y^*),\quad \forall (x^{**},x^*)\in X^{**}\times X^*,$$

where the $\liminf$ *is taken over all nets in* $X\times X^*$ *converging to* (x^{**},x^*) *in the* $\sigma(X^{**},X^*)\times$*strong topology of* $X^{**}\times X^*$.

Proof. First use Theorem 2.2 to conclude that f^{**} and $\tilde f$ coincides on $X\times X^*$. Since $f^{**}(x^{**},x^*)\le \tilde f(x^{**},x^*)$ for all $(x^{**},x^*)\in X^{**}\times X^*$ and $(x^{**},x^*)\mapsto f^{**}(x^{**},x^*)$ is lower semicontinuous in the $\sigma(X^{**},X^*)\times$ strong topology of $X^{**}\times X^*$, we have

$$\operatorname{cl}\operatorname{conv}_{w-*\times s}\tilde f(x^{**},x^*)\ge f^{**}(x^{**},x^*),\quad \forall (x^{**},x^*)\in X^{**}\times X^*.$$

Suppose that the above inequality is strict at some $(\hat x^{**},\hat x^*)\in X^{**}\times X^*$. Define $g: X^{**}\times X^*\to\bar{\mathbb{R}}$ by $g(x^{**},x^*)=\operatorname{cl}\operatorname{conv}_{w-*\times s}\tilde f(x^{**},x^*)$. Then, $\operatorname{epi}(g)$ is convex, nonempty, and closed in the $\sigma(X^{**},X^*)\times$strong $\times$ strong topology of $X^{**}\times X^*\times\mathbb{R}$ and

$$\big((\hat x^{**},\hat x^*),f^{**}(\hat x^{**},\hat x^*)\big)\notin \operatorname{epi}(g).$$

Using the Hahn-Banach separation theorem in the linear space $X^{**}\times X^*\times\mathbb{R}$, endowed with the locally convex topology $\sigma(X^{**},X^*)\times$strong$\times$ strong, we have that there exists a nonzero vector $(z^*,z^{**},\beta)\in X^*\times X^{**}\times\mathbb{R}$ and $\mu\in\mathbb{R}$ such that

$$\langle x^{**},z^*\rangle+\langle x^*,z^{**}\rangle-\beta\lambda<\mu<\langle \hat x^{**},z^*\rangle+\langle \hat x^*,z^{**}\rangle-\beta f^{**}(\hat x^{**},\hat x^*), \tag{7}$$

for all $(x^{**},x^*,\lambda)\in\operatorname{epi}(g)$. Since $\operatorname{epi}(g)$ is nonempty and $g(x,x^*)\le f(x,x^*)$ for all $(x,x^*)\in X\times X^*$ we conclude, from the above inequality, that $\beta\ge 0$ and

$$\langle x,z^*\rangle+\langle x^*,z^{**}\rangle-\beta f(x,x^*)<\mu \tag{8}$$

for all $(x,x^*)\in\operatorname{dom} f$. Now, take $(\hat z^*,\hat z^{**})\in\operatorname{dom}(f^*)$ and note that

$$\langle x,\hat z^*\rangle+\langle x^*,\hat z^{**}\rangle-f(x,x^*)\le f^*(\hat z^*,\hat z^{**})<\infty \tag{9}$$

for all $(x,x^*)\in X\times X^*$. Multiplying the first inequality in the above equation by $\theta>0$ and adding the resulting inequality to the inequality of (8) we get

$$\langle x,z^*+\theta\hat z^*\rangle+\langle x^*,z^{**}+\theta\hat z^{**}\rangle-(\beta+\theta)f(x,x^*)<\mu+\theta f^*(\hat z^*,\hat z^{**})$$

for all $(x,x^*)\in\operatorname{dom} f$. Dividing the above inequality by $\beta+\theta>0$, and defining

$$z^*_\theta=\frac{1}{\beta+\theta}(z^*+\theta\hat z^*),\qquad z^{**}_\theta=\frac{1}{\beta+\theta}(z^{**}+\theta\hat z^{**}),\qquad \mu_\theta=\frac{\mu+\theta f^*(\hat z^*,\hat z^{**})}{\beta+\theta}$$

we conclude that

$$f^*(z^*_\theta,z^{**}_\theta)\le\mu_\theta.$$

Therefore,

$$f^{**}(\hat x^{**},\hat x^*)\ge\langle z^*_\theta,\hat x^{**}\rangle+\langle z^{**}_\theta,\hat x^*\rangle-\mu_\theta.$$

