Isospectral Alexandrov spaces

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Outline

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Δ on manifolds

Definition

Let M be a Riemannian manifold.

- The Laplace Operator is given by

$$\Delta : C^\infty(M) \ni f \mapsto \delta df = -\operatorname{div}\operatorname{grad} f \in C^\infty(M).$$

Theorem

Let M be a compact Riemannian manifold. Then:

- *The eigenspaces of Δ are finite-dimensional.*
- *The eigenvalues (and dim. of eigenspaces) of Δ determine:*
  - $\operatorname{vol}(M)$, $\dim(M)$
  - $\int_M \operatorname{scal}$ *and other curvature integrals*

The Isospectrality Problem

Question (L. Green, M. Kac 1960s)

Are compact Riem. manifolds with the same Δ-eigenvalues (and multiplicities) isometric?

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Counterexamples (selection)

- Milnor (1964): Isospectral flat tori (dim=16)
- Sunada (1985): Condition for isospectrality of properly discontinuous quotients M/Γ_1, M/Γ_2
- Gordon (1993): Isospectral, not locally isometric manifolds via “torus method”
- Schüth (1999): Isospectral metrics on simply connected manifolds

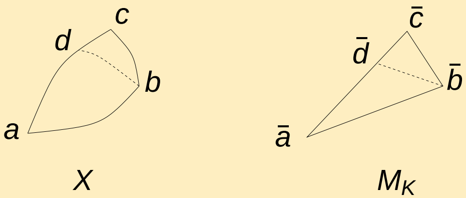
Outline

1. Introduction
   - The Isospectrality Problem
   - Alexandrov spaces
   - The Laplacian on Alexandrov spaces
2. The Torus Method
   - The Manifold Case
   - Generalization to Alexandrov Spaces
   - Non-Isometry
3. Applications
   - Isospectral non-isometric Alexandrov spaces
   - Isospectral Metrics on Orbifolds

Definition of Alexandrov spaces

Definition

Let $K \in \mathbb{R}$. An Alexandrov space of curvature $\geq K$ is a complete locally compact intrinsic metric space X such that every point in X has a neighbourhood in which every geodesic triangle has a comparison triangle $\Delta \bar{a}\bar{b}\bar{c}$ in M_K such that $|ad| = |\bar{a}\bar{d}|$ implies $|bd| \geq |\bar{b}\bar{d}|$.



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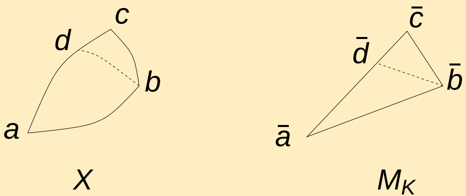

Example (Topogonov's Theorem)

Complete Riemannian manifold with sect. curv. bounded below

Examples and Properties of Alexandrov spaces

Example (Burago, Gromov, Perel'man 1992)

Given:

- (M, g) complete Riem. manifold with sect. curv. $\geq K \in \mathbb{R}$
- G compact group of isometries on (M, g)

Then: $(M/G, d_g)$ Alexandrov space of curvature $\geq K$.

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Then: $(M/G, d_g)$ Alexandrov space of curvature $\geq K$.

- Special case: If all stabilizers $G_x, x \in M$, are finite, then M/G is a Riemannian orbifold.
- A good orbifold is a quotient M/Γ with $\Gamma \subset \mathrm{Isom}(M)$ discrete.

Remark

Within the class of compact metric spaces we have:
manifolds $\subset$ good orbifolds $\subset$ orbifolds $\subset$ Alexandrov spaces

Outline

The set of regular points as a manifold

Let X be an Alexandrov space of dimension $n \geq 2$ and consider the dense subset

$$X^{\text{reg}} := \{x \in X;\ \frac{1}{\varepsilon} B_\epsilon(x) \to B_1(0) \subset \mathbb{R}^n \text{ for } \varepsilon \to 0\}.$$

Assume that X^{reg} is open in X (automatically satisfied in our special case M/G with G a torus). Then

Theorem (Kuwae, Machigashira, Shioya 2001)

- *There is a unique continuous Riemannian metric g on X^{reg} such that the distance function d_g induced by g coincides with the original metric on X^{reg}.*
- *X^{reg} carries a unique C^∞-manifold structure such that g is a limit of C^∞-Riemannian metrics on X^{reg}*

These observations enable us to define the Laplacian on X...

