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A New Old Class of Maximal Monotone Operators

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Dedicated to Stephen Simons on the occasion of his 70th birthday.

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In a recent paper in Journal of Convex Analysis the authors studied, in non-reflexive Banach spaces, a class of maximal monotone operators, characterized by the existence of a function in Fitzpatrick's family of the operator which conjugate is above the duality product. This property was used to prove that such operators satisfies a strict version of Brøndsted-Rockafellar property.

In this work we will prove that if a single Fitzpatrick function of a maximal monotone operator has a conjugate above the duality product, then all Fitzpatrick function of the operator have a conjugate above the duality product. As a consequence, the family of maximal monotone operators with this property is just the class NI, previously defined and studied by Simons.

We will also prove that an auxiliary condition used by the authors to prove the strict Brøndsted-Rockafellar property is equivalent to the assumption of the conjugate of the Fitzpatrick function to majorize the duality product.

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1. Introduction

Let X be a real Banach space. We use the notation X^* for the topological dual of X and $\langle \cdot, \cdot \rangle$ for the duality product in $X \times X^*$:

$$\langle x, x^* \rangle = x^*(x).$$

In $X \times X^*$ we use the norm-topology. Whenever necessary, we will identify X with its image under the canonical injection of X into X^{**}. To simplify the notation, from now

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on π and π_* stand for the duality product in $X \times X^*$ and $X^* \times X^{**}$ respectively:

$$\begin{aligned} &\pi : X \times X^* \to \mathbb{R}, && \pi_* : X^* \times X^{**} \to \mathbb{R} \\ &\pi(x, x^*) = \langle x, x^* \rangle, && \pi_*(x^*, x^{**}) = \langle x^*, x^{**} \rangle. \end{aligned} \tag{1}$$

The *indicator function* of $A \subset X$ is $\delta_A : X \to \bar{\mathbb{R}}$,

$$\delta_A(x) := \begin{cases} 0, & x \in A \\ \infty, & \text{otherwise.} \end{cases}$$

For $f : X \to \bar{\mathbb{R}}$, the lower semicontinuous convex closure of f is $\operatorname{cl\,conv} f : X \to \bar{\mathbb{R}}$, the largest lower semicontinuous convex function majorized by f. The conjugate of f is $f^* : X^* \to \bar{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

It is trivial to check that $f^* = (\operatorname{cl\,conv} f)^*$.

A point-to-set operator $T : X \rightrightarrows X^*$ is a relation on $X \times X^*$:

$$T \subset X \times X^*$$

and $x^* \in T(x)$ means $(x, x^*) \in T$. An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T.$$

and it is *maximal monotone* if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of X to X^*. Maximal monotone operators in Banach spaces arise, for example, in the study of PDE's, equilibrium problems and calculus of variations.

Given a maximal monotone operator $T : X \rightrightarrows X^*$, Fitzpatrick defined [15] the family $\mathcal{F}_T$ as those convex, lower semicontinuous functions in $X \times X^*$ which are bounded below by the duality product and coincide with it at T:

$$\mathcal{F}_T = \left\{ h \in \bar{\mathbb{R}}^{X \times X^*} \;\middle|\; \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ \langle x, x^* \rangle \leq h(x, x^*), \quad \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right\}. \tag{2}$$

Fitzpatrick found an explicit formula for the minimal element of $\mathcal{F}_T$, from now on *Fitzpatrick function* of T, $\varphi_T : X \times X^* \to \bar{\mathbb{R}}$

$$\varphi_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y^*, y \rangle. \tag{3}$$

Moreover, he also proved that if $h \in \mathcal{F}_T$ then h represents T in the following sense:

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle.$$

Note that

$$\varphi_T(x, x^*) = (\pi + \delta_T)^*(x^*, x).$$

The supremum of the Fitzpatrick family is the $\mathcal{S}$-function, originally defined and studied by Burachik and Svaiter in [12, Eq. (35)], [11, Eq. (29)]

$$\mathcal{S}_T : X \times X^* \to \bar{\mathbb{R}}, \qquad \mathcal{S}_T = \operatorname{cl}\operatorname{conv}(\pi + \delta_T). \tag{4}$$

Beside being the point-wise supremum of the Fitzpatrick Family, the $\mathcal{S}$-function also belongs to the Fitzpatrick family [12, 11]:

$$\mathcal{S}_T = \sup_{h \in \mathcal{F}_T} h, \qquad \mathcal{S}_T \in \mathcal{F}_T. \tag{5}$$

Some authors [4, 25, 6] attribute the $\mathcal{S}$-function to [18] although [18] was submitted after the publication of [12]. Moreover, the content of [12], and specifically the $\mathcal{S}$-function, was presented on Erice workshop on July 2001, by R. S. Burachik [9]. A list of the talks of this congress, which includes [20], is available on the www[1]. It shall also be noted that [11], the preprint of [12], was published (and available on www) at IMPA preprint server in August 2001.

