On the Surjectivity Properties of
Perturbations of Maximal Monotone Operators
in Non-Reflexive Banach Spaces

M. Marques Alves
IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil
maicon@impa.br

B. F. Svaiter
IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil
benar@impa.br

Received: July 27, 2008

We are concerned with surjectivity of perturbations of maximal monotone operators in non-reflexive Banach spaces. While in a reflexive setting, a classical surjectivity result due to Rockafellar gives a necessary and sufficient condition to maximal monotonicity, in a non-reflexive space we characterize maximality using an “enlarged” version of the duality mapping, introduced previously by Gossez.

Keywords: Maximal monotone operators, Fitzpatrick functions, duality mapping, non-reflexive Banach spaces

2000 Mathematics Subject Classification: 47H05, 47H14, 49J52, 47N10

1. Introduction

Let $X$ be a real Banach space and $X^*$ its topological dual. We use the notation $\pi$ and $\pi_*$ for the duality product in $X \times X^*$ and in $X^* \times X^{**}$, respectively:

$$\pi : X \times X^* \to \mathbb{R}, \quad \pi_* : X^* \times X^{**} \to \mathbb{R}$$

$$\pi(x, x^*) = \langle x, x^* \rangle, \quad \pi_*(x^*, x^{**}) = \langle x^*, x^{**} \rangle. \quad (1)$$

The norms on $X$, $X^*$ and $X^{**}$ will be denoted by $\| \cdot \|$. We also use the notation $\bar{\mathbb{R}}$ for the extended real numbers:

$$\bar{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

Whenever necessary, we will identify $X$ with its image under the canonical injection of $X$ into $X^{**}$.

A point to set operator $T : X \rightrightarrows X^*$ is a relation on $X \times X^*$:

$$T \subset X \times X^*$$

*Partially supported by Brazilian CNPq scholarship 140525/2005-0.
†Partially supported by CNPq grants 300755/2005-8, 475647/2006-8 and by PRONEX-Optimization.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag
and $T(x) = \{x^* \in X^* \mid (x, x^*) \in T\}$. An operator $T : X \rightrightarrows X^*$ is \textit{monotone} if
\[
\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T
\]
and it is \textit{maximal monotone} if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of $X$ into $X^*$. The conjugate of $f$ is $f^* : X^* \to \mathbb{R}$,
\[
f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x).
\]
Note that $f^*$ is always convex and lower semicontinuous.

The \textit{subdifferential} of $f$ is the point to set operator $\partial f : X \rightrightarrows X^*$ defined at $x \in X$ by
\[
\partial f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle, \forall y \in X\}.
\]
For each $x \in X$, the elements $x^* \in \partial f(x)$ are called \textit{subgradients} of $f$. The concept of $\varepsilon$-\textit{subdifferential} of a convex function $f$ was introduced by Brøndsted and Rockafellar [4]. It is a point to set operator $\partial_\varepsilon f : X \rightrightarrows X^*$ defined at each $x \in X$ as
\[
\partial_\varepsilon f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle - \varepsilon, \forall y \in X\},
\]
where $\varepsilon \geq 0$. Note that $\partial f = \partial_0 f$ and $\partial f(x) \subseteq \partial_\varepsilon f(x)$, for all $\varepsilon \geq 0$.

A convex function $f : X \to \mathbb{R}$ is said to be proper if $f > -\infty$ and there exists a point $\hat{x} \in X$ for which $f(\hat{x}) < \infty$. Rockafellar proved that if $f$ is proper, convex and lower semicontinuous, then $\partial f$ is maximal monotone on $X$ [18]. If $f : X \to \mathbb{R}$ is proper, convex and lower semicontinuous, then $f^*$ is proper and $f$ satisfies the \textit{Fenchel-Young inequality}: for all $x \in X$, $x^* \in X^*$,
\[
f(x) + f^*(x^*) \geq \langle x, x^* \rangle, \quad f(x) + f^*(x^*) = \langle x, x^* \rangle \iff x^* \in \partial f(x).
\]
Moreover, in this case, $\partial_\varepsilon f$ (and $\partial f = \partial_0 f$) may be characterized using $f^*$:
\[
\partial f(x) = \{x^* \in X^* \mid f(x) + f^*(x^*) = \langle x, x^* \rangle\},
\]
\[
\partial_\varepsilon f(x) = \{x^* \in X^* \mid f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \varepsilon\}. \quad (3)
\]

The subdifferential and the $\varepsilon$-subdifferential of the function $\frac{1}{2} \| \cdot \|^2$ will be of special interest in this paper, and will be denoted by $J : X \rightrightarrows X^*$ and $J_\varepsilon : X \rightrightarrows X^*$ respectively
\[
J(x) = \partial \frac{1}{2} \| x \|^2, \quad J_\varepsilon(x) = \partial_\varepsilon \frac{1}{2} \| x \|^2.
\]
Using $f(x) = (1/2) \| x \|^2$ in (3), it is trivial to verify that
\[
J(x) = \left\{x^* \in X^* \mid \frac{1}{2} \| x \|^2 + \frac{1}{2} \| x^* \|^2 = \langle x, x^* \rangle\right\}
\]
\[
= \left\{x^* \in X^* \mid \| x \|^2 = \| x^* \|^2 = \langle x, x^* \rangle\right\}
\]
and
\[
J_\varepsilon(x) = \left\{x^* \in X^* \mid \frac{1}{2} \| x \|^2 + \frac{1}{2} \| x^* \|^2 \leq \langle x, x^* \rangle + \varepsilon\right\}.
\]
The operator $J$ is widely used in Convex Analysis in Banach spaces and it is called the duality mapping of $X$. The operator $J$ was introduced by Gossez [11] to generalize some results concerning maximal monotonicity in reflexive Banach spaces to non-reflexive Banach spaces. It was also used in [10] to the study of locally maximal monotone operators in non-reflexive Banach spaces.

If $X$ is a real reflexive Banach space and $T : X \rightrightarrows X^*$ is monotone, then $T$ is maximal monotone if and only if

$$ R(T(\cdot + z_0) + J) = X^*, \quad \forall z_0 \in X. $$

We shall prove a similar result for a class of maximal monotone operators in non-reflexive Banach spaces.

2. Basic definitions and theory

In this section we present the tools and results which will be used to prove the main results of this paper.

For $f : X \rightarrow \mathbb{R}$, convex $f : X \rightarrow \mathbb{R}$ is the largest convex function majorized by $f$, and cl $f : X \rightarrow \mathbb{R}$ is the largest lower semicontinuous function majorized by $f$. It is trivial to verify that

$$ \text{cl} f(x) = \liminf_{y \rightarrow x} f(y), \quad f^* = (\text{conv} f)^* = (\text{cl conv} f)^*. $$

The functions cl $f$ and cl conv $f$ are usually called the (lower semicontinuous) closure of $f$ and the convex lower semicontinuous closure of $f$, respectively.

