Cálculo C (2011/2): Lista 1: Soluções

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1. (a) Primeiro note que

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} \overset{\text{L'Hôpital}}{=} \lim_{t\to 0^+} \frac{t^2}{t} = \lim_{t\to 0^+} t = 0$$

e portanto

$$\lim_{t\to 0^+} (\cos t, \operatorname{sen} t, t\ln t) = (\cos 0, \operatorname{sen} 0, 0) = (1, 0, 0)$$

(b) Primeiro note que

$$\lim_{t\to 0} \frac{e^t - 1}{t} \overset{\text{L'Hôpital}}{=} \lim_{t\to 0} \frac{e^t}{1} = \frac{1}{1} = 1,$$

$$\lim_{t\to 0} \frac{\sqrt{1+t} - 1}{t} \overset{\text{L'Hôpital}}{=} \lim_{t\to 0} \frac{\frac{1}{2}(1+t)^{-1/2}}{1} = \frac{1}{2}$$

e portanto

$$\lim_{t\to 0} \left(\frac{e^t - 1}{t}, \frac{\sqrt{1+t} - 1}{t}, \frac{3}{t+1} \right) = \left(1, \frac{1}{2}, 3\right)$$

2. A curva (b) parece a única com imagem interessante (quase uma hélice).

3. (a)

$$\mathbf{R}'(t) = (\operatorname{sen} t + t\cos t, 2t, -2\operatorname{sen} 2t)$$

(b)

$$\mathbf{R}'(t) = 4e^{4t}\mathbf{k}$$

(c)

$$\mathbf{R}'(t) = -(\operatorname{sen} t)^{-2}\cos t\,\mathbf{i} + (-2t)\frac{1}{2}(1-t^2)^{-\frac{1}{2}}\mathbf{j} = -\frac{\cos t}{\operatorname{sen}^2 t}\mathbf{i} - \frac{t}{\sqrt{1-t^2}}\mathbf{j}$$

(d) Aplicando a regra para a derivada do produto vetorial de duas curvas, obtemos

$$\mathbf{R}'(t) = \mathbf{a} \times (\mathbf{b} + t\mathbf{c}) + t\mathbf{a} \times \mathbf{c}$$

4. (a)
$$\int_0^1 (16t^3\mathbf{i} - 9t^2\mathbf{j} + 25t^4)\,\mathrm{d}t = 4t^4\big|_0^1\mathbf{i} - 3t^3\big|_0^1\mathbf{j} + 5t^5\big|_0^1\mathbf{k}$$

(b)
$$\int_0^1 \left(\frac{4}{1+t^2}\mathbf{j} + \frac{2t}{1+t^2}\mathbf{k}\right)\,\mathrm{d}t = 4\,\mathrm{arctg}\,t\big|_0^1\mathbf{j} + \ln(1+t^2)\big|_0^1$$

(Para a primitiva na componente $\mathbf{j}$ veja uma tabela (ou use substituição trigonométrica), para a integral na componente $\mathbf{k}$ substituir $s = 1 + t^2$.)

(c) Usamos integração parcial:

$$\int_0^{\pi/2} \mathrm{sen}^2\,t\cos t\,\mathrm{d}t = \mathrm{sen}^3\,t\big|_0^{\pi/2} - \int_0^{\pi/2} 2\,\mathrm{sen}\,t\cos t\,\mathrm{sen}\,t\,\mathrm{d}t$$
$$\Rightarrow 3\int_0^{\pi/2} \mathrm{sen}^2\,t\cos t\,\mathrm{d}t = \mathrm{sen}^3\,t\big|_0^{\pi/2} = 1 - 0 = 1;$$
$$\int_0^{\pi/2} \mathrm{sen}\,t\cos^2 t\,\mathrm{d}t = -\cos t\cos^2 t\big|_0^{\pi/2} + \int_0^{\pi/2} \cos t 2\cos t(-\,\mathrm{sen}\,t)\,\mathrm{d}t$$
$$\Rightarrow 3\int_0^{\pi/2} \mathrm{sen}\,t\cos^2 t\,\mathrm{d}t = -\cos^3 t\big|_0^{\pi/2} = 1;$$
$$\int_0^{\pi/2} \mathrm{sen}\,t\cos t\,\mathrm{d}t = \mathrm{sen}^2\,t\big|_0^{\pi/2} - \int_0^{\pi/2} \cos t\,\mathrm{sen}\,t\,\mathrm{d}t$$
$$\Rightarrow 2\int_0^{\pi/2} \mathrm{sen}\,t\cos t\,\mathrm{d}t = \mathrm{sen}^2\,t\big|_0^{\pi/2} = 1.$$

