Cálculo C (2011/2): Lista 2: Soluções

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1.

$$\mathbf{R}(t) = \underbrace{2\cos t}_{x(t)}\,\mathbf{i} + \underbrace{\operatorname{sen} t}_{y(t)}\,\mathbf{j} \text{ e portanto}$$

$$\frac{x(t)^2}{4} + y(t)^2 = \frac{4\cos^2 t}{4} + \operatorname{sen}^2 t = 1, \text{i.e. } \mathbf{R}(t) \in E.$$

Queremos achar $t_0 \in [0, 2\pi)$ tal que $\mathbf{R}(t_0) = P = (-1, \frac{\sqrt{3}}{2})$. Em particular $-1 = x(t_0) = 2\cos t_0$ e portanto $\cos t_0 = -\frac{1}{2}$. Uma solução desta equação é dada por $t_0 = \arccos(-\frac{1}{2}) = \frac{5}{8}\pi$ e de fato obtemos $y(t_0) = \frac{\sqrt{3}}{2}$, i.e. $\mathbf{R}(t_0) = P$.

$$\mathbf{R}'(t_0) = -2\operatorname{sen} t_0\mathbf{i} + \cos t_0\mathbf{j} = -2\frac{\sqrt{3}}{2}\mathbf{i} + \left(-\frac{1}{2}\right)\mathbf{j} = -\sqrt{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

e portanto $\mathbf{S}(t) = P + t\mathbf{R}'(t_0) = (-1 - \sqrt{3}t)\mathbf{i} + \left(\frac{\sqrt{3}}{2} - \frac{t}{2}\right)\mathbf{j}$ é uma parametrização da reta tangente a E no ponto P.

(Note que nós também poderiamos calcular uma equação da reta tangente usando a derivada da função $f(x) = \sqrt{1 - \frac{x^2}{4}}$ em $x_0 = 1$ (Cálculo A).)

2. Primeiro calculamos

$$\begin{aligned}\mathbf{V}(t) &= \mathbf{V}(0) + \int_0^t \mathbf{a}(u)\,\mathrm{d}u = \mathbf{i} + \int_0^t (u\mathbf{i} + e^u\mathbf{j} + e^{-u}\mathbf{k})\,\mathrm{d}u \\ &= \mathbf{i} + \frac{1}{2}u^2\Big|_0^t\mathbf{i} + e^u\Big|_0^t\mathbf{j} - e^{-u}\Big|_0^t\mathbf{k} = \left(1 + \frac{t^2}{2}\right)\mathbf{i} + (e^t - 1)\mathbf{j} - (e^{-t} - 1)\mathbf{k}\end{aligned}$$

Usando este $\mathbf{V}$ e o $\mathbf{R}(0)$ dado, agora obtemos

$$\begin{aligned}
\mathbf{R}(t) &= \mathbf{R}(0) + \int_0^t \mathbf{V}(u)\,\mathrm{d}u \\
&= \mathbf{j} + \int_0^t \left[\left(1+\frac{u^2}{2}\right)\mathbf{i} + (e^u - 1)\,\mathbf{j} + \left(1 - e^{-u}\right)\mathbf{k}\right]\mathrm{d}u \\
&= \mathbf{j} + \left(u + \frac{u^3}{6}\right)\Big|_0^t \mathbf{i} + (e^u - u)\Big|_0^t \mathbf{j} + (u + e^u)\Big|_0^t \mathbf{k} \\
&= \left(t + \frac{t^3}{6}\right)\mathbf{i} + \left(1 + e^t - t - 1\right)\mathbf{j} + \left(t + e^t - 1\right)\mathbf{k} \\
&= \left(t + \frac{t^3}{6}\right)\mathbf{i} + \left(e^t - t\right)\mathbf{j} + \left(t - 1 + e^t\right)\mathbf{k}
\end{aligned}$$

3. Nós vamos usar o Sistema Internacional de Unidades (SI), i.e. trabalhar com metros e segundos. Assim a posição inicial da bola é dada por $\mathbf{R}(0) = 0,9144\mathbf{j}$. A velocidade (escalar) inicial é $v_0 = |\mathbf{V}(0)| = 35,052$ e a velocidade (vetorial) inicial é $\mathbf{V}(0) = 35,052(\cos\alpha\mathbf{i} + \operatorname{sen}\alpha\mathbf{j})$, onde $\alpha = 50°$. A aceleração da gravidade é $\mathbf{a} = -9,81\mathbf{j}$ (em m/s^2). Com esses dados podemos calcular:

$$\begin{aligned}
\mathbf{V}(t) &= \mathbf{V}(0) + \int_0^t (-9,81)\mathbf{j}\,\mathrm{d}u = v_0\cos\alpha\mathbf{i} + (v_0\operatorname{sen}\alpha - 9,81t)\mathbf{j} \Rightarrow \\
\mathbf{R}(t) &= \mathbf{R}(0) + \int_0^t (v_0\cos\alpha\mathbf{i} + (v_0\operatorname{sen}\alpha - 9,81u)\mathbf{j}\,\mathrm{d}u \\
&= v_0\cos\alpha\cdot t\mathbf{i} + (0,9144 + v_0\operatorname{sen}\alpha\cdot t - 4,405t^2)\mathbf{j} =: x(t)\mathbf{i} + y(t)\mathbf{j}
\end{aligned}$$