Fitzpatrick proved constructively that maximal monotone operators are representable by convex functions. Let $T : X \rightrightarrows X^*$ be maximal monotone. The Fitzpatrick function of $T$ [9] is $\varphi_T : X \times X^* \rightarrow \mathbb{R}$

$$ \varphi_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \tag{4} $$

and Fitzpatrick family associated with $T$ is

$$ \mathcal{F}_T = \left\{ h \in \mathbb{R}^{X \times X^*} \middle| h \text{ is convex and lower semicontinuous} \right\} \left\{ \begin{array}{l} (x, x^*) \leq h(x, x^*), \quad \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right\}. \tag{5} $$

Theorem 2.1 ([9, Theorem 3.10]). Let $X$ be a real Banach space and $T : X \rightrightarrows X^*$ be maximal monotone. Then for any $h \in \mathcal{F}_T$ (5)

$$ (x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^* $$

and $\varphi_T$ (4) is the smallest element of the family $\mathcal{F}_T$.

Fitzpatrick’s results described above were rediscovered by Martínez-Legaz and Théra [15], and Burachik and Svaiter [7]. Since then, this area has been the subject of intense research.
The **indicator function** of \( A \subset X \) is \( \delta_A : X \to \bar{\mathbb{R}} \),

\[
\delta_A(x) := \begin{cases} 
0, & x \in A \\
\infty, & \text{otherwise.}
\end{cases}
\]

Using the indicator function we have another expression for Fitzpatrick function:

\[
\varphi_T(x, x^*) = (\pi + \delta_T^*) (x^*, x).
\]

The supremum of Fitzpatrick family is the \( S \)-function, defined and studied by Burachik and Svaiter in [7], \( S_T : X \times X^* \to \bar{\mathbb{R}} \)

\[
S_T(x, x^*) = \sup \left\{ h(x, x^*) \mid h : X \times X^* \to \bar{\mathbb{R}} \text{ convex lower semicontinuous} \right. \\
h(x, x^*) \leq \langle x, x^* \rangle, \forall (x, x^*) \in T \right\}
\]

or, equivalently (see [7, Eq. (35)], [6, Eq. 29])

\[
S_T = \text{cl conv}(\pi + \delta_T).
\]

Some authors [2, 21, 3] attribute the \( S \)-function to [16] although this work was submitted after the publication of [7]. Moreover, the content of [7], and specifically the \( S \) function, was presented on Erice workshop on July 2001, by R. S. Burachik [5]. A list of the talks of this congress, which includes [17], is available on the www1.http://www.polyu.edu.hk/∼ama/events/conference/EriceItaly-OCA2001/Abstract.html.

Burachik and Svaiter also proved that the family \( \mathcal{F}_T \) is invariant under the mapping

\[
\mathcal{J} : \mathbb{R}^{X \times X^*} \to \mathbb{R}^{X \times X^*}, \quad \mathcal{J} h(x, x^*) = h^*(x^*, x).
\]

If \( T : X \rightrightarrows X^* \) is maximal monotone, then [7]

\[
\mathcal{J}(\mathcal{F}_T) \subset \mathcal{F}_T, \quad \mathcal{J} S_T = \varphi_T.
\]

In particular, for any \( h \in \mathcal{F}_T \),

\[
h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \forall (x, x^*) \in X \times X^*.
\]

A partial converse of this fact was proved in [8]: in a reflexive Banach space, if \( h \) is convex, lower semicontinuous and satisfy (8) then

\[
T := \{(x, x^*) \mid h(x, x^*) = \langle x, x^* \rangle \}
\]

is maximal monotone and \( h \in \mathcal{F}_T \) [8]. In order to extend this result to non-reflexive Banach spaces, Marques Alves and Svaiter considered an extension of condition (8) to non-reflexive Banach spaces:

\[
h(x, x^*) \geq \langle x, x^* \rangle, \forall (x, x^*) \in X \times X^*,

h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \forall (x^*, x^{**}) \in X^* \times X^{**}.
\]

We shall prefer the synthetic notation \( h \geq \pi, h^* \geq \pi_* \) for the above condition. The following result will be fundamental in our analysis

\[1\text{http://www.polyu.edu.hk/∼ama/events/conference/EriceItaly-OCA2001/Abstract.html} \]
Theorem 2.2 ([12, Theorem 3.4]). Let \( h : X \times X^* \to \mathbb{R} \) be a convex and lower semicontinuous function. If
\[
h \geq \pi, \quad h^* \geq \pi^* \quad \text{and} \quad h(x, x^*) < \langle x, x^* \rangle + \epsilon, \ 
then \text{for any } \lambda > 0 \text{ there exists } x_\lambda, x^*_\lambda \text{ such that} \\
h(x_\lambda, x^*_\lambda) = \langle x_\lambda, x^*_\lambda \rangle, \quad \|x_\lambda - x\| < \lambda, \quad \|x^*_\lambda - x^*\| < \epsilon/\lambda.
\]
Using Theorem 2.2, the authors proved [12] that condition (9) ensures that \( h \) represents a maximal monotone operator. Here we will be interested also in the case where the lower semicontinuity assumption is removed.

Theorem 2.3 ([12, Theorem 4.2, Corollary 4.4]). Let \( h : X \times X^* \to \bar{\mathbb{R}} \) be a convex function. If \( h \geq \pi, \ h^* \geq \pi^* \)
then
\[
T = \{(x, x^*) \in X \times X^* \mid h(x^*, x) = \langle x, x^* \rangle\}
\]
is maximal monotone and satisfy the restricted Brøndsted-Rockafellar property. Additionally, if \( h \) is also lower semicontinuous, then
\[
T = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}.
\]
We will need the following immediate consequence of the above theorem:

Corollary 2.4. Let \( h : X \times X^* \to \bar{\mathbb{R}} \). If
\[
\text{conv } h \geq \pi, \quad h^* \geq \pi^*
\]
then
\[
T = \{(x, x^*) \in X \times X^* \mid h(x^*, x) = \langle x, x^* \rangle\}
\]
\[
= \{(x, x^*) \in X \times X^* \mid \text{cl conv } h(x, x^*) = \langle x, x^* \rangle\}
\]
is maximal monotone,
\[
T = \{(x, x^*) \in X \times X^* \mid \text{cl conv } h(x, x^*) = \langle x, x^* \rangle\}
\]
\[
\text{cl conv } h \in \mathcal{F}_T \text{ and } \partial h \in \mathcal{F}_T, \ 
where \partial h(x, x^*) = h^*(x^*, x).
\]

**Proof.** As the duality product is continuous in \( X \times X^* \), \( \text{cl conv } h \geq \pi \). As conjugation is invariant under the conv operation and the (lower semicontinuous) closure, \( (\text{cl conv } h)^* = h^* \geq \pi^* \). To end the proof, apply Theorem 2.3 to \( \text{cl conv } h \), observe that \( \partial h \) is convex, lower semicontinuous, \( \partial h \geq \pi \) and use definition (5).