Definition (Kuwae, Machigashira, Shioya 2001)

Let X be an Alexandrov space. Then set

$$H^1(X,\mathbb{R}) := \{u : X \to \mathbb{R};\ u \text{ measurable, } u_{|X^{\mathrm{reg}}} \in H^1(X^{\mathrm{reg}},\mathbb{R})$$

$\Delta : \mathcal{D}(\Delta) \to L^2(X,\mathbb{R})$ maximal self-adjoint operator such that

- $\mathcal{D}(\Delta) \subset H^1(X,\mathbb{R})$
- $\int_X u\Delta v = \int_X \langle \nabla u, \nabla v\rangle\ \forall u \in H^1(X,\mathbb{R}), v \in \mathcal{D}(\Delta)$

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Theorem

For a compact Alexandrov space one has as for manifolds:

1. *Spectrum of Δ: eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \ldots \nearrow \infty$, $L^2(\mathcal{O})$ has ON base (ϕ_i) with $\Delta\phi_i = \lambda_i\phi_i$ (Kuwae et al.).*
2. *(1) implies Min-Max-Principle:*
   $\lambda_k = \inf_{U\in L_k} \sup_{f\in U\setminus\{0\}} \frac{\int_X \|\nabla f\|^2}{\int_X |f|^2}$
   with L_k the set of k-dim. subspaces of $H^1(X)$.

Special Case: Orbifolds

- Heat kernel expansion shows: The spectrum on orbifolds determines volume, dimension and certain curvature integrals. (Not known for general Alexandrov spaces.)

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- Heat kernel expansion shows: The spectrum on orbifolds determines volume, dimension and certain curvature integrals. (Not known for general Alexandrov spaces.)
- This motivates constructions of isospectral orbifolds:

Theorem (Sunada 1985, Bérard 1992)

Let M be a compact Riem. manifold, $\Gamma_1, \Gamma_2 \subset \mathrm{Isom}(M)$ finite subgroups with a bijection $\phi\colon \Gamma_1 \to \Gamma_2$ such that $\phi(\gamma) = g_\gamma \gamma g_\gamma^{-1}$. Then M/Γ_1 and M/Γ_2 are isospectral orbifolds.

Theorem (Rossetti, Schüth, Weilandt 2007)

Let G be a compact Lie group with bi-invariant metric, let $\Gamma_1, \Gamma_2 \subset G$ satisfy the conditions above and set $M := G/\Gamma_1$. Then $\Gamma_1 \backslash M$, $\Gamma_2 \backslash M$ are isospectral orbifolds and have different maximal isotropy orders.

Other Possibilities

- Formula for dimensions of eigenspaces on compact flat orbifolds $\mathbb{R}^n/\Gamma$ (Miatello, Rossetti 2001)
- Given "equivariantly isospectral" G-manifolds M_1, M_2, one can construct isospectral orbifolds M_1/Γ_1, M_2/Γ_2 (with $\Gamma_i \subset G$) (Sutton 2006 based on Pesce).

Note: All these constructions yield isospectral good orbifolds, i.e., orbifolds isometric to quotients M/Γ with M a Riem. manifold and $\Gamma \subset \mathrm{Isom}(M)$ discrete.

Open Question

Central Problem

Can a singular space (sing. orbifold/sing. Alexandrov space) be isospectral to a manifold?

Still open. However, there are some obstructions, e.g.:

Theorem (Gordon, Rossetti 2003)

Let Γ_i act properly discontinuously on M and assume that M/Γ_1, M/Γ_2 are isospectral. Then M/Γ_1 is a manifold if and only if M/Γ_2 is.

- Together with a similar obstruction this implies: No known construction of isospectral orbifolds can provide a (positive) answer to the question above.
- Therefore: Study bad orbifolds - or more generally Alexandrov spaces

Outline

The Torus Method on Manifolds

Theorem (Schüth 2001)

- *Let T be a (non-trivial) torus acting effectively and isometrically on two connected Riem. manifolds M_1, M_2. Let $\widehat{M}_i \subset M_i$ denote the (dense) subset of points in which T acts freely.*
- *Assume: For every subtorus $W \subset T$ of codimension 1 there is a T-equivariant diffeo $F_W : M_1 \to M_2$ satisfying:*
  - $F_W^* \operatorname{dvol}_2 = \operatorname{dvol}_1$, $\overline{F}_W : \widehat{M}_1/W \to \widehat{M}_2/W$ *is an isometry.*

Then: M_1 and M_2 isospectral.

Proof.

Idea: Use Min-Max-Principle, representation theory and denseness $\widehat{M}_i \subset M_i$ (based on ideas of Gordon 1993). □

The Torus Method on Manifolds in Practice

- Let T act isometrically on a fixed Riem. manifold (M, g_0)
- For λ a T-invariant and T-horizontal $\mathfrak{t}$-valued 1-form on M define [with $\lambda(X)^*_x := \frac{d}{dt}_{|t=0} \exp(t\lambda_x(X))x$]

$$g_\lambda(X, Y) := g_0\,(X + \lambda(X)^*, Y + \lambda(Y)^*)$$

- $\mathcal{L} := \ker(\exp(\mathfrak{t} \to T))$ and $\mathcal{L}^* := \{\mu \in \mathfrak{t}^*; \mu(Z) \in \mathbb{Z}\ \forall Z \in \mathcal{L}\}$

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Theorem (λ-torus method, Schüth 2001)

Let λ_1, λ_2 be 1-forms as above such that: For every $\mu \in \mathcal{L}^*$ there is a T-equivariant $F_\mu \in \mathrm{Isom}(M, g_0)$ with $\mu \circ \lambda_1 = F_\mu^*(\mu \circ \lambda_2)$. Then: (M, g_{λ_1}) and (M, g_{λ_2}) are isospectral.