Burachik and Svaiter defined [12], for $h : X \times X^* \to \bar{\mathbb{R}}$,

$$\mathcal{J}h : X \times X^* \to \bar{\mathbb{R}}, \qquad \mathcal{J}h(x, x^*) = h^*(x^*, x)$$

and proved that if T is maximal monotone, then $\mathcal{J}$ maps $\mathcal{F}_T$ into itself and $\mathcal{J}\,\mathcal{S}_T = \varphi_T$:

$$\mathcal{S}_T^*(x^*, x) = \varphi_T(x, x^*).$$

Note that any $h \in \mathcal{F}_T$ satisfies the condition below:

$$\begin{array}{l} h(x, x^*) \geq \langle x, x^* \rangle \\ h^*(x^*, x) \geq \langle x, x^* \rangle \end{array} \quad \forall\, (x, x^*) \in X \times X^*. \tag{6}$$

What about the converse? Burachik and Svaiter proved in [13, Theorem 3.1] that if a lower semicontinuous convex function h satisfies (6) in a *reflexive* Banach space, then h represents a maximal monotone operator and h belongs to the Fitzpatrick family of this operator. This result has been used for ensuring maximal monotonicity in reflexive Banach spaces [23, 18, 19, 5, 7, 1, 2, 3, 6, 22].

For the case of a non-reflexive Banach space, Marques-Alves and Svaiter proved [16] that if h is a convex lower semicontinuous function in $X \times X^*$ and

$$\begin{array}{ll} h(x, x^*) \geq \langle x, x^* \rangle, & \forall\, (x, x^*) \in X \times X^* \\ h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, & \forall\, (x^*, x^{**}) \in X^* \times X^{**} \end{array} \tag{7}$$

then again h and $\mathcal{J}h$ represent a maximal monotone operator and belong to Fitzpatrick family of this operator. Moreover, the operator T satisfies a strict version of the Brøndsted-Rockafellar property. In particular, Marques-Alves and Svaiter proved that if T is maximal monotone and *one* function in Fitzpatrick family of T satisfies (7), then T satisfies a strict Brøndsted-Rockafellar property. The case of h convex (but not lower semicontinuous) and satisfying (7) was also examined in [16]. The “strong” version of Brøndsted-Rockafellar property for maximal monotone operators in reflexive Banach

[1] http://www.polyu.edu.hk/∼ama/events/conference/EriceItaly-OCA2001/Abstract.html

spaces was established by Torralba [24] and independently rediscovered in Hilbert spaces and in reflexive Banach spaces in [14, 10] respectively.

A maximal monotone operator $T : X \rightrightarrows X^*$ satisfies the Brøndsted-Rockafellar property whenever, for any $x \in X$, $x^* \in X^*$ and $\varepsilon \geq 0$, the condition

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in T \tag{8}$$

implies that for any $\lambda > 0$ there exists $x_\lambda \in X$, $x^*_\lambda \in X^*$ such that

$$(x_\lambda, x^*_\lambda) \in T, \qquad \|x_\lambda - x\| \leq \lambda, \qquad \|x^*_\lambda - x^*\| \leq \frac{\varepsilon}{\lambda}.$$

This condition is an extension for maximal monotone operators of a property of the subdifferential, proved by Brøndsted and Rockafellar in the seminal paper [8]. For Marques Alves and Svaiter, a maximal monotone operator has the strict Brøndsted-Rockafellar property [16] whenever, for any $x \in X$, $x^* \in X^*$ and $\varepsilon \geq 0$, the condition (8) implies that for any $\lambda > 0$ and $\eta > \varepsilon$ there exists $x_\lambda \in X$, $x^*_\lambda \in X^*$ such that

$$(x_\lambda, x^*_\lambda) \in T, \qquad \|x_\lambda - x\| < \lambda, \qquad \|x^*_\lambda - x^*\| < \frac{\eta}{\lambda}.$$