Multiplication of the above inequality by $\beta+\theta$ and direct algebraic manipulation of the result yields

$$\langle z^* + \theta\hat{z}^*, \hat{x}^{**}\rangle + \langle z^{**} + \theta\hat{z}^{**}, \hat{x}^*\rangle \le (\mu + \theta f^*(\hat{z}^*, \hat{z}^{**})) + (\beta+\theta) f^{**}(\hat{x}^{**}, \hat{x}^*).$$

Taking the limit as $\theta \to 0^+$ on the above inequality we get

$$\langle z^*, \hat{x}^{**}\rangle + \langle z^{**}, \hat{x}^*\rangle \le \mu + \beta f^{**}(\hat{x}^{**}, \hat{x}^*)$$

which contradicts the second inequality of (7).

The last part of the lemma follows from the first part and the topological characterization of lower semicontinuous closure by nets. □

Theorem 4.2. *Let $f : X \times X^* \to \bar{\mathbb{R}}$ be a proper convex lower semicontinuous function. Then, for any $(x^{**}, x^*) \in X^{**} \times X^*$ there exists a bounded net $\{(z_i, z_i^*)\}_{i\in I}$ in $X \times X^*$ which converges to (x^{**}, x^*) in the $\sigma(X^{**}, X^*)\times$strong topology of $X^{**} \times X^*$ and*

$$f^{**}(x^{**}, x^*) = \lim_{i\in I} f(z_i, z_i^*).$$

Proof. Suppose that $f^{**}(x^{**}, x^*) \in \mathbb{R}$. First use Lemma 2.3 to conclude that there exists $\alpha > \|(x^{**}, x^*)\|$ such that $f + \delta_\alpha$ is proper, convex, lower semicontinuous and

$$f^{**}(x^{**}, x^*) = (f + \delta_\alpha)^{**}(x^{**}, x^*).$$

Therefore, using now Lemma 4.1 we conclude that there exists a net $\{(z_i, z_i^*)\}_{i\in I}$ in $X \times X^*$, converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*)\times$strong topology of $X^{**} \times X^*$, such that

$$\lim_{i\in I} (f + \delta_\alpha)(z_i, z_i^*) = f^{**}(x^{**}, x^*).$$

As the above limit is finite, the net can be chosen with $(f + \delta_\alpha)(z_i, z_i^*)$ finite, which readily implies that the net is bounded.

Suppose now that $f^{**}(x^{**}, x^*) = \infty$. Existence of a bounded net $\{z_i\}_{i\in I}$ in X converging to x^{**} in the $\sigma(X^{**}, X^*)$ topology follows from Goldstine theorem. Defining $z_i^* = x^*$ for all $i \in I$ we conclude, using again Lemma 4.1, that $\liminf f(z_i, z_i^*)$ is ∞. Hence, the net $\{(z_i, z_i^*)\}_{i\in I}$ has the desired properties. □

Recall that Fitzpatrick functions are defined in $X \times X^*$ and are bounded bellow by the duality product. The next ressult, which is an immediate consequence of Theorem 4.2, deals with convex function bounded below by the duality product.

Corollary 4.3. *If $f : X \times X^* \to \bar{\mathbb{R}}$ is convex lower semicontinuous,*

$$f(x, x^*) \ge \langle x, x^*\rangle, \quad \forall (x, x^*) \in X \times X^*,$$

and $\tilde{f} : X^{**} \times X^* \to \bar{\mathbb{R}}$ is defined as

$$\tilde{f}(x^{**}, x^*) = \begin{cases} f(x^{**}, x^*), & x^{**} \in X, \\ \infty & \textit{otherwise}, \end{cases}$$

then

$$\operatorname{cl}\operatorname{conv}_{w-*\times s} \tilde{f}(x^{**}, x^*) = f^{**}(x^{**}, x^*) \ge \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**} \tag{10}$$

where $\operatorname{cl}\operatorname{conv}_{w-*\times s}$ *is defined as in Lemma* 4.1.