Proof.

For $W \subset T$ from theorem above choose $\mu \in \mathcal{L}^*$ with $\ker \mu = T_eW$ and use $F_W := F_\mu$. □

Outline

The Torus Method on (Special) Alexandrov Spaces

- Fix a manifold M with an action of a torus G and two metrics g_1, g_2 on M. Under which condition are $(M/G, d_{g_1})$ and $(M/G, d_{g_2})$ isospectral Alexandrov spaces?

The Torus Method on (Special) Alexandrov Spaces

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Theorem (M.W.)

Let T be a non-triv. torus acting on M, isom. w.r.t. g_1 and g_2. Assume that the T- and G-actions commute and that the T-action on M/G is effective and set
$\widehat{M} := \{x \in M;\ G_x = \{\mathrm{Id}_M\}, T_{[x]} = \{\mathrm{Id}_{M/G}\}\} \subset M.$
Assume that for every $W \subset T$ of codim. 1 there is a G- and T-equiv. diffeo E_W on M which satisfies $E_W^* \operatorname{dvol}_{g_2} = \operatorname{dvol}_{g_1}$ and induces an isometry $\overline{F_W} : ((\widehat{M}/G)/W, g_1^W) \to ((\widehat{M}/G)/W, g_2^W)$. Then the Alexandrov spaces $(M/G, d_{g_1}), (M/G, d_{g_2})$ are isospectral.

Proof.

(following Schüth 2001)

- Consider the complex Sobolev spaces $H^i := H^1(M/G, d_{g_i}), i = 1, 2$.
- Decompose $H^i = \bigoplus_{\mu \in \mathcal{L}^*} H^i_\mu$ with $\mathcal{L} = \ker \exp_T$ and $H^i_\mu = \{f \in H^i;\ [Z]f = e^{2\pi i \mu(Z)} f\ \forall Z \in \mathfrak{t}\}$.
- With $S^i_W := \bigoplus_{\substack{\mu \in \mathcal{L}^* \setminus \{0\} \\ T_e W = \ker \mu}} H^i_\mu$ we obtain $H^i = H^i_0 \oplus \bigoplus_W S^i_W$.
- Assumptions imply that $F^*_W : S^2_W \to S^1_W$ is an L^2-preserving isometry.
- Patch together to L^2-preserving isometry $H^2 \to H^1$.
- Min-Max-Principle gives isospectrality of $(M/G, d_{g_1})$ and $(M/G, d_{g_2})$.

$\square$

In analogy to the manifold setting the theorem above yields the λ-torus method on Alexandrov spaces (of the form M/G):

- Fix (M, g_0), commuting tori $G, T \subset \mathrm{Isom}(M)$.
- A $\mathfrak{t}$-valued 1-form λ on M is called admissible if it is invariant and horizontal w.r.t. G and T.
- $g_\lambda(X, Y) := g_0(X + \lambda(X)^*, Y + \lambda(Y)^*)$

Theorem

Let λ_1, λ_2 be admissible 1-forms on M such that for every $\mu \in \mathcal{L}^*$ there is a G- and T-equivariant $E_\mu \in \mathrm{Isom}(M, g_0)$ satisfying

$$\mu \circ \lambda_1 = E_\mu^*(\mu \circ \lambda_2)$$

Then $(M/G, d_{\lambda_1})$ and $(M/G, d_{\lambda_2})$ are isospectral Alexandrov spaces.

Outline

When are $(M/G, d_{\lambda_1}), (M/G, d_{\lambda_2})$ not isometric?

- **Idea:** Use criterion from Schüth 2001. Gives criterion on λ_1, λ_2 for non-isometry of $(\widehat{M}/G, g_{\lambda_1}), (\widehat{M}/G, g_{\lambda_2})$.