Martínez-Legaz and Svaiter [17] defined (with a different notation), for $h : X \times X^* \to \bar{\mathbb{R}}$ and $(x_0, x^*_0) \in X \times X^*$

$$\begin{aligned} &h_{(x_0,x^*_0)} : X \times X^* \to \bar{\mathbb{R}}, \\ &h_{(x_0,x^*_0)}(x, x^*) := h(x + x_0, x^* + x^*_0) - [\langle x, x^*_0\rangle + \langle x_0, x^*\rangle + \langle x_0, x^*_0\rangle]. \end{aligned} \tag{9}$$

The operation $h \mapsto h_{(x_0,x^*_0)}$ preserves many properties of h, as convexity, lower semicontinuity and can be seen as the action of the group $(X \times X^*, +)$ on $\bar{\mathbb{R}}^{X\times X^*}$, because

$$\left(h_{(x_0,x^*_0)}\right)_{(x_1,x^*_1)} = h_{(x_0+x_1,x^*_0+x^*_1)}.$$

Moreover

$$\left(h_{(x_0,x^*_0)}\right)^* = (h^*)_{(x^*_0,x_0)} ,$$

where the rightmost x_0 is identified with its image under the canonical injection of X into X^{**}. Therefore,

1. $h \geq \pi \iff h_{(x_0,x^*_0)} \geq \pi$,
2. $\left(h_{(x_0,x^*_0)}\right)^* \geq \pi_* \iff (h^*)_{(x^*_0,x_0)} \geq \pi_*$,

and finally,

$$h \in \mathcal{F}_T \iff h_{(x_0,x^*_0)} \in \mathcal{F}_{T-\{(x_0,x^*_0)\}}.$$

Marques-Alves and Svaiter work [16] was heavily based on these nice properties of the map $h \mapsto h_{(x_0,x^*_0)}$. These authors also used the fact that if h satisfies condition (7), then it also satisfies the following *auxiliary condition*:

$$\inf_{(x,x^*)\in X\times X^*} h_{(x_0,x^*_0)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x^*_0) \in X \times X^*. \tag{10}$$

A possible generalization of [16] would be to require only the auxiliary condition (10) for one Fitzpatrick function of a maximal monotone operator T and then conclude that this operator satisfies the strict Brøndsted-Rockafellar property. Unfortunately, condition (10) is not more general than condition (7), as we will prove.

The class of operators studied in [16] is the class of maximal **M**onotone operators for which there exists a function in Fitzpatrick family with a conjugate **A**bove the duality product. So, for the time being, we will call these operators type MA. We will also prove that MA condition is equivalent to NI condition. A maximal monotone $T : X \rightrightarrows X^*$ is type (NI) [21] if

$$\inf_{(y,y^*)\in T} \langle x^{**} - y, x^* - y^* \rangle \le 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

For proving this equivalence we will show that if *some* $h \in \mathcal{F}_T$ satisfies condition (7), then *all* functions in Fitzpatrick family of T satisfy condition (7). Observe again that, for a function in Fitzpatrick family, $h \ge \pi$ holds by definition.

The main results of this work are the two theorems below:

Theorem 1.1. *Let X be a real Banach space and h be a convex function on $X \times X^*$. Then h satisfies the condition* [16, *Eq.* (4)]

$$h \ge \pi \qquad h^* \ge \pi_* \tag{11}$$

if, and only if, h satisfies the auxiliary condition [16, *Eq. immediately below Eq.* (29)],

$$\inf_{(x,x^*)\in X\times X^*} h_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*. \tag{12}$$

Theorem 1.2. *Let X be a real Banach space and $T : X \rightrightarrows X^*$. The following conditions are equivalent:*

1. *T is type* MA*, that is, T is maximal monotone and there exists some $h \in \mathcal{F}_T$ such that $h^* \ge \pi_*$ (and $h \ge \pi$),*
2. *T is maximal monotone and* all *$h \in \mathcal{F}_T$, satisfies the condition $h^* \ge \pi_*$ (and $h \ge \pi$),*
3. *T is maximal monotone and* some *$h \in \mathcal{F}_T$ satisfies the condition*

$$\inf_{(x,x^*)\in X\times X^*} h_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*.$$

4. *T is maximal monotone and* all *$h \in \mathcal{F}_T$ satisfies the condition*

$$\inf_{(x,x^*)\in X\times X^*} h_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*.$$

5. *T is type* NI*:*

$$\inf_{(y,y^*)\in T} \langle x^{**} - y, x^* - y^* \rangle \le 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

where π and π_* are the duality products in $X \times X^*$ and $X^* \times X^{**}$, as described in (1).