Proof. The corollary holds trivially if $f \equiv \infty$. If f is proper, apply Lemma 4.1 to conclude that the equality in (10) holds. To prove the inequality in (10) use Theorem 4.2. $\square$

For $h : X \times X^* \to \bar{\mathbb{R}}$ and $(z, z^*) \in X \times X^*$ define $h_{(z,z^*)} : X \times X^* \to \bar{\mathbb{R}}$ by [15, 14]

$$h_{(z,z^*)}(x, x^*) = h(x + z, x^* + z^*) - [\langle x, z^*\rangle + \langle z, x^*\rangle + \langle z, z^*\rangle].$$

By $R(T)$ we denote the *range* of $T : X \rightrightarrows X^*$ and by $(x, x^*) \in T(\cdot + z_0)$ we means $(x + z_0, x^*) \in T$. Moreover, $\overline{A}$ denotes the closure (in the strong topology of X^*) of a set $A \subset X^*$.

The subdifferential and the ε-subdifferential of the function $\frac{1}{2}\|\cdot\|^2$ will be denoted by $J : X \rightrightarrows X^*$ and $J_\varepsilon : X \rightrightarrows X^*$ respectively

$$J(x) = \partial\, \frac{1}{2}\|x\|^2, \qquad J_\varepsilon(x) = \partial_\varepsilon\, \frac{1}{2}\|x\|^2.$$

The operator J is known by the *duality mapping* of X. The operator J_ε was introduced by Gossez in [7].

Theorem 4.4. *Let $T : X \rightrightarrows X^*$ be maximal monotone. Then all the conditions below are equivalent and they hold if and only if T is of Gossez type* (D)*:*

1. *T is of type* (NI)*;*
2. $(\mathcal{S}_T)^*(x^*, x^{**}) \geq \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}$*;*
3. *For all $h \in \mathcal{F}_T$,*
$$h^*(x^*, x^{**}) \geq \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**};$$
4. *There exists $h \in \mathcal{F}_T$ such that*
$$h^*(x^*, x^{**}) \geq \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**};$$
5. *There exists $h \in \mathcal{F}_T$ such that*
$$\inf_{(x,x^*)\in X\times X^*} h_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*;$$
6. *For* all *$h \in \mathcal{F}_T$,*
$$\inf_{(x,x^*)\in X\times X^*} h_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*;$$
7. $\overline{R(T(\cdot + z_0) + J)} = X^*$*, for all $z_0 \in X$;*
8. $\overline{R(T(\cdot + z_0) + J_\varepsilon)} = X^*$*, for all $\varepsilon > 0$ and $z_0 \in X$;*
9. $R(T(\cdot + z_0) + J_\varepsilon) = X^*$*, for all $\varepsilon > 0$ and $z_0 \in X$.*

Moreover, if any of these conditions hold, then

a) *T satisfies the strict Brøndsted-Rockafellar property;*
b) *T has a unique maximal monotone extension to an operator from X^{**} into X^* and this extension is $\overline{T}^g : X^{**} \rightrightarrows X^*$.*

c) $(\mathcal{S}_T)^* = \varphi_{\Lambda\overline{T}^g}$;

d) *For any* $h \in \mathcal{F}_T$, $h^* \in \mathcal{F}_{\Lambda\overline{T}^g}$.

Proof. Equivalence between conditions *1.–9.* follows from [13, Proposition 1.3], [14, Theorem 1.2] and [11, Theorem 3.6]. Statements *a)*, *b)*, *c)* and *d)* have been proved in [13, Theorem 1.1]. The fact that a Gossez type (D) operator is of type (NI) has been proved in [18, Lemma 15].

For finish the proof, we will prove that condition *1.* implies that T is of Gossez type (D). For this aim, suppose T is of type (NI) and

$$(y^{**}, y^*) \in \overline{T}^g.$$

Take $h \in \mathcal{F}_T$. Using *d)* we have $h^* \in \mathcal{F}_{\Lambda\overline{T}^g}$ and, since by Theorem 3.4, $\mathcal{J}_*$ maps $\mathcal{F}_{\Lambda\overline{T}^g}$ into itself, $\mathcal{J}_*(h^*)(y^*, y^{**}) = \langle y^*, y^{**}\rangle$. Therefore

$$h^{**}(y^{**}, y^*) = \langle y^*, y^{**}\rangle. \tag{11}$$

By Theorem 4.2, there exists a bounded net $\{(z_i, z_i^*)\}_{i\in I}$ which converges to (y^{**}, y^*) in the $\sigma(X^{**}, X^*)\times$strong topology of $X^{**} \times X^*$ and

$$h^{**}(y^{**}, y^*) = \lim_{i\in I} h(z_i, z_i^*). \tag{12}$$

Moreover, since $h^{**}(y^{**}, y^*)$ is finite, the net can be chosen so that $\{h(z_i, z_i^*)\}_{i\in I}$ is bounded. Let