In a non-reflexive Banach Space \( X \), if \( T : X \rightrightarrows X^* \) is maximal monotone and for some \( h \in \mathcal{F}_T \) it holds that \( h \geq \pi, \ h^* \geq \pi^* \), then \( T \) behaves similarly to a maximal monotone operator in a reflexive Banach space. A natural question is: what is the class of maximal monotone operators (in non-reflexive Banach spaces) which have some function in Fitzpatrick family satisfying (9)? To answer this question, first let us recall the definition of maximal monotone operators of type NI [20].
Definition 2.5. A maximal monotone operator $T : X \rightrightarrows X^*$ is of type NI if
\[
\inf_{(y, y^*) \in T} \langle y^* - x^*, x^{**} - y \rangle \leq 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.
\]

In [22] it was observed that if $T$ is a maximal monotone operators of type NI, then $S_T$ satisfies condition (9).

Martínez-Legaz and Svaiter defined [14] (with a different notation), for $h : X \times X^* \to \bar{\mathbb{R}}$ and $(x_0, x^*_0) \in X \times X^*$
\[
h_{(x_0, x^*_0)} : X \times X^* \to \bar{\mathbb{R}},
\]
\[
h_{(x_0, x^*_0)}(x, x^*) := h(x + x_0, x^* + x^*_0) - [\langle x, x^*_0 \rangle + \langle x_0, x^* \rangle + \langle x_0, x^*_0 \rangle]. \tag{10}
\]

The operation $h \mapsto h_{(x_0, x^*_0)}$ preserves many properties of $h$, as convexity, lower semi-continuity and can be seen as the action of the group $(X \times X^*, +)$ on $\bar{\mathbb{R}}^{X \times X^*}$, because
\[
(h_{(x_0, x^*_0)})(x_1, x^*_1) = h_{(x_0 + x_1, x^*_0 + x^*_1)}.
\]

Moreover
\[
(h_{(x_0, x^*_0)})^* = (h^*)_{(x^*_0, x_0)},
\]
where the rightmost $x_0$ is identified with its image under the canonical injection of $X$ into $X^{**}$. Therefore,

1. $h \geq \pi \iff h_{(x_0, x^*_0)} \geq \pi$,
2. $(h_{(x_0, x^*_0)})^* \geq \pi^* \iff (h^*)_{(x^*_0, x_0)} \geq \pi^*$.

We shall need the following theorem. Its proof is heavily based on these nice properties of the map $h \mapsto h_{(x_0, x^*_0)}$ and it is presented on the Appendix A.

Theorem 2.6 ([13, Theorem 1.2]). Let $T : X \rightrightarrows X^*$ be maximal monotone. The following conditions are equivalent

1. $T$ is of type NI,
2. there exists $h \in F_T$ such that $h \geq \pi$ and $h^* \geq \pi^*$,
3. for all $h \in F_T$, $h \geq \pi$ and $h^* \geq \pi^*$,
4. there exists $h \in F_T$ such that
\[
\inf h_{(x_0, x^*_0)} + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 = 0, \quad \forall (x_0, x^*_0) \in X \times X^*,
\]
5. for all $h \in F_T$,
\[
\inf h_{(x_0, x^*_0)} + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 = 0, \quad \forall (x_0, x^*_0) \in X \times X^*.
\]

3. Surjectivity and maximal monotonicity in non-reflexive Banach spaces

We begin with two elementary technical results which will be useful.

Proposition 3.1. The following statements holds:
1. For any \( \varepsilon \geq 0 \), if \( y^* \in J_\varepsilon(x) \), then \( |\|x\| - \|y^*\|| \leq \sqrt{2\varepsilon} \).

2. Let \( T : X \rightrightarrows X^* \) be a monotone operator and \( \varepsilon, M > 0 \). Then,

\[
(T + J_\varepsilon)^{-1}(B_{X^*}[0,M])
\]

is bounded.

**Proof.** To prove item 1., let \( \varepsilon \geq 0 \) and \( y^* \in J_\varepsilon(x) \). The desired result follows from the following inequalities:

\[
\frac{1}{2}(|\|x\| - \|y^*\|)^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y^*\|^2 - \langle x, y^* \rangle \leq \varepsilon.
\]

To prove item 2., take \((z, z^*) \in T\). If \( x \in (T + J_\varepsilon)^{-1}(B_{[0,M]}) \) then there exists \( x^*, y^* \) such that

\[
x^* \in T(x), \quad y^* \in J_\varepsilon(x), \quad \|x^* + y^*\| \leq M.
\]

Therefore, using Fenchel Young inequality (2), the monotonicity of \( T \) and the definition of \( J_\varepsilon \) we obtain

\[
\frac{1}{2}\|x - z\|^2 + \frac{1}{2}\|x^* + y^* - z^*\|^2 \geq \langle x - z, x^* + y^* - z^* \rangle \geq \langle x - z, y^* \rangle \geq \left[ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y^*\|^2 - \varepsilon \right] - \|z\||y^*||.
\]

Note also that

\[
\|x - z\|^2 \leq \|x\|^2 + 2\|x\||z|| + \|z||^2, \quad \|x^* + y^* - z^*\|^2 \leq (M + ||z^*||)^2.
\]

Combining the above equations we obtain

\[
\frac{1}{2}\|z\|^2 + \frac{1}{2}(M + ||z^*||)^2 \geq \frac{1}{2}\|y^*\|^2 - \|x\||z|| - ||z||\|y^*|| - \varepsilon.
\]

As \( y^* \in J_\varepsilon(x) \), by item 1., we have \( \|x\| \leq \|y^*\| + \sqrt{2\varepsilon} \). Therefore

\[
\frac{1}{2}\|z\|^2 + \frac{1}{2}(M + ||z^*||)^2 \geq \frac{1}{2}\|y^*\|^2 - 2\|y^*\||z|| - ||z||\sqrt{2\varepsilon} - \varepsilon.
\]

Hence, \( y^* \) is bounded. In fact,

\[
\|y^*\| \leq \|z\| + \sqrt{4\|z\|^2 + 2\left[\|z\|\sqrt{2\varepsilon} + \varepsilon\right] + \|z\|^2 + (M + ||z^*||)^2}.
\]

As we already observed, \( \|x\| \leq \|y^*\| + \sqrt{2\varepsilon} \) and so, \( x \) is also bounded.