2. Proof of the main results

Proof of Theorem 1.1. Let $\bar{h} := \operatorname{cl}\operatorname{conv} h$. As h is convex,

$$\bar{h}(x,x^*) = \lim \inf_{(y,y^*)\to(x,x^*)} h(y,y^*),$$

and, for any $(x_0,x_0^*) \in X \times X^*$,

$$\bar{h}_{(x_0,x_0^*)}(x,x^*) = \lim \inf_{(y,y^*)\to(x,x^*)} h_{(x_0,x_0^*)}(y,y^*).$$

As the duality product is continuous and $(\operatorname{cl}\operatorname{conv} h)^* = h^*$, condition (11) holds for h if, and only if, it holds for $\bar{h}$. As the norms are continuous (this is indeed trivial), condition (12) holds for h if, and only if, it holds for $\bar{h}$. So, it suffices to prove the theorem for the case where h is lower semicontinuous, and we assume it from now on in this proof.

For the sake of completeness, we discuss the implication (11) $\Rightarrow$ (12). Take $(x_0,x_0^*) \in X \times X^*$. If condition (11) holds for h, then it holds for $h_{(x_0,x_0^*)}$ and using [16, Theorem 3.1, Eq. (12)] we conclude that condition (12) holds.

For proving the implication (12) $\Rightarrow$ (11), first note that, for any $(z,z^*) \in X \times X^*$,

$$h_{(z,z^*)}(0,0) \geq \inf_{(x,x^*)} h_{(z,z^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2.$$

Therefore, using also (12) we obtain

$$h(z,z^*) - \langle z,z^*\rangle = h_{(z,z^*)}(0,0) \geq 0.$$

Since (z,z^*) is an arbitrary element of $X \times X^*$ we conclude that $h \geq \pi$.

For proving that, under assumption (12), $h^* \geq \pi_*$, take some $(y^*,y^{**}) \in X^* \times X^{**}$. First, use Fenchel-Young inequality to conclude that for any $(x,x^*),(z,z^*) \in X \times X^*$,

$$h_{(z,z^*)}(x,x^*) \geq \langle x, y^*-z^*\rangle + \langle x^*, y^{**}-z\rangle - \left(h_{(z,z^*)}\right)^*(y^*-z^*, y^{**}-z).$$

As $\left(h_{(z,z^*)}\right)^* = (h^*)_{(z^*,z)}$,

$$\begin{aligned}\left(h_{(z,z^*)}\right)^*(y^*-z^*,y^{**}-z) &= h^*(y^*,y^{**}) - \langle z, y^*-z^*\rangle - \langle z^*, y^{**}-z\rangle - \langle z,z^*\rangle \\ &= h^*(y^*,y^{**}) - \langle y^*,y^{**}\rangle + \langle y^*-z^*, y^{**}-z\rangle.\end{aligned}$$

Combining the two above equations we obtain

$$\begin{aligned}&h_{(z,z^*)}(x,x^*) \\ &\geq \langle x,y^*-z^*\rangle + \langle x^*,y^{**}-z\rangle - \langle y^*-z^*,y^{**}-z\rangle + \langle y^*,y^{**}\rangle - h^*(y^*,y^{**}).\end{aligned}$$

Adding $(1/2)\|x\|^2 + (1/2)\|x^*\|^2$ in both sides of the above inequality we have

$$\begin{aligned}h_{(z,z^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq \langle x,y^*-z^*\rangle + \langle x^*,y^{**}-z\rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \\ - \langle y^*-z^*,y^{**}-z\rangle + \langle y^*,y^{**}\rangle - h^*(y^*,y^{**}).\end{aligned}$$

Note that

$$\langle x,y^*-z^*\rangle + \frac{1}{2}\|x\|^2 \geq -\frac{1}{2}\|y^*-z^*\|^2, \qquad \langle x^*,y^{**}-z\rangle + \frac{1}{2}\|x^*\|^2 \geq -\frac{1}{2}\|y^{**}-z\|^2.$$

Therefore, for any $(x,x^*),(z,z^*) \in X \times X^*$,

$$\begin{aligned} &h_{(z,z^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \\ \geq\ & -\frac{1}{2}\|y^* - z^*\|^2 - \frac{1}{2}\|y^{**} - z\|^2 - \langle y^* - z^*, y^{**} - z\rangle + \langle y^*, y^{**}\rangle - h^*(y^*, y^{**}). \end{aligned}$$