Outline

Applications of the λ-torus method

Schüth 2001: M.W.:

- $T := S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ acts effectively and isometrically on $M := S^{2n+1}$ via $z(u,v) = (u, zv)$ with $u \in \mathbb{C}^{n-1}, v \in \mathbb{C}^2$.
- Let $G := S^1$ act on S^{2n+1} via $\sigma(u,v) := (\sigma u, v)$. T-action above induces an effective isometric action of T on S^{2n+1}/S^1.
- For $j : \mathfrak{t} \to \mathfrak{su}(n-1)$ linear define $\mathfrak{t} \simeq \mathbb{R}^2$-valued 1-form $\lambda = (\lambda^1, \lambda^2)$ on S^{2n+1}:

$$\lambda^k_{(u,v)}(U,V) := \|u\|^2 \langle j_{Z_k} u, U\rangle - \langle U, iu\rangle\langle j_{Z_k} u, iu\rangle$$

  with $k = 1,2$ and $Z_1 = (i,0), Z_2 = (0,i)$ basis of $\mathfrak{t} = T_{(1,1)}(S^1 \times S^1) \subset \mathbb{C} \times \mathbb{C}$.

Observe:

- λ is T-invariant and -horizontal (Schüth 2001).
- λ is also S^1-invariant und -horizontal.

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Theorem (M.W.)

Let $j_1, j_2 : \mathfrak{t} \to \mathfrak{su}(n-1)$ be isospectral linear maps, i.e.:

> *For every $Z \in \mathfrak{t}$ there is $A_Z \in SU(n-1)$ such that $j_2(Z) = A_Z j_1(Z) A_Z^{-1}$.*

Define 1-forms λ_1, λ_2 on M as on the preceding slide. Then the Alexandrov spaces $(M/S^1, g_{\lambda_1}), (M/S^1, g_{\lambda_2})$ are isospectral.

Proof.

Use λ-torus-method: Let $\mu \in \mathcal{L}^* \subset \mathfrak{t}^*$. Set $Z := \mu(Z_1)Z_1 + \mu(Z_2)Z_2$ and choose A_Z as in the assumption. $E_\mu := (A_Z, \mathrm{Id}) \in SU(n-1) \times SU(2)$ satisfies $\mu \circ \lambda_1 = E_\mu^*(\mu \circ \lambda_2)$ on S^{2n+1} and is T-equivariant (Schüth 2001). E_μ is is also S^1-equivariant. $\square$

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Define 1-forms λ_1, λ_2 *on* M *as on the preceding slide. Then the Alexandrov spaces* $(M/S^1, g_{\lambda_1}), (M/S^1, g_{\lambda_2})$ *are isospectral.*

Proof.

Use λ-torus-method: Let $\mu \in \mathcal{L}^* \subset \mathfrak{t}^*$. Set $Z := \mu(Z_1)Z_1 + \mu(Z_2)Z_2$ and choose A_Z as in the assumption. $E_\mu := (A_Z, \mathrm{Id}) \in SU(n-1) \times SU(2)$ satisfies $\mu \circ \lambda_1 = E_\mu^*(\mu \circ \lambda_2)$ on S^{2n+1} and is T-equivariant (Schüth 2001). E_μ is is also S^1-equivariant. □

Theorem (Schüth 2001)

For $n \geq 4$ there are continuous families $j(t) : \mathfrak{t} \to \mathfrak{su}(n-1)$ of isospectral 1-forms which via $\lambda(t)$ induce isospectral pairwise non-isometric metrics on S^{2n+1}.

The same reasoning applied to S^{2n+1}/S^1 shows:

Theorem (M.W.)

For the families $j(t)$ from the theorem above the Alexandrov spaces $(S^{2n+1}/S^1, d_{\lambda(t)})$ are pairwise non-isometric.

Moreover, results from foliation theory (due to Lytchak, Alexandrino) show:

Theorem

None of our Alexandrov spaces $(S^{2n+1}/S^1, d_{\lambda(t)})$ can be isometric to a Riemannian orbifold.

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Isospectral Metrics on Weighted Projective Spaces

- Let $p, q \in \mathbb{N}$ pairwise prime, $n \geq 4$.
- Again consider $M = S^{2n+1}$, $G = S^1$, but with the action

$$\sigma(u, v) := (\sigma^p u, \sigma^q v)$$

- All stabilizers are finite ($G_{(0,v)} \simeq \mathbb{Z}_q, G_{(u,0)} \simeq \mathbb{Z}_p$), hence M/G is an orbifold (a weighted projective space $\mathbb{W}(p, q)$)
- The same construction as before applies $\rightsquigarrow$ isospectral metrics on any fixed $\mathbb{W}(p, q)$.
- The special case $\mathbb{CP}^n$ (i.e. $p = q = 1$) has been treated by Rückriemen (2006).
- For the spectrum of weighted projective spaces with their standard metrics see work by Abreu, Freitas, Godinho, Dryden (2008) and Guillemin, Uribe, Wang (2008).

Summary

Theorem

For every $n \geq 4$ there are families of isospectral non-isometric bad Riemannian orbifolds (weighted projective spaces) or even more general Alexandrov spaces of dimension $2n$.

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