Now we will prove that under monotonicity, dense range of some perturbation of a monotone operator is equivalent to surjectivity of that perturbation.
Lemma 3.2. Let $T : X \Rightarrow X^*$ be monotone and $\mu > 0$. Then the conditions below are equivalent

1. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$, for any $\varepsilon > 0$ and $z_0 \in X$,
2. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$, for any $\varepsilon > 0$ and $z_0 \in X$.

Proof. It suffices to prove the lemma for $\mu = 1$ and then, for the general case, consider $T' = \mu^{-1}T$. Now note that for any $z_0 \in X$ and $z_0^* \in X^*$, $T - \{(z_0, z_0^*)\}$ is also monotone. Therefore, it suffices to prove that $0 \in R(T + J_\varepsilon)$, for any $\varepsilon > 0$ if and only if $0 \in R(T + J_\varepsilon)$, for any $\varepsilon > 0$. The "if" is easy to check. To prove the "only if", suppose that

$$0 \in R(T + J_\varepsilon), \ \forall \varepsilon > 0.$$

First use item 2. of Proposition 3.1 with $M = 1/2$ to conclude that there exists $\rho > 0$ such that

$$(T + J_{1/2})^{-1}(B_X \cdot [0, 1/2]) \subset B_X [0, \rho].$$

By assumption, for any $0 < \eta < \frac{1}{2}$ there exists $x_\eta \in X$, $x_\eta^*, y_\eta^* \in X^*$ such that

$$x_\eta^* \in T(x_\eta), \ y_\eta^* \in J_\eta(x_\eta) \text{ and } \|x_\eta^* + y_\eta^*\| < \eta < \frac{1}{2}. \quad (11)$$

As $J_\eta(x_\eta) \subset J_{1/2}(x_\eta)$, $x_\eta \in (T + J_{1/2})^{-1}(x_\eta^* + y_\eta^*)$ and so,

$$\|x_\eta\| \leq \rho, \quad \|y_\eta^*\| \leq \rho + 1,$$

where the second inequality follows from the first one and item 1. of Proposition 3.1. Therefore

$$\frac{1}{2}\|x_\eta^*\|^2 \leq \frac{1}{2} \left(\|x_\eta^* + y_\eta^*\| + \|y_\eta^*\|\right)^2 \leq \frac{1}{2} \eta^2 + \eta(\rho + 1) + \frac{1}{2}\|y_\eta^*\|^2,$$

$$\langle x_\eta, x_\eta^* \rangle = \langle x_\eta, x_\eta^* + y_\eta^* \rangle - \langle x_\eta, y_\eta^* \rangle \leq \rho \eta - \langle x_\eta, y_\eta^* \rangle.$$

Combining the above inequalities we obtain

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \langle x_\eta, x_\eta^* \rangle \leq \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|y_\eta^*\|^2 - \langle x_\eta, y_\eta^* \rangle + \eta(2\rho + 1) + \frac{1}{2}\eta^2.$$

The inclusion $y_\eta^* \in J_\eta(x_\eta)$ means that,

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|y_\eta^*\|^2 - \langle x_\eta, y_\eta^* \rangle \leq \eta. \quad (12)$$

Hence, using the two above inequalities we conclude that

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \langle x_\eta, x_\eta^* \rangle \leq 2\eta(\rho + 1) + \frac{1}{2}\eta^2.$$

To end the prove, take an arbitrary $\varepsilon > 0$. Choosing $0 < \eta < 1/2$ such that,

$$2\eta(\rho + 1) + \frac{1}{2}\eta^2 < \varepsilon,$$
we have
\[ \frac{1}{2} \| x_\eta \|^2 + \frac{1}{2} \| x^*_\eta \|^2 + \langle x_\eta, x^*_\eta \rangle < \varepsilon, \ x^*_\eta \in T(x_\eta). \]

According to the above inequality, \(-x^*_\eta \in J_\varepsilon(x_\eta)\). Hence \(0 \in (T + J_\varepsilon)(x_\eta)\).

In a reflexive Banach space, surjectivity of a monotone operator plus the duality mapping is equivalent to maximal monotonicity. This is a classical result of Rockafellar [19]. To obtain a partial extension of this result to non-reflexive Banach spaces, we must consider the “enlarged” duality mapping.

**Lemma 3.3.** Let \( T : X \Rightarrow X^* \) be monotone and \( \mu > 0 \). If
\[ R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*, \ \forall \varepsilon > 0, z_0 \in X \]
then \( \overline{T}, \) the closure of \( T \) in the norm-topology of \( X \times X^* \), is maximal monotone and of type NI.

**Proof.** Note that \( T + \mu J_\varepsilon = \mu(\mu^{-1}T + J_\varepsilon) \). Therefore, it suffices to prove the lemma for \( \mu = 1 \) and then, for the general case, consider \( T' = \mu^{-1}T \). The monotonicity of \( \overline{T} \) follows from the continuity of the duality product.

Using the assumptions on \( T \) and Lemma 3.2 we conclude that \( T(\cdot + z_0) + J_\varepsilon \) is onto, for any \( \varepsilon > 0 \) and \( z_0 \in X \). Therefore, for any \((z_0, z^*_0) \in X \times X^* \) and \( \varepsilon > 0 \), there exists \( x_\varepsilon, x^*_\varepsilon \) such that
\[ x^*_\varepsilon + z^*_0 \in T(x_\varepsilon + z_0) \quad \text{and} \quad -x^*_\varepsilon \in J_\varepsilon(x_\varepsilon). \quad (13) \]

Note that the second inclusion in the above equation is equivalent to
\[ \frac{1}{2} \| x_\varepsilon \|^2 + \frac{1}{2} \| x^*_\varepsilon \|^2 \leq \langle x_\varepsilon, -x^*_\varepsilon \rangle + \varepsilon. \quad (14) \]

To prove maximal monotonicity of \( \overline{T} \), suppose that \((z_0, z^*_0) \in X \times X^* \) is monotonically related to \( \overline{T} \). As \( T \subset \overline{T} \)
\[ \langle z - z_0, z^* - z^*_0 \rangle \geq 0, \ \forall (z, z^*) \in T. \]

So, taking \( \varepsilon > 0 \) and \( x_\varepsilon \in X, x^*_\varepsilon \in X^* \) as in (13) we conclude that
\[ \langle x_\varepsilon, x^*_\varepsilon \rangle = \langle x_\varepsilon + z_0 - z_0, x^*_\varepsilon + z^*_0 - z^*_0 \rangle \geq 0, \]
which, combined with (14) yields
\[ \frac{1}{2} \| x_\varepsilon \|^2 + \frac{1}{2} \| x^*_\varepsilon \|^2 \leq \varepsilon. \]