$$\varepsilon_i = h(z_i, z_i^*) - \langle z_i, z_i^*\rangle, \quad i \in I.$$

The boundedness of the net $\{(z_i, z_i^*)\}_{i\in I}$, together with its convergence in the $\sigma(X^{**}, X^*)$ $\times$strong topology implies that

$$\lim_{i\in I} \langle z_i, z_i^*\rangle = \langle y^{**}, y^*\rangle.$$

Combining this result with (11), (12) we conclude that the real (non-negative) net $\{\varepsilon_i\}_{i\in I}$ is bounded and converges to 0.

Since $h \in \mathcal{F}_T$ and $h^* \in \mathcal{F}_{\Lambda\overline{T}^g}$, the functions h and h^* majorize the duality product in their respective domains. Therefore, using the above definition of ε_i and Theorem 3.5 for each $i \in I$, with $\eta_i = \sqrt{2}$ and $\mu_i = \sqrt{2}$, we have that for each $i \in I$, there exists $(y_i, y_i^*) \in X \times X^*$ such that

$$h(y_i, y_i^*) = \langle y_i, y_i^*\rangle, \qquad \|z_i - y_i\| \le \sqrt{2\varepsilon_i}, \qquad \|z_i^* - y_i^*\| \le \sqrt{2\varepsilon_i}. \tag{13}$$

The inclusion $h \in \mathcal{F}_T$ and the first equality in the above equation readily implies

$$(y_i, y_i^*) \in T, \quad \forall i \in I.$$

Using the boundedness of $\{(z_i, z_i^*)\}_{i\in I}$ and $\{\varepsilon_i\}_{i\in I}$ together with the two inequalities in (13) and we conclude that $\{(y_i, y_i^*)\}_{i\in I}$ is bounded. Moreover, since the nets $\{\|y_i - z_i\|\}_{i\in I}$ and $\{\|y_i^* - z_i^*\|\}_{i\in I}$ converge to 0, the net $\{(y_i, y_i^*)\}_{i\in I}$ also converges to (y^{**}, y^*) in the $\sigma(X^{**}, X^*)\times$strong topology of $X^{**} \times X^*$. $\square$

Note that in items *8.* and *9.* of the above theorem, for all $\varepsilon > 0$ and $z_0 \in X$ stand for $\forall(\varepsilon, z_0) \in (0, \infty) \times X$.

Theorem 4.5. *If $T : X \rightrightarrows X^*$ is maximal monotone and has a unique maximal monotone extension $\tilde{T} : X^{**} \rightrightarrows X^*$ to the bidual, then one of the following conditions holds:*

1) *T is of Gossez type (D);*

2) *T is affine and non-enlargeable, that is,*

$$\varphi_T(x, x^*) = \begin{cases} \langle x, x^* \rangle, & (x, x^*) \in T, \\ \infty & \text{otherwise.} \end{cases} \tag{14}$$

and $\mathcal{F}_T = \{\varphi_T\}$.

Proof. Suppose T is maximal monotone and has a unique maximal monotone extension $\tilde{T} : X^{**} \rightrightarrows X^*$ to the bidual. Then from [13, Theorem 1.3] we have either T is of type (NI), or T is an affine manifold and $T = \text{dom}(\varphi_T)$. Using Theorem 4.4 we obtain either condition *1.* holds or T is an affine manifold and $T = \text{dom}(\varphi_T)$.

Now, suppose that T is an affine manifold and $T = \text{dom}(\varphi_T)$. Since φ_T coincides with the duality product in T, we conclude that (14) holds. Let $h \in \mathcal{F}_T$. Since $\varphi_T \leq h$ and h coincides with the duality product in T, we have $h = \varphi_T$. $\square$

A direct consequence of the above theorem is that, for nonlinear maximal monotone operators, being of Gossez type (D) is a necessary and sufficient condition for uniqueness of the maximal monotone extension to the bidual.

Corollary 4.6. *Suppose that $T : X \rightrightarrows X^*$ is maximal monotone and T is nonlinear, that is, T is not an affine subspace of $X \times X^*$. Then T has a unique maximal monotone extension to the bidual if, and only if, T is of Gossez type (D).*

5. Gossez contributions

Since operators of type (NI) have been studied by many authors, it seems relevant to stress some of Gossez's main contributions to the study of type (D) operators. This because the class of type (NI) is equal to the class of type (D).

