Cálculo C (03220): Prova 1 Soluções

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24 de Setembro de 2011(Prova de 20/09/11)

1. (a) Nós derivamos $\mathbf{R}$ e obtemos:

$$\mathbf{V}(t) = \frac{2}{t}\mathbf{i} + 2\mathbf{j} + 6t^2\mathbf{k}$$
$$\mathbf{a}(t) = -\frac{2}{t^2}\mathbf{i} + 12t\mathbf{k}$$

   (b) Olhamos para a segunda componente $y(t)$ de $\mathbf{R}(t)$. Para ter $2t_0 = y(t_0) = 1$ temos de escolher $t_0 = \frac{1}{2}$. Depois verificamos que realmente

$$\mathbf{R}(t_0) = \ln(1)\mathbf{i} + \mathbf{j} + 2\left(\frac{1}{2}\right)^3\mathbf{k} = (0,1,\frac{1}{4}).$$

   Agora podemos calcular

$$\left|\mathbf{a}\left(\frac{1}{2}\right)\right| = \sqrt{(2\cdot 4)^2 + (12\cdot\frac{1}{2})^2} = \sqrt{8^2+6^2} = 10.$$

2. (a)

$$\mathbf{R}'(t) = 2\cos(2t)\mathbf{i} + 2\operatorname{sen}(2t)\mathbf{j} - 2\sqrt{3}\mathbf{k}$$

   e portanto

$$\phi_L(t) = \int_0^t \sqrt{4\cos^2(2u) + 4\operatorname{sen}^2(2u) + 4\cdot 3}\,\mathrm{d}u = \sqrt{4+12}(t-0) = 4t.$$

   (b) $\phi_L^{-1}(s) = s/4$ e portanto

$$\widehat{R}(s) = \operatorname{sen}(s/2)\mathbf{i} - \cos(s/2)\mathbf{j} - \frac{\sqrt{3}s}{2}\mathbf{k}.$$

   Agora calculamos

$$\widehat{\mathbf{R}}'(s) = \frac{1}{2}\cos(s/2)\mathbf{i} + \frac{1}{2}\cos(s/2)\mathbf{j} - \frac{\sqrt{3}}{2}$$

   e obtemos

$$|\widehat{\mathbf{R}}'(s)| = \sqrt{\frac{1}{4}(\cos^2(s/2) + \operatorname{sen}^2(s/2)) + \frac{3}{4}} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

   para todo $s \geq 0$.

3. O Teorema Fundamental de integrais curvilíneas dá:

$$\begin{aligned}\mathbf{V}(t) &= \int_0^t \mathbf{a}(u)\,\mathrm{d}u + \mathbf{V}(0) = \int_0^t (\cos u\mathbf{i} + e^{-u}\mathbf{j})\,\mathrm{d}u - \mathbf{j} \\ &= (\operatorname{sen} t - \operatorname{sen} 0)\mathbf{i} + (-e^{-t} + e^0)\mathbf{j} - \mathbf{j} = \operatorname{sen} t\mathbf{i} - e^{-t}\mathbf{j} \\ \mathbf{R}(t) &= \int_0^t \mathbf{V}(u)\,\mathrm{d}u + \mathbf{R}(0) = \int_0^t (\operatorname{sen} u\mathbf{i} - e^{-u}\mathbf{j})\,\mathrm{d}u + \mathbf{k} \\ &= -\cos(u)\Big|_0^t + e^{-u}\Big|_0^t + \mathbf{k} = (1-\cos t)\mathbf{i} + (e^{-t}-1)\mathbf{j} + \mathbf{k}\end{aligned}$$

Alternativa: Usar integrais indefinidas e determinar a constante de integração usando $\mathbf{V}(0)$ e $\mathbf{R}(0)$.

4. (a)

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = \left(\frac{-8x}{(4x^2+y^2)^2}, \frac{-2y}{(4x^2+y^2)^2}\right)$$

(b)

$$\mathbf{R}'(t) = -2\operatorname{sen} t\mathbf{i} + \cos t\mathbf{j}$$

e

$$\nabla f(\mathbf{R}(t)) = \frac{1}{(16\cos^2 t + \operatorname{sen}^2 t)^2}(-16\cos t, -2\operatorname{sen} t)$$

implicam

$$\langle \mathbf{R}'(t), \nabla f(\mathbf{R}(t))\rangle = \frac{30\operatorname{sen} t\cos t}{(16\cos^2 t + \operatorname{sen}^2 t)^2}.$$

(Erro na prova, vou pensar nas consequencias para a pontuação.)

5. (a) $\mathbf{R}(t) = \cos t\mathbf{i} + \operatorname{sen} t\mathbf{j}$, $t \in [0,2\pi]$, é uma parametrização do círculo dado. Calculamos $\mathbf{R}'(t) = -\operatorname{sen} t\mathbf{i} + \cos t\mathbf{j}$ e obtemos

$$\begin{aligned}\int_C^{\tan} \mathbf{F} &= \int_0^{2\pi} \langle \mathbf{F}(\mathbf{R}(t)), \mathbf{R}'(t)\rangle\,\mathrm{d}t \\ &= \int_0^{2\pi} (\cos^2 t\operatorname{sen} t(-\operatorname{sen} t) - \cos^3 t\cos t)\,\mathrm{d}t \\ &= -\int_0^{2\pi} \cos^2 t(\operatorname{sen}^2 t + \cos^2 t)\,\mathrm{d}t \\ &= -\int_0^{2\pi} \cos^2 t\,\mathrm{d}t = -\pi.\end{aligned}$$

(b) Escrevemos $P(x,y) = x^2y$, $Q(x,y) = -x^3$ e aplicamos o Teorema de Green (com $D = \{(x,y) \in \mathbb{R}^2 | x^2+y^2 \le 1\}$):

$$\begin{aligned}\int_C^{\tan} \mathbf{F} &= \int_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right)\mathrm{dA} = \int_D (-3x^2 - x^2)\,\mathrm{dA}(x,y) \\ &= -4\int_0^1\int_0^{2\pi} r^2\cos^2(\phi) r\,\mathrm{d}\phi\mathrm{d}r = -\pi,\end{aligned}$$

onde no último passo nós usamos coordenadas polares $x(r,\phi) = r\cos\phi$, $y(r,\phi) = r \operatorname{sen}\phi$ em D.

(c) **F** não é conservativo porque a integral em (a) e (b) é diferente de zero e C é um caminho fechado. (Tem outros argumentos.)