As \((x_\varepsilon + z_0, x^*_\varepsilon + z^*_0) \in T, \) and \( \varepsilon \) is an arbitrary strictly positive number, we conclude that \((z_0, z^*_0) \in \overline{T}, \) and \( \overline{T} \) is maximal monotone.
It remains to prove that $\bar{T}$ is of type NI. Consider an arbitrary $(z_0, z_0^*) \in X \times X^*$ and $h \in \mathcal{F}_T$. Then, using (13), (14) we conclude that for any $\varepsilon > 0$, there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$h(x_\varepsilon + z_0, x_\varepsilon^* + z_0^*) = \langle x_\varepsilon + z_0, x_\varepsilon^* + z_0^* \rangle, \quad \frac{1}{2} \|x_\varepsilon\|^2 + \frac{1}{2} \|x_\varepsilon^*\|^2 \leq \langle x_\varepsilon, -x_\varepsilon^* \rangle + \varepsilon.$$

The first equality above is equivalent to $h_{(z_0, z_0^*)}(x_\varepsilon, x_\varepsilon^*) = \langle x_\varepsilon, x_\varepsilon^* \rangle$. Therefore,

$$h_{(z_0, z_0^*)}(x_\varepsilon, x_\varepsilon^*) + \frac{1}{2} \|x_\varepsilon\|^2 + \frac{1}{2} \|x_\varepsilon^*\|^2 < \varepsilon,$$

that is,

$$\inf h_{(z_0, z_0^*)}(x, x^*) + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 = 0.$$

Now, use item 5. of Theorem 2.6 to conclude that $\bar{T}$ is of type NI.

Direct application of Lemma 3.3 gives the next corollary.

**Corollary 3.4.** If $T : X \rightrightarrows X^*$ is monotone, closed, $\mu > 0$ and $\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*$, $\forall \varepsilon > 0, z_0 \in X$ then $T$, is maximal monotone and of type NI.

**Proof.** Use Lemma 3.3 and the assumption $T = \bar{T}$.

**Lemma 3.5.** Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone and of type NI. Take

$$h_1 \in \mathcal{F}_{T_1}, \quad h_2 \in \mathcal{F}_{T_2}$$

and define

$$h : X \times X^* \to \bar{\mathbb{R}},$$

$$h(x, x^*) = (h_1(x, \cdot) \square h_2(x, \cdot))(x^*) = \inf_{y^* \in X^*} h_1(x, y^*) + h_2(x, x^* - y^*),$$

$$D_X(h_i) = \{x \in X \mid \exists x^*, h_i(x, x^*) < \infty\}, \quad i = 1, 2.$$  

If

$$\bigcup_{\lambda > 0} \lambda(D_X(h_1) - D_X(h_2))$$

is a closed subspace then

$$h \geq \pi, h^* \geq \pi^*, \quad \forall h \geq \pi, (\lambda h)^* \geq \pi^*,$$

$$T_1 + T_2 = \{(x, x^*) \mid \lambda h(x, x^*) = \langle x, x^* \rangle\}$$

$$= \{(x, x^*) \mid h(x, x^*) = \langle x, x^* \rangle\}$$

and $T_1 + T_2$ is maximal monotone of type NI and

$$\lambda h, \text{cl } h \in \mathcal{F}_{T_1 + T_2}.$$
Proof. Since $h_1 \in \mathcal{F}_{T_1}$ and $h_2 \in \mathcal{F}_{T_2}$, we have $h_1 \geq \pi$ and $h_2 \geq \pi$. So

$$h_1(x, y^*) + h_2(x, x^* - y^*) \geq \langle x, y^* \rangle + \langle x, x^* - y^* \rangle = \langle x, x^* \rangle.$$  

Taking the inf in $y^*$ at the left-hand side of the above inequality we conclude that $h \geq \pi$.

Let $(x^*, x^{**}) \in X^* \times X^{**}$. Using the definition of $h$ we have

$$h^*(x^*, x^{**}) = \sup_{(z, z^*) \in X \times X^*} \langle z, x^* \rangle + \langle z^*, x^{**} \rangle - h(z, z^*) \tag{16}$$

$$= \sup_{(z, z^*, y^*) \in X \times X^* \times X^{**}} \langle z, x^* \rangle + \langle z^*, x^{**} \rangle - h_1(z, y^*) - h_2(z, z^* - y^*) \tag{17}$$

$$= \sup_{(z, y^*, w^*) \in X \times X^* \times X^{**}} \langle z, x^* \rangle + \langle y^*, x^{**} \rangle + \langle w^*, x^{**} \rangle - h_1(z, y^*) - h_2(z, w^*) \tag{18}$$

where we used the substitution $z^* = w^* + y^*$ in the last term. So, defining $H_1, H_2 : X \times X^* \times X^* \to \mathbb{R}$

$$H_1(x, y^*, z^*) = h_1(x, y^*), \quad H_2(x, y^*, z^*) = h_2(x, z^*), \tag{19}$$

we have

$$h^*(x^*, x^{**}) = (H_1 + H_2)^*(x^*, x^{**}).$$

Using (15), the Attouch-Brezis extension [1, Theorem 1.1] of Fenchel-Rockafellar duality theorem and (19) we conclude that the conjugate of the sum at the right hand side of the above equation is the exact inf-convolution of the conjugates. Therefore,

$$h^*(x^*, x^{**}) = \min_{(u^*, y^{**}, z^{**})} H_1^*(u^*, y^{**}, z^{**}) + H_2^*(x^* - u^*, x^{**} - y^{**}, x^{**} - z^{**}).$$

Direct use of definition (19) yields

$$H_1^*(u^*, y^{**}, z^{**}) = h_1^*(u^*, y^{**}) + \delta_0(z^{**}), \quad \forall (u^*, y^{**}, z^{**}) \in X^* \times X^{**} \times X^{**}, \tag{20}$$

$$H_2^*(u^*, y^{**}, z^{**}) = h_2^*(u^*, z^{**}) + \delta_0(y^{**}), \quad \forall (u^*, y^{**}, z^{**}) \in X^* \times X^{**} \times X^{**}. \tag{21}$$

Hence,

$$h^*(x^*, x^{**}) = \min_{u^* \in X^*} h_1^*(u^*, x^{**}) + h_2^*(x^* - u^*, x^{**}). \tag{22}$$

Therefore, using that $h_1^* \geq \pi_*, h_2^* \geq \pi_*$, (22) and the same reasoning used to show that $h \geq \pi$ we have

$$h^* \geq \pi^*.$$

Up to now, we proved that $h \geq \pi$ and $h^* \geq \pi_*(\text{ and } \partial h \geq \pi)$. So, using Theorem 2.3 we conclude that $S : X \rightrightarrows X^*$, defined as

$$S = \{(x, x^*) \in X \times X^* \mid \partial h(x, x^*) = \langle x, x^* \rangle\},$$

is maximal monotone. As $\partial h$ is convex and lower semicontinuous, $\partial h \in \mathcal{F}_S$.  