Gossez proved [7, 8] that if $T : X \rightrightarrows X^*$ is of type (D):

1. T has a unique maximal monotone extension to the bidual;
2. $\overline{R(\lambda J_\varepsilon + T)} = X^*$, for all $\lambda, \varepsilon > 0$;
3. $\overline{R(T)}$ is convex;
4. if $D(T) = X$ and T is coercive, then $R(T)$ is dense in X^* in the $\sigma(X^*, X)$ topology,

where

$$D(T) = \{x \in X \mid T(x) \neq \emptyset\}.$$

Gossez also supplied an example [8] of a *linear* maximal monotone operator, which is not of type (D), with a unique maximal monotone extension to the bidual. This shows that Theorem 4.5 can not be strengthened.

A. Proof of Lemma 2.3

To prove Lemma 2.3 we will use the Fenchel-Rockafellar duality formula:

Theorem A.1. *Let $f, g : X \to \bar{\mathbb{R}}$ be proper convex functions. Suppose that f (or g) is continuous at a point $\hat{x} \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$. Then*

$$\inf_{x\in X} f(x) + g(x) = \max_{x^*\in X^*} -f^*(x^*) - g^*(-x^*). \tag{15}$$

For a proof of Theorem A.1, see [3]. This theorem has been generalized by Attouch and Brezis [1], but we only need its classical version.

Proof of Lemma 2.3. For $\alpha > 0$, define $\delta_\alpha : X \to \bar{\mathbb{R}}$ by

$$\delta_\alpha(x) = \begin{cases} 0, & \|x\| \le \alpha, \\ \infty & \text{otherwise.} \end{cases}$$

The lower semicontinuity of $f + \delta_\alpha$ follows from the lower semicontinuity of f and δ_α. Direct calculations yields, for any $x^* \in X^*$ and $\alpha > 0$,

$$\begin{aligned} (f+\delta_\alpha)^*(x^*) &= \sup_{x\in X} \langle x, x^*\rangle - (f+\delta_\alpha)(x) \\ &= \sup_{x\in X} (x^* - \delta_\alpha)(x) - f(x) \\ &= -\inf_{x\in X} f(x) + (\delta_\alpha - x^*)(x). \end{aligned} \tag{16}$$

There exists $\hat{x} \in X$ such that $f(\hat{x}) \in \mathbb{R}$. Take

$$\alpha > \|\hat{x}\|. \tag{17}$$

In this case, $\hat{x} \in \operatorname{dom}(f) \cap \operatorname{dom}(\delta_\alpha)$ and also δ_α is continuous at $\hat{x}$. Now, let $x^* \in X^*$. Using (16) and Theorem A.1, for f and $g = \delta_\alpha - x^*$ we get

$$\begin{aligned} (f+\delta_\alpha)^*(x^*) &= -\inf_{x\in X} f(x) + (\delta_\alpha - x^*)(x) \\ &= -\max_{y^*\in X^*} -f^*(y^*) - (\delta_\alpha - x^*)^*(-y^*) \\ &= \min_{y^*\in X^*} f^*(y^*) + (\delta_\alpha)^*(x^* - y^*) \\ &= f^* \square\, \delta^*_\alpha(x^*). \end{aligned}$$

Thus, $(f + \delta_\alpha)^* = f^* \square\, \delta^*_\alpha$ and so that

$$(f+\delta_\alpha)^{**} = (f^*\square\, \delta^*_\alpha)^* = f^{**} + \delta^{**}_\alpha, \tag{18}$$

the second equality being well known (and it can be easily seen from direct calculations). It is also known (and not dificult to check) that, for $x^* \in X^*$, $x^{**} \in X^{**}$,

$$\delta^*_\alpha(x^*) = \alpha\|x^*\|, \qquad \delta^{**}_\alpha(x^{**}) = \begin{cases} 0, & \|x^{**}\| \le \alpha, \\ \infty & \text{otherwise.} \end{cases} \tag{19}$$

To end the proof, let $x^{**} \in X^{**}$. Taking $\alpha > \max\{\|x^{**}\|, \|\hat{x}\|\}$ and using (18) and the expression in (19) for δ^{**}_α we have

$$f^{**}(x^{**}) = f^{**}(x^{**}) + \delta^{**}_\alpha(x^{**}) = (f+\delta_\alpha)^{**}(x^{**}).$$

□

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