Finalmente obtemos

$$\int_0^{\pi/2} (3\,\mathrm{sen}^2\,t\cos t\mathbf{i} + 3\,\mathrm{sen}\,t\cos^2 t\mathbf{j} + 2\,\mathrm{sen}\,t\cos t\mathbf{k})\,\mathrm{d}t = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

(d) Para determinar $\int \ln t\,\mathrm{d}t$ olhamos para uma tabela ou usamos integração parcial:

$$\int \ln t\mathrm{d}t = \int 1\cdot\ln t\mathrm{d}t = t\ln t - \int t\cdot\frac{1}{t}\,\mathrm{d}t = t\ln t - t + C.$$

E portanto

$$\int (e^t\mathbf{i} + 2t\mathbf{j} + \ln t\mathbf{k})\,\mathbf{d}t = e^t\mathbf{i} + t^2\mathbf{j} + (t\ln t - t)\mathbf{k} + \mathbf{D}$$

onde $\mathbf{D} \in \mathbb{R}^3$ é algum vetor constante.

5. (a)

$$\begin{aligned}
\mathbf{R}'(t) &= (2\cos t, 5, -2\operatorname{sen} t) \\
\Rightarrow |\mathbf{R}'(t)| &= \sqrt{4(\cos^2 t + \operatorname{sen}^2 t) + 25} = \sqrt{29} \\
\Rightarrow L(\mathbf{R}, [-10, 10]) &= \int_{-10}^{10} \sqrt{29}\,\mathrm{d}t = 20\sqrt{29}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{R}'(t) &= \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \\
\Rightarrow |\mathbf{R}'(t)| &= \sqrt{2 + e^{2t} + e^{-2t}} \\
\Rightarrow L(\mathbf{R}, [0, 1]) &= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\,\mathrm{d}t = \int_0^1 \sqrt{(e^t + e^{-t})^2}\,\mathrm{d}t \\
&= \int_0^1 (e^t + e^{-t})\,\mathrm{d}t = (e^t - e^{-t})\big|_0^1 = e - \frac{1}{e}
\end{aligned}$$

6.

$$\begin{aligned}
\mathbf{R}'(t) &= \left(1, \frac{1}{t}, \ln t + t \cdot \frac{1}{t}\right) \\
\Rightarrow |\mathbf{R}'(t)| &= \sqrt{1 + \frac{1}{t^2} + 1 + 2\ln t + \ln^2 t} = \sqrt{2 + \frac{1}{t^2} + 2\ln t + \ln^2 t} \\
\Rightarrow L(\mathbf{R}, [1, 3]) &= \int_1^3 \sqrt{2 + \frac{1}{t^2} + 2\ln t + \ln^2 t}\,\mathrm{d}t \approx 4.0531
\end{aligned}$$