We will prove that $T_1 + T_2 = S$. Take $(x, x^*) \in S$, that is, $\langle h(x, x^*) \rangle = \langle x, x^* \rangle$. Using (22) we conclude that there exists $u^* \in X^*$ such that
\[
h_1^*(u^*, x) + h_2^*(x^* - u^*, x) = \langle x, x^* \rangle.
\]
We know that
\[
h_1^*(u^*, x) \geq \langle x, u^* \rangle, \quad h_2^*(x^* - u^*, x) \geq \langle x, x^* - u^* \rangle.
\]
Combining these inequalities with the previous equation we conclude that these inequalities hold as equalities, and so
\[
u^* \in T_1(x), \quad x^* - u^* \in T_2(x), \quad x^* \in (T_1 + T_2)(x).
\]
We proved that $S \subset T_1 + T_2$. Since $T_1 + T_2$ is monotone and $S$ is maximal monotone, we have $T_1 + T_2 = S$ (and $\partial h \in \mathcal{F}_{T_1+T_2}$). Note also that $h(x, x^*) \leq \langle x, x^* \rangle$ for any $(x, x^*) \in T_1 + T_2 = S$. As $h \geq \pi$, we have equality in $T_1 + T_2$. Therefore,
\[
T_1 + T_2 \subset \{(x, x^*) \mid h(x, x^*) = \langle x, x^* \rangle\} \subset \{(x, x^*) \mid \text{cl} h(x, x^*) \leq \langle x, x^* \rangle\}.
\]
Since $h \geq \pi$ and the duality product $\pi$ is continuous in $X \times X^*$, we also have $\text{cl} h \geq \pi$. Hence, using the above inclusion we conclude that $\text{cl} h$ coincides with $\pi$ in $T_1 + T_2$. Therefore, $\text{cl} h \in \mathcal{F}_{T_1+T_2}$ and the rightmost set in the above inclusions is $T_1 + T_2$. Hence
\[
T_1 + T_2 = \{(x, x^*) \mid h(x, x^*) = \langle x, x^* \rangle\}.
\]
Conjugation is invariant under the (lower semicontinuous) closure operation. Therefore,
\[
(\text{cl} h)^* = h^* \geq \pi^*_s
\]
and so $T_1 + T_2$ is NI. We proved already that $\partial h \in \mathcal{F}_{T_1+T_2}$. Using item 3. of Theorem 2.6 we conclude that $(\partial h)^* \geq \pi^*_s$. \qed

**Theorem 3.6.** If $T : X \rightrightarrows X^*$ is a closed monotone operator, then the conditions below are equivalent

1. $R(T(\cdot + z_0) + J) = X^*$ for all $z_0 \in X$,
2. $R(T(\cdot + z_0) + J_\varepsilon) = X^*$ for all $\varepsilon > 0$, $z_0 \in X$,
3. $R(T(\cdot + z_0) + J_\varepsilon) = X^*$ for all $\varepsilon > 0$, $z_0 \in X$,
4. $T$ is maximal monotone and of type NI.

**Proof.** Item 1. trivially implies item 2.. Using Lemma 3.2 we conclude that, in particular, item 2. implies item 3.. Now use Corollary 3.4 to conclude that item 3. implies item 4.. Up to now we have $1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 4.$.

To complete the proof we will show that item 4. implies item 1.. So, assume that item 4. holds, that is, $T$ is of type NI. Take $z^*_0 \in X^*$ and $z_0 \in X$. Define $T_0 = T - \{(z_0, z^*_0)\}$. Trivially
\[
z^*_0 \in R(T(\cdot + z_0) + J) \iff 0 \in R(T_0 + J).
\]
As the class NI is invariant under translations, in order to prove item 1., it is sufficient to prove that if $T$ is of type NI, then $0 \in R(T + J)$. Let $h \in \mathcal{F}_T$ and $\varepsilon > 0$. Define $p : X \times X^* \to \mathbb{R}$,

$$p(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2.$$  \hspace{1cm} (23)

Item 5. of Theorem 2.6 ensure us that there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$h(x_\varepsilon, x_\varepsilon^*) + p(x_\varepsilon, -x_\varepsilon^*) < \varepsilon^2.$$  \hspace{1cm} (24)

Direct calculations yields $p \geq \pi$ and $p^* \geq \pi_*$. We also know that $p \in \mathcal{F}_J$ and so $J$ is of type NI. Define $H : X \times X^* \to \mathbb{R}$,

$$H(x, x^*) = \inf_{y^* \in X^*} h(x, y^*) + p(x, x^* - y^*).$$

As $D(p) = X \times X^*$, we may apply Lemma 3.5 to conclude that $T + J$ is NI and $\text{cl} \ H \in \mathcal{F}_{T+J}$. Using (24) we have

$$H(x_\varepsilon, 0) \leq h(x_\varepsilon, x_\varepsilon^*) + p(x_\varepsilon, -x_\varepsilon^*) < \varepsilon^2.$$  

So, $\text{cl} \ H(x_\varepsilon, 0) \leq H(x_\varepsilon, 0) < (x_\varepsilon, 0) + \varepsilon^2$. Now use Theorem 2.2 to conclude that there exists $\bar{x}, \bar{x}^*$ such that

$$(\bar{x}, \bar{x}^*) \in T + J, \quad \|\bar{x} - x_\varepsilon\| < \varepsilon, \quad \|\bar{x}^* - 0\| < \varepsilon.$$  

So, $\bar{x}^* \in R(T + J)$ and $\|\bar{x}^*\| < \varepsilon$. As $\varepsilon > 0$ is arbitrary, $0$ is in the closure of $R(T + J)$. \hfill $\square$

**Corollary 3.7.** If $T : X \rightrightarrows X^*$ is a closed monotone operator then the conditions bellow are equivalent

a. $R(T(\cdot + z_0) + \mu J) = X^*$ for all $z_0 \in X$ and some $\mu > 0$,

b. $R(T(\cdot + z_0) + \mu J) = X^*$ for all $z_0 \in X$, $\mu > 0$,

c. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$ for all $\varepsilon > 0$, $z_0 \in X$ and some $\mu > 0$,

d. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$ for all $\varepsilon > 0$, $z_0 \in X$, $\mu > 0$,

e. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$ for all $\varepsilon > 0$, $z_0 \in X$, and some $\mu > 0$,

f. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$ for all $\varepsilon > 0$, $z_0 \in X$, $\mu > 0$,

g. $T$ is maximal monotone and of type NI.