Para saber se a bola passou por cima da cerca (que fica $121,92$ metros da base do lançamento), temos de determinar $t_0 \geq 0$ tal que $v_0\cos\alpha\cdot t_0 = x(t_0) = 121,92$, i.e. $t_0 = \frac{121,92}{v_0\cos\alpha} = \frac{121,92}{35,052\cos(50°)} \approx 5,41$. Finalmente calculamos

$$\begin{aligned}
y(t_0) &= 0,9144 + v_0\operatorname{sen}\alpha\cdot t_0 - 4,405t_0^2 \\
&\approx 0,9144 + 35,052\operatorname{sen}(50°)\cdot 5,41 - 4,405\cdot 5,41^2 \\
&\approx 0,9144 + 145,266 - 128,926 = 17,2544 \\
&> 3,048.
\end{aligned}$$

Em outras palavras a bola passa por cima da cerca (que tem altura de 3,048 metros).

4. (a)

$$\begin{aligned}
\mathbf{R}(t) &= r\cos\left(\frac{t}{r}\right)\mathbf{i} + r\operatorname{sen}\left(\frac{t}{r}\right)\mathbf{j} \Rightarrow \\
\mathbf{R}'(t) &= -\operatorname{sen}\left(\frac{t}{r}\right)\mathbf{i} + \cos\left(\frac{t}{r}\right)\mathbf{j}
\end{aligned}$$

e portanto $|\mathbf{R}'(t)| \equiv 1$, i.e. $\mathbf{R}$ é uma parametrização por comprimento de arco e $\mathbf{T} = \mathbf{R}$.

$$\mathbf{N}(t) = \mathbf{T}'(t) = \mathbf{R}''(t) = \frac{1}{r}\left[-\cos\left(\frac{t}{r}\right)\mathbf{i} - \operatorname{sen}\left(\frac{t}{r}\right)\mathbf{j}\right] \Rightarrow$$

$$\mathbf{N}'(t) = \frac{1}{r^2}\left[\operatorname{sen}\left(\frac{t}{r}\right)\mathbf{i} - \cos\left(\frac{t}{r}\right)\mathbf{j}\right] \quad \text{e}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\operatorname{sen}(t/r) & \cos(t/r) & 0 \\ -\frac{1}{r}\cos(t/r) & -\frac{1}{r}\operatorname{sen}(t/r) & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + \frac{1}{r}\left[\operatorname{sen}^2\left(\frac{t}{r}\right) + \cos^2\left(\frac{t}{r}\right)\right]\mathbf{k} = \frac{1}{r}\mathbf{k}$$

e portanto $\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle = 0$.

(b) Primeiro usamos três vezes a fórmula para a derivada do produto escalar de curvas (e o fato que $\langle \mathbf{B}(s), \mathbf{T}(s)\rangle$, $\langle \mathbf{B}(s), \mathbf{N}(s)\rangle$ e $\langle \mathbf{B}(s), \mathbf{B}(s)\rangle$ são constantes (0 ou 1), i.e. não dependem do parâmetro s):

$$0 = \frac{d}{ds}\langle \mathbf{B}(s), \mathbf{T}(s)\rangle = \langle \mathbf{B}'(s), \mathbf{T}(s)\rangle + \langle \mathbf{B}(s), \mathbf{T}'(s)\rangle$$
$$\Rightarrow \langle \mathbf{B}'(s), \mathbf{T}(s)\rangle = -\langle \mathbf{B}, \mathbf{T}'\rangle = -\langle \mathbf{B}, \mathbf{N}\rangle = 0, \qquad (1)$$

$$0 = \frac{d}{ds}\langle \mathbf{B}(s), \mathbf{N}(s)\rangle = \langle \mathbf{B}'(s), \mathbf{N}(s)\rangle + \langle \mathbf{B}(s), \mathbf{N}'(s)\rangle$$
$$\Rightarrow \langle \mathbf{B}'(s), \mathbf{N}(s)\rangle = -\langle \mathbf{B}(s), \mathbf{N}'(s)\rangle = -\tau(s), \qquad (2)$$

$$0 = \frac{d}{ds}\langle \mathbf{B}(s), \mathbf{B}(s)\rangle = 2\langle \mathbf{B}'(s), \mathbf{B}(s)\rangle. \qquad (3)$$