7. Primeiro determinamos a intersecção do cilíndro parabólico $x^2 = 2y$ e da superfície $3z = xy$. Colocando $y = x^2/2$ na equação da superfície obtemos $3z = x \cdot \frac{x^2}{2}$. Portanto a intersecção pode ser parametrizada por $\mathbf{R}(t) = \left(t, \frac{t^2}{2}, \frac{t^3}{6}\right)$. Agora note que $(0, 0, 0) = \mathbf{R}(0)$ e $(6, 18, 36) = \mathbf{R}(6)$, i.e. o comprimento é dado por

$$\begin{aligned}
L(\mathbf{R}, [0, 6]) &= \int_0^6 |\mathbf{R}'(t)|\,\mathrm{d}t = \int_0^6 \sqrt{1 + t^2 + \frac{t^4}{4}}\,\mathrm{d}t = \int_0^6 \sqrt{(1 + \frac{t^2}{4})^2}\,\mathrm{d}t \\
&= \int_0^6 \left(1 + \frac{t^2}{2}\right)\,\mathrm{d}t = \left(t + \frac{t^3}{6}\right)\big|_0^6 = 6 + 36 = 42.
\end{aligned}$$

8.

$$\begin{aligned}
\mathbf{R}'(u) &= 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \\
\Rightarrow \phi_L(t) &= \int_0^t |\mathbf{R}'(u)|\,\mathrm{d}u = \int_0^t \sqrt{4 + 9 + 16}\,\mathrm{d}t = \sqrt{29}t
\end{aligned}$$

e portanto a inversa de ϕ_L é dada por $\phi_L^{-1}(s) = \frac{s}{\sqrt{29}}$ para $s \in [0, \infty)$. A reparametrização por comprimento de arco agora é:

$$\widehat{\mathbf{R}}'(s) = \mathbf{R} \circ \phi_L^{-1}(s) = \mathbf{R}\left(\frac{s}{\sqrt{29}}\right) = \frac{2}{\sqrt{29}}\mathbf{i} + \left(1 - \frac{3}{\sqrt{29}}s\right)\mathbf{j} + \left(5 + \frac{4}{\sqrt{29}}s\right)\mathbf{k}$$

9. Note que $\mathbf{R}(t) = (3 \operatorname{sen} t, 4t, 3\cos t)$ é uma parametrização duma hélice circular. Nós queremos saber onde estamos depois de nos movermos 5 unidades na curva partindo do ponto $(0, 0, 3) = \mathbf{R}(0)$.

   Um jeito: Achar a parametrização por comprimento de arco $\widehat{\mathbf{R}} : [0, \infty) \to \mathbb{R}^3$ partindo do ponto $\mathbf{R}(0)$ e calcular $\widehat{\mathbf{R}}(5)$. (Compare Ex. 8.)

   Outro jeito: Usamos $\phi_L(t) := L(\mathbf{R}, [0, t])$ e determinamos t_0 tal que $\phi_L(t_0) = 5$:

$$\phi_L(t) = \int_0^t |\mathbf{R}'(u)|\, \mathrm{d}u = \int_0^t \sqrt{9\cos^2 u + 16 + 9 \operatorname{sen}^2 u}\, \mathrm{d}u = \int_0^t \sqrt{25}\, \mathrm{d}u = 5t,$$

   i.e., $t_0 = 1$. $\Rightarrow$ Depois de nos movermos 5 unidades pelo arco estamos no ponto $\mathbf{R}(1) = (3 \operatorname{sen} 1, 4, 3 \cos 1) \approx (2.52, 4, 1.62)$.

10.

$$\begin{aligned}
\mathbf{T}(t) = \frac{\mathbf{R}'(t)}{|\mathbf{R}'(t)|} &\overset{5(\mathrm{a})}{=} \frac{1}{\sqrt{29}}(2\cos t, 5, -2\operatorname{sen} t) \\
\mathbf{T}'(t) &= \frac{1}{\sqrt{29}}(-2\operatorname{sen} t, 0, -2\cos t) \;\Rightarrow\; |\mathbf{T}'| \equiv \frac{2}{\sqrt{29}} \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = (-\operatorname{sen} t, 0, -\cos t) \\
\kappa(t) &= \frac{|\mathbf{T}'(t)|}{|\mathbf{R}'(t)|} = \frac{\frac{2}{\sqrt{29}}}{\sqrt{29}} = \frac{2}{29}
\end{aligned}$$

Para mais informações veja `http://mtm.ufsc.br/~martin/calc-c/index.html`.