**Proof.** Suppose that item a. holds. Define $T' = \mu^{-1}T$ and use Theorem 3.6 to conclude that $T'$ is maximal monotone and of type NI. Therefore, $T = \mu T'$ is maximal monotone and of type NI, which means that g. holds.

Now assume that item g. holds, that is, $T$ is maximal monotone and of type NI. Then, for all $\mu > 0$, $\mu^{-1}T$ is maximal monotone and of type NI, which implies item b..

As the implication $b. \Rightarrow a.$ is trivial, we conclude that items a., b., g. are equivalent. The same reasoning shows that items c., d., g. are equivalent and so on. \hfill $\square$
\textbf{A. Proof of Theorem 2.6}

\textbf{Proof.} First let us prove that item 2. and item 4. are equivalent. So, suppose item 2. holds and let \((x_0, x_0^*) \in X \times X^*\). Direct calculations yield

\[ h_{(x_0, x_0^*)} \geq \pi, \quad (h_{(x_0, x_0^*)})^* \geq \pi^* . \]

Using [12, Theorem 3.1, Eq. (12)] we conclude that condition item 4. holds. For proving that item 4. \(\Rightarrow\) item 2., first note that, for any \((z, z^*) \in X \times X^*\),

\[ h_{(z, z^*)}(0, 0) \geq \inf_{(x, x^*)} h_{(z, z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 . \]

Therefore, using item 4. we obtain

\[ h(z, z^*) - \langle z, z^* \rangle = h_{(z, z^*)}(0, 0) \geq 0 . \]

Since \((z, z^*)\) is an arbitrary element of \(X \times X^*\) we conclude that \(h \geq \pi\).

For proving that, \(h^* \geq \pi^*\), take some \((y^*, y^*) \in X^* \times X^{**}\). First, use Fenchel-Young inequality to conclude that for any \((x, x^*)\), \((z, z^*) \in X \times X^*\),

\[ h_{(z, z^*)}(x, x^*) \geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle - (h_{(z, z^*)})^*(y^* - z^*, y^{**} - z) . \]

As \((h_{(z, z^*)})^* = (h^*)_{(z^*, z)}\),

\[ (h_{(z, z^*)})^*(y^* - z^*, y^{**} - z) = h^*(y^*, y^{**}) - \langle z, y^* - z^* \rangle - \langle z^*, y^{**} - z \rangle - \langle z, z^* \rangle . \]

Combining the two above equations we obtain

\[ h_{(z, z^*)}(x, x^*) \]
\[ \geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}) . \]

Adding \((1/2)\|x\|^2 + (1/2)\|x^*\|^2\) in both sides of the above inequality we have

\[ h_{(z, z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \]
\[ \geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \]
\[ - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}) . \]

Note that

\[ \langle x, y^* - z^* \rangle + \frac{1}{2}\|x\|^2 \geq -\frac{1}{2}\|y^* - z^*\|^2 , \quad \langle x^*, y^{**} - z \rangle + \frac{1}{2}\|x^*\|^2 \geq -\frac{1}{2}\|y^{**} - z\|^2 . \]

Therefore, for any \((x, x^*), (z, z^*) \in X \times X^*\),

\[ h_{(z, z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \]
\[ \geq -\frac{1}{2}\|y^* - z^*\|^2 - \frac{1}{2}\|y^{**} - z\|^2 \]
\[ - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}) . \]
Using now the assumption we conclude that the infimum, for \((x, x^*) \in X \times X^*\), at the left hand side of the above inequality is 0. Therefore, taking the infimum on \((x, x^*) \in X \times X^*\) at the left hand side of the above inequality and rearranging the resulting inequality we have

\[
h^*(y^*, y^{**}) - \langle y^*, y^{**} \rangle \geq -\frac{1}{2} \|y^* - z^*\|^2 - \frac{1}{2} \|y^{**} - z\|^2 - \langle y^* - z^*, y^{**} - z \rangle.
\]

Note that

\[
\sup_{z^* \in X^*} -\langle y^* - z^*, y^{**} - z \rangle - \frac{1}{2} \|y^* - z^*\|^2 = \frac{1}{2} \|y^{**} - z\|^2.
\]

Hence, taking the sup in \(z^* \in X^*\) at the right hand side of the previous inequality we obtain

\[
h^*(y^*, y^{**}) - \langle y^*, y^{**} \rangle \geq 0
\]

and item 4. holds. Now, using that item 2. and item 4. are equivalent it is trivial to verify that item 3. and item 5. are equivalent.

The second step is to prove that item 4. and item 5. are equivalent. So, assume that item 4. holds, that is, for some \(h \in \mathcal{F}_T\),

\[
\inf_{(x, x^*) \in X \times X^*} h(x_0, x^*_0)(x, x^*) + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 = 0, \quad \forall (x_0, x^*_0) \in X \times X^*.
\]

Take \(g \in \mathcal{F}_T\), and \((x_0, x^*_0) \in X \times X^*\). First observe that, for any \((x, x^*) \in X \times X^*\),

\[
g(x_0, x^*_0)(x, x^*) \geq \langle x, x^* \rangle
\]

and

\[
g(x_0, x^*_0)(x, x^*) + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 \geq \langle x, x^* \rangle + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 \geq 0.
\]

Therefore,

\[
\inf_{(x, x^*) \in X \times X^*} g(x_0, x^*_0)(x, x^*) + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 \geq 0. \tag{25}
\]

As the square of the norm is coercive, there exist \(M > 0\) such that

\[
\left\{(x, x^*) \in X \times X^* \mid h(x_0, x^*_0)(x, x^*) + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 < 1\right\} \subset B_{X \times X^*}(0, M),
\]

where

\[
B_{X \times X^*}(0, M) = \left\{(x, x^*) \in X \times X^* \mid \sqrt{\|x\|^2 + \|x^*\|^2} < M\right\}.
\]

For any \(\varepsilon > 0\), there exists \((\tilde{x}, \tilde{x}^*)\) such that

\[
\min \{1, \varepsilon^2\} > h(x_0, x^*_0)(\tilde{x}, \tilde{x}^*) + \frac{1}{2} \|\tilde{x}\|^2 + \frac{1}{2} \|\tilde{x}^*\|^2.
\]