Using now assumption (12) we conclude that the infimum, for $(x,x^*) \in X\times X^*$, at the left hand side of the above inequality is 0. Therefore, taking the infimum on $(x,x^*) \in X\times X^*$ at the left hand side of the above inequality and rearranging the resulting inequality we have

$$h^*(y^*, y^{**}) - \langle y^*, y^{**}\rangle \geq -\frac{1}{2}\|y^* - z^*\|^2 - \frac{1}{2}\|y^{**} - z\|^2 - \langle y^* - z^*, y^{**} - z\rangle.$$

Note that

$$\sup_{z^* \in X^*} -\langle y^* - z^*, y^{**} - z\rangle - \frac{1}{2}\|y^* - z^*\|^2 = \frac{1}{2}\|y^{**} - z\|^2.$$

Hence, taking the sup over $z^* \in X^*$ at the right hand side of the previous inequality we obtain

$$h^*(y^*, y^{**}) - \langle y^*, y^{**}\rangle \geq 0$$

and condition (11) holds. □

Proof of Theorem 1.2. First use Theorem 1.1 to conclude that item *1.* and *3.* are equivalent. The same theorem also shows that items *2.* and *4.* are equivalent.

Now assume that item *3.* holds, that is, for some $h \in \mathcal{F}_T$,

$$\inf_{(x,x^*)\in X\times X^*} h_{(x_0,x_0^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0,x_0^*) \in X \times X^*.$$

Take $g \in \mathcal{F}_T$, and $(x_0,x_0^*) \in X \times X^*$. First observe that, for any $(x,x^*) \in X \times X^*$, $g_{(x_0,x_0^*)}(x,x^*) \geq \langle x, x^*\rangle$ and

$$g_{(x_0,x_0^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq \langle x,x^*\rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq 0.$$

Therefore,

$$\inf_{(x,x^*)\in X\times X^*} g_{(x_0,x_0^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq 0. \tag{13}$$

Note that the square of the norm is coercive, that is, it grows faster than any scalar multiple of the norm, as the norm tends to infinity. Therefore, there exists $M > 0$ such that

$$\left\{(x,x^*) \in X \times X^* \mid h_{(x_0,x_0^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 < 1\right\} \subset B_{X\times X^*}(0,M),$$

where

$$B_{X\times X^*}(0,M) = \left\{(x,x^*) \in X \times X^* \mid \sqrt{\|x\|^2 + \|x^*\|^2} < M\right\}.$$

For any $\varepsilon > 0$, there exists $(\tilde{x},\tilde{x}^*)$ such that

$$\min\left\{1,\varepsilon^2\right\} > h_{(x_0,x_0^*)}(\tilde{x},\tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2.$$

Therefore

$$\begin{gathered}\varepsilon^2 > h_{(x_0,x_0^*)}(\tilde{x},\tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 \geq h_{(x_0,x_0^*)}(\tilde{x},\tilde{x}^*) - \langle \tilde{x},\tilde{x}^*\rangle \geq 0,\\ M^2 \geq \|\tilde{x}\|^2 + \|\tilde{x}^*\|^2.\end{gathered} \tag{14}$$

In particular,

$$\varepsilon^2 > h_{(x_0,x_0^*)}(\tilde{x},\tilde{x}^*) - \langle \tilde{x},\tilde{x}^*\rangle.$$

Using now the fact that operators type MA satisfies the strict Brøndsted-Rockafellar property [16, Theorem 3.4] we conclude that there exists $(\bar{x},\bar{x}^*)$ such that

$$h_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) = \langle \bar{x},\bar{x}^*\rangle, \qquad \|\tilde{x}-\bar{x}\| < \varepsilon, \qquad \|\tilde{x}^*-\bar{x}^*\| < \varepsilon. \tag{15}$$

Therefore,

$$h(\bar{x}+x_0,\bar{x}^*+x_0^*) - \langle \bar{x}+x_0,\bar{x}^*+x_0^*\rangle = h_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) - \langle \bar{x},\bar{x}^*\rangle = 0,$$

and $(\bar{x}+x_0,\bar{x}^*+x_0^*) \in T$. As $g \in \mathcal{F}_T$,

$$g(\bar{x}+x_0,\bar{x}^*+x_0^*) = \langle \bar{x}+x_0,\bar{x}^*+x_0^*\rangle,$$

i.e.,

$$g_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) = \langle \bar{x},\bar{x}^*\rangle. \tag{16}$$