Agora aplicamos equações (1),(2),(3) à representação de $\mathbf{B}'(s)$ como combinação linear dos vetores ortogonais e unitários $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$ e obtemos:

$$\mathbf{B}'(s) = \langle \mathbf{B}'(s), \mathbf{T}(s)\rangle\mathbf{T}(s) + \langle \mathbf{B}'(s), \mathbf{N}(s)\rangle\mathbf{N}(s) + \langle \mathbf{B}'(s), \mathbf{B}(s)\rangle\mathbf{B}(s)$$
$$= 0\mathbf{T}(s) - \tau(s)\mathbf{N}(s) + 0\mathbf{B}(s) = -\tau(s)\mathbf{N}(s).$$

5. (a) Nós denotamos a matriz Jacobiana de g por J_g e obtemos, pela Regra da Cadeia:

$$\begin{pmatrix} \frac{\partial h}{\partial r}(r,\theta) & \frac{\partial h}{\partial \theta}(r,\theta) \end{pmatrix} = \nabla h(r,\theta) = \nabla f(g(r,\theta)) \cdot J_g(r,\theta)$$
$$= \begin{pmatrix} \frac{\partial f}{\partial x}(r\cos\theta, r\operatorname{sen}\theta) & \frac{\partial f}{\partial y}(r\cos\theta, r\operatorname{sen}\theta) \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial r}(r,\theta) & \frac{\partial g_1}{\partial \theta}(r,\theta) \\ \frac{\partial g_2}{\partial r}(r,\theta) & \frac{\partial g_2}{\partial \theta}(r,\theta) \end{pmatrix},$$

onde $g_1(r,\theta) = r\cos\theta,\ g_2(r,\theta) = r\,\mathrm{sen}\,\theta$. Usando a equação acima, podemos calcular:

$$\begin{aligned}\frac{\partial h}{\partial r}(r,\theta) &= \frac{\partial f}{\partial x}(g(r,\theta))\frac{\partial g_1}{\partial r}(r,\theta) + \frac{\partial f}{\partial y}(g(r,\theta))\frac{\partial g_2}{\partial r}(r,\theta)\\ &= \cos\theta\frac{\partial f}{\partial x}(g(r,\theta)) + \mathrm{sen}\,\theta\frac{\partial f}{\partial y}(g(r,\theta))\\ \frac{\partial h}{\partial \theta}(r,\theta) &= \frac{\partial f}{\partial x}(g(r,\theta))\frac{\partial g_1}{\partial \theta}(r,\theta) + \frac{\partial f}{\partial y}(g(r,\theta))\frac{\partial g_2}{\partial \theta}(r,\theta)\\ &= -r\,\mathrm{sen}\,\theta\frac{\partial f}{\partial x}(g(r,\theta)) + r\cos\theta\frac{\partial f}{\partial y}(g(r,\theta)).\end{aligned}$$

(b) Agora note que para nossa função f temos $\frac{\partial f}{\partial x}(x,y) = 2xy - 2x$ e $\frac{\partial f}{\partial y}(x,y) = x^2 + 3y^2$. Portanto (a) implica

$$\begin{aligned}\frac{\partial h}{\partial r}(r,\theta) &= \cos\theta(2r\cos\theta\cdot r\,\mathrm{sen}\,\theta - 2r\cos\theta) + \mathrm{sen}\,\theta((r\cos\theta)^2 + 3(r\,\mathrm{sen}\,\theta)^2)\\ &= 3r^2\cos^2\theta\,\mathrm{sen}\,\theta - 2r\cos^2\theta + 3r^2\,\mathrm{sen}^3\,\theta = 3r^2\,\mathrm{sen}\,\theta - 2r\cos^2\theta\\ \frac{\partial h}{\partial \theta}(r,\theta) &= -r\,\mathrm{sen}\,\theta(2r\cos\theta\cdot r\,\mathrm{sen}\,\theta - 2r\cos\theta) + r\cos\theta((r\cos\theta)^2 + 3(r\,\mathrm{sen}\,\theta)^2)\\ &= r^3\cos\theta\,\mathrm{sen}^2\,\theta + 2r^2\cos\theta\,\mathrm{sen}\,\theta + r^3\cos^3\theta\end{aligned}$$