Therefore

\[
\varepsilon^2 > h(x_0, x^*_0)(\tilde{x}, \tilde{x}^*) + \frac{1}{2} \|\tilde{x}\|^2 + \frac{1}{2} \|\tilde{x}^*\|^2 \geq h(x_0, x^*_0)(\tilde{x}, \tilde{x}^*) - \langle \tilde{x}, \tilde{x}^* \rangle \geq 0,
\]

\[
M^2 \geq \|\tilde{x}\|^2 + \|\tilde{x}^*\|^2. \tag{26}
\]
In particular,
\[ \varepsilon^2 > h(x_0, x_0^*)(\bar{x}, \bar{x}^*) - \langle \bar{x}, \bar{x}^* \rangle. \]

Now using Theorem 2.2 we conclude that there exists \((\bar{x}, \bar{x}^*)\) such that
\[ h(x_0, x_0^*)(\bar{x}, \bar{x}^*) = \langle \bar{x}, \bar{x}^* \rangle, \quad \|\bar{x} - \bar{x}\| < \varepsilon, \quad \|\bar{x}^* - \bar{x}^*\| < \varepsilon. \quad (27) \]

Therefore,
\[ h(\bar{x} + x_0, \bar{x}^* + x_0^*) - \langle \bar{x} + x_0, \bar{x}^* + x_0^* \rangle = h(x_0, x_0^*)(\bar{x}, \bar{x}^*) - \langle \bar{x}, \bar{x}^* \rangle = 0, \]
and \((\bar{x} + x_0, \bar{x}^* + x_0^*) \in T\). As \(g \in \mathcal{F}_T\),
\[ g(\bar{x} + x_0, \bar{x}^* + x_0^*) = \langle \bar{x} + x_0, \bar{x}^* + x_0^* \rangle, \]
and
\[ g(x_0, x_0^*)(\bar{x}, \bar{x}^*) = \langle \bar{x}, \bar{x}^* \rangle. \quad (28) \]

Using the first line of (26) we have
\[ \varepsilon^2 > h(x_0, x_0^*)(\bar{x}, \bar{x}^*) + \left[ \frac{1}{2} \|\bar{x}\|^2 + \frac{1}{2} \|\bar{x}^*\|^2 + \langle \bar{x}, \bar{x}^* \rangle \right] - \langle \bar{x}, \bar{x}^* \rangle \geq \frac{1}{2} \|\bar{x}\|^2 + \frac{1}{2} \|\bar{x}^*\|^2 + \langle \bar{x}, \bar{x}^* \rangle. \]

Therefore,
\[ \varepsilon^2 > \frac{1}{2} \|\bar{x}\|^2 + \frac{1}{2} \|\bar{x}^*\|^2 + \langle \bar{x}, \bar{x}^* \rangle. \quad (29) \]

Direct use of (27) gives
\[ \langle \bar{x}, \bar{x}^* \rangle = \langle \bar{x}, \bar{x}^* \rangle + \langle \bar{x} - \bar{x}, \bar{x}^* \rangle + \langle \bar{x}, \bar{x}^* - \bar{x}^* \rangle + \langle \bar{x} - \bar{x}, \bar{x}^* - \bar{x}^* \rangle \leq \langle \bar{x}, \bar{x}^* \rangle + \|\bar{x} - \bar{x}\| \|\bar{x}^*\| + \|\bar{x}\| \|\bar{x}^* - \bar{x}^*\| + \|\bar{x} - \bar{x}\| \|\bar{x}^* - \bar{x}^*\| \]
\[ \leq \langle \bar{x}, \bar{x}^* \rangle + \varepsilon (\|\bar{x}^*\| + \|\bar{x}\|) + \varepsilon^2 \]
and
\[ \|\bar{x}\|^2 + \|\bar{x}^*\|^2 \leq (\|\bar{x}\| + \|\bar{x} - \bar{x}\|)^2 + (\|\bar{x}^*\| + \|\bar{x}^* - \bar{x}^*\|)^2 \]
\[ \leq \|\bar{x}\|^2 + \|\bar{x}^*\|^2 + 2\varepsilon (\|\bar{x}\| + \|\bar{x}^*\|) + 2\varepsilon^2. \]

Combining the two above equations with (28) we obtain
\[ g(x_0, x_0^*)(\bar{x}, \bar{x}^*) + \frac{1}{2} \|\bar{x}\|^2 + \frac{1}{2} \|\bar{x}^*\|^2 \leq \langle \bar{x}, \bar{x}^* \rangle + \frac{1}{2} \|\bar{x}\|^2 + \frac{1}{2} \|\bar{x}^*\|^2 + 2\varepsilon (\|\bar{x}\| + \|\bar{x}^*\|) + 2\varepsilon^2 \]

Using now (29) and the second line of (26) we conclude that
\[ g(x_0, x_0^*)(\bar{x}, \bar{x}^*) + \frac{1}{2} \|\bar{x}\|^2 + \frac{1}{2} \|\bar{x}^*\|^2 \leq 2\varepsilon M \sqrt{2} + 3\varepsilon^2. \]

As \(\varepsilon\) is an arbitrary strictly positive number, using also (25) we conclude that
\[ \inf_{(x, x^*) \in X \times X^*} g(x_0, x_0^*)(x, x^*) + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x^*\|^2 = 0. \]
Altogether, we conclude that if item 4. holds then item 5. holds. The converse item 5. ⇒ item 4. is trivial to verify. Hence item 4. and item 5. are equivalent. As item 2. is equivalent to item 4. and item 3. is equivalent to 5., we conclude that items 2., 3., 4. and 5. are equivalent.

Now we will prove that item 1. is equivalent to item 3. and conclude the proof of the theorem. First suppose that item 3. holds. Since $S_T \in \mathcal{F}_T$

$$(S_T)^* \geq \pi_*.$$ 

As has already been observed, for any proper function $h$ it holds that $(cl\ conv h)^* = h^*$. Therefore

$$(S_T)^* = (\pi + \delta_T)^* \geq \pi_*,$$

that is,

$$\sup_{(y, y^*) \in T} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - \langle y, y^* \rangle \geq \langle x^*, x^{**} \rangle, \ \forall (x^*, x^{**}) \in X^* \times X^{**} \quad (30)$$

After some algebraic manipulations we conclude that (30) is equivalent to

$$\inf_{(y, y^*) \in T} \langle x^{**} - y, x^* - y^* \rangle \leq 0, \ \forall (x^*, x^{**}) \in X^* \times X^{**},$$

that is, $T$ is of type (NI) and so item 1. holds. If item 1. holds, by the same reasoning we conclude that (30) holds and therefore $(S_T)^* \geq \pi_*$. As $S_T \in \mathcal{F}_T$, we conclude that item 2. holds. As has been proved previously item 2. ⇒ item 3.. \hfill $\Box$

References