Using the first line of (14) we have

$$\varepsilon^2 > h_{(x_0,x_0^*)}(\tilde{x},\tilde{x}^*) + \left[\frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x},\tilde{x}^*\rangle\right] - \langle \tilde{x},\tilde{x}^*\rangle \geq \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x},\tilde{x}^*\rangle.$$

Therefore,

$$\varepsilon^2 > \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x},\tilde{x}^*\rangle. \tag{17}$$

Direct use of (15) gives

$$\begin{aligned}\langle \bar{x},\bar{x}^*\rangle &= \langle \tilde{x},\tilde{x}^*\rangle + \langle \bar{x}-\tilde{x},\tilde{x}^*\rangle + \langle \tilde{x},\bar{x}^*-\tilde{x}^*\rangle + \langle \bar{x}-\tilde{x},\bar{x}^*-\tilde{x}^*\rangle\\ &\leq \langle \tilde{x},\tilde{x}^*\rangle + \|\bar{x}-\tilde{x}\|\,\|\tilde{x}^*\| + \|\tilde{x}\|\,\|\bar{x}^*-\tilde{x}^*\| + \|\bar{x}-\tilde{x}\|\,\|\bar{x}^*-\tilde{x}^*\|\\ &\leq \langle \tilde{x},\tilde{x}^*\rangle + \varepsilon[\|\tilde{x}^*\| + \|\tilde{x}\|] + \varepsilon^2\end{aligned}$$

and

$$\begin{aligned}\|\bar{x}\|^2 + \|\bar{x}^*\|^2 &\leq (\|\tilde{x}\| + \|\bar{x}-\tilde{x}\|)^2 + (\|\tilde{x}^*\| + \|\bar{x}^*-\tilde{x}^*\|)^2\\ &\leq \|\tilde{x}\|^2 + \|\tilde{x}^*\|^2 + 2\varepsilon[\|\tilde{x}\| + \|\tilde{x}^*\|] + 2\varepsilon^2\,.\end{aligned}$$

Combining the two above equations with (16) we obtain

$$g_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) + \frac{1}{2}\|\bar{x}\|^2 + \frac{1}{2}\|\bar{x}^*\|^2 \leq \langle \tilde{x},\tilde{x}^*\rangle + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + 2\varepsilon[\|\tilde{x}\| + \|\tilde{x}^*\|] + 2\varepsilon^2$$

Using now (17) and the second line of (14) we conclude that

$$g_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) + \frac{1}{2}\|\bar{x}\|^2 + \frac{1}{2}\|\bar{x}^*\|^2 \leq 2\varepsilon\, M\sqrt{2} + 3\varepsilon^2.$$

As ε is an arbitrary strictly positive number, using also (13) we conclude that

$$\inf_{(x,x^*)\in X\times X^*} g_{(x_0,x_0^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0.$$

Altogether, we conclude that if item *3.* holds then item *4.* holds. The converse item *4.* $\Rightarrow$ item *3.* is trivial to verify. Hence item *3.* and item *4.* are equivalent. As item *1.* is equivalent to *3.* and item *2.* is equivalent to *4.*, we conclude that items *1.*, *2.*, *3.* and *4.* are equivalent.

Now we will deal with item *5.* First suppose that item *2.* holds. Since, from (5), $\mathcal{S}_T \in \mathcal{F}_T$

$$(\mathcal{S}_T)^* \geq \pi_*.$$

As has already been observed, for any proper function h it holds that $(\operatorname{cl}\operatorname{conv} h)^* = h^*$. Therefore, using (4)

$$(\mathcal{S}_T)^* = (\pi + \delta_T)^* \geq \pi_*,$$

that is

$$\sup_{(y,y^*)\in T} \langle y, x^*\rangle + \langle y^*, x^{**}\rangle - \langle y, y^*\rangle \geq \langle x^*, x^{**}\rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}. \tag{18}$$

After some algebraic manipulations we conclude that (18) is equivalent to

$$\inf_{(y,y^*)\in T} \langle x^{**} - y, x^* - y^*\rangle \leq 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**},$$

that is, T is type (NI). If item *5.* holds, by the same reasoning we conclude that (18) holds and therefore $(\mathcal{S}_T)^* \geq \pi_*$. As $\mathcal{S}_T \in \mathcal{F}_T$, we conclude that item *5.* $\Rightarrow$ item *1.* As has been proved previously item *1.* $\Rightarrow$ item *2.* □

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