Sem Regra da Cadeia (i.e., sem (a)): Primeiro temos de calcular a composta $h = f\circ g$:

$$\begin{aligned}h(r,\theta) &= f\circ g(r,\theta) = f(r\cos\theta, r\,\mathrm{sen}\,\theta)\\ &= (r\cos\theta)^2 r\,\mathrm{sen}\,\theta - (r\cos\theta)^2 + (r\,\mathrm{sen}\,\theta)^3\\ &= r^3\cos^2\theta\,\mathrm{sen}\,\theta - r^2\cos^2\theta + r^3\,\mathrm{sen}^3\,\theta)\end{aligned}$$

Agora usamos o termo acima para calcular as derivadas parciais:

$$\begin{aligned}\frac{\partial h}{\partial r}(r,\theta) &= 3r^2\cos^2\theta\,\mathrm{sen}\,\theta - 2r\cos^2\theta + 3r^2\,\mathrm{sen}^3\,\theta = 3r^2\,\mathrm{sen}\,\theta - 2r\cos^2\theta\\ \frac{\partial h}{\partial \theta}(r,\theta) &= r^3[2\cos\theta(-\,\mathrm{sen}\,\theta)\,\mathrm{sen}\,\theta + \cos^3\theta] - r^2 2\cos\theta(-\,\mathrm{sen}\,\theta) + r^3 3\,\mathrm{sen}^2\,\theta\cos\theta\\ &= r^3\cos\theta\,\mathrm{sen}^2\,\theta + 2r^2\cos\theta\,\mathrm{sen}\,\theta + r^3\cos^3\theta\end{aligned}$$

Em outras palavras, nós obtemos o mesmo resultado que usando (a).

6. (a)

$$\begin{aligned}\nabla f(x,y) &= \begin{pmatrix} e^x\cos y & -e^x\,\mathrm{sen}\,y\end{pmatrix}\\ \frac{\partial f}{\partial \mathbf{u}}(x,y) &= \langle\begin{pmatrix} e^x\cos y & -e^x\,\mathrm{sen}\,y\end{pmatrix}, \begin{pmatrix} 4 & -3\end{pmatrix}\rangle = 4e^x\cos y + 3e^x\,\mathrm{sen}\,y\\ \frac{\partial f}{\partial \mathbf{u}}(P) &= 4e^0\cos(\pi/2) + 3e^0\,\mathrm{sen}(\pi/2) = 4\cdot 0 + 3\cdot 1 = 3\end{aligned}$$

(b)

$$
\begin{aligned}
f(x,y) &= \sqrt{x^2+(y-3)^2} \\
\nabla f(x,y) &= \left(\tfrac{x}{\sqrt{x^2+(y-3)^2}} \quad \tfrac{y-3}{\sqrt{x^2+(y-3)^2}} \right) \\
\frac{\partial f}{\partial \mathbf{u}}(x,y) &= \left\langle \left(\tfrac{x}{\sqrt{x^2+(y-3)^2}} \quad \tfrac{y-3}{\sqrt{x^2+(y-3)^2}} \right), (-1 \quad 2) \right\rangle \\
&= -\frac{x}{\sqrt{x^2+(y-3)^2}} + 2\frac{y-3}{\sqrt{x^2+(y-3)^2}} \\
\frac{\partial f}{\partial \mathbf{u}}(P) &= -\frac{1}{\sqrt{1+(1-3)^2}} + 2\frac{1-3}{\sqrt{1+(1-3)^2}} = -\frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{5}{\sqrt{5}} = -\sqrt{5}
\end{aligned}
$$

7. (a) Círculos de raio 4, 2, $\frac{4}{3}$, 1.

(b)

$$
\nabla f(x,y) = \left(-\tfrac{x}{(x^2+y^2)^{\frac{3}{2}}} \quad -\tfrac{y}{(x^2+y^2)^{\frac{3}{2}}} \right) = -\frac{(x,y)}{(x^2+y^2)^{\frac{3}{2}}}
$$

$$
\begin{aligned}
\nabla f(P) &= \frac{(1,3)}{(1+\sqrt{3}^2)^{\frac{3}{2}}} = \frac{1}{8}(1,\sqrt{3}) \\
|\nabla f(P)| &= \frac{1}{8}\sqrt{1+3} = \frac{1}{4} \\
\mathbf{u} &= \frac{\nabla f(PI)}{|\nabla f(P)|} = 4 \cdot \frac{1}{8}(1,\sqrt{3}) = \frac{1}{2}(1,\sqrt{3})
\end{aligned}
$$

8. O fato que f é constante ao longo de C implica $(f \circ \mathbf{R})' \equiv 0$ e portanto $\frac{\partial f}{\partial \mathbf{u}}(P) = (f \circ \mathbf{R})'(c) = 0$.

Para mais informações veja `http://mtm.ufsc.br/~martin/calc-c/index.html`.