14/02/07

Cálculo A

Exercícios resolvidos: Integração p/ partes

1. $\int dx\ x \sin x$

2. $\int dx\ x \sec^2 x$

3. $\int dx\ x \ln x$

4. $\int dx\ x \ln x^2$

5. $\int dx\ (\ln x)^2$

6. $\int dx\ x^2 \ln x$

7. $\int dx\ x^3 \ln x$

8. $\int dx\ x e^{-x}$

9. $\int dx\ x^2 e^{4x}$

10. $\int dx\ x^2 \sin x$

11. $\int dx\ x^3 \cos x$

12. $\int dx\ e^{3x} \cos 3x$

13. $\int dx\ \frac{\sin x}{e^x}$

14. $\int dt\ t\, 2^t$

15. $\int dt\ t\, 3^{-t}$

Solução

1. $\int dx\, x \sin x$

$$\begin{cases} u = x \longrightarrow du = dx \\ dv = \sin x\, dx \longrightarrow v = -\cos x \end{cases}$$

$$\int dx\, x \sin x = -x\cos x - \int -\cos x\, dx$$
$$= -x\cos x + \int \cos x\, dx$$
$$= -x\cos x + \sin x + C$$

2. $\int dx\, x \sec^2 x$

$$\begin{cases} u = x, \quad du = dx \qquad \left(\int u\,dv = uv - \int v\,du\right) \\ dv = \sec^2 x\, dx, \quad v = \operatorname{tg} x \end{cases}$$

$$\int dx\, x \sec^2 x = x \operatorname{tg} x - \int \operatorname{tg} x\, dx$$
$$= x \operatorname{tg} x - \int \operatorname{tg} x\, dx \quad (*)$$

Mas tem-se por substituição que:

$$\int \operatorname{tg} x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

Faz-se $u = \cos x \longrightarrow du = -\sin x\, dx$, daí

$$\int \operatorname{tg} x\, dx = \int -\frac{du}{u} = -\ln|u| = -\ln|\cos x| \quad (**) \longrightarrow$$

Subst. (**) → (*) temos:

$$\int dx\; x \sec^2 x = x\, tg\, x - (-\ln|\cos x|)$$

$$= x\, tg\, x + \ln|\cos x| + C \;//$$

3. $\int dx\; x \ln x$

$u = \ln x \quad , \quad du = \frac{1}{x}\, dx$

$dv = x\, dx \quad , \quad v = \frac{x^2}{2}$

$$\int dx\; x \ln x = \frac{x^2}{2} \ln x - \int dx\; \frac{1}{x} \frac{x^2}{2}$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int dx\; x$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2}$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C \;//$$

4. $\int dx\; x \ln x^2\, dx$

Note que $\ln x^2 = 2 \ln x$ logo temos

$$\int dx\; x \ln x^2 = \int dx\; 2x \ln x = 2 \underbrace{\int dx\; x \ln x}_{\text{questão anterior}}$$

$$= 2\left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right)$$

$$= x^2 \ln x - \frac{x^2}{2} + C \;//$$

5. $\int dx\,(\ln x)^2$

$$\begin{cases} u = (\ln x)^2, & du = 2\ln x\,\frac{1}{x}\,dx \\ dv = dx, & v = x \end{cases}$$

$$\int dx\,(\ln x)^2 = x(\ln x)^2 - \int dx\, 2\ln x\,\frac{1}{\cancel{x}}\cancel{x}$$

$$= x(\ln x)^2 - 2\int dx\,\ln x \quad (*)$$

Agar..

$\int dx\,\ln x$

$$\begin{cases} u = \ln x, & du = \frac{1}{x}\,dx \\ dv = dx, & v = x \end{cases}$$

$$\int dx\,\ln x = x\ln x - \int dx$$

$$= x\ln x - x \quad (**)$$

Subst. $(**) \rightarrow (*)$:

$$\int dx\,(\ln x)^2 = x(\ln x)^2 - 2(x\ln x - x)$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

6. $\int x^2 \ln x \, dx$

$$\begin{cases} u = \ln x \to du = \frac{1}{x} dx \\ dv = x^2 dx \to v = \frac{x^3}{3} \end{cases}$$

$\therefore$

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^{\cancel{3}\,2}}{3} \frac{1}{\cancel{x}} dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C \quad //$$

7. $\int x^3 \ln x \, dx$

$u = \ln x \to du = \frac{1}{x} dx$

$dv = x^3 dx \to v = \frac{x^4}{4}$

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{1}{4} x^4 \frac{1}{x} dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + C \quad //$$

8. $\int x e^{-x} dx =$

$$\begin{cases} u = x \longrightarrow du = dx \\ dv = e^{-x} dx \longrightarrow v = -e^{-x} \end{cases}$$

$\therefore$

$$\int x e^{-x} dx = -x e^{-x} - \int -e^{-x} dx$$

$$= -x e^{-x} + \int e^{-x} dx$$

$$= -x e^{-x} - e^{-x}$$

$$= -e^{-x}(x+1) + C \,//$$

9. $\int x^2 e^{4x} dx$

$u = x^2 \longrightarrow du = 2x\,dx$

$dv = e^{4x} dx \longrightarrow v = \frac{1}{4} e^{4x}$

$$(\star) \begin{cases} \int x^2 e^{4x} dx = \frac{1}{4} x^2 e^{4x} - \int \frac{1}{\cancel{4}} e^{4x} \cancel{2} x \, dx \\ = \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \underbrace{\int e^{4x} x \, dx}_{(*)} \end{cases}$$

$(*) = \int e^{4x} x \, dx$

$u = x \longrightarrow du = dx$

$dv = e^{4x} dx \longrightarrow v = \frac{1}{4} e^{4x}$

$\longrightarrow$

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$$(*) = \int e^{4x} x \, dx = \frac{1}{4} x e^{4x} - \int \frac{1}{4} e^{4x} dx$$

$$= \frac{1}{4} x e^{4x} - \frac{1}{4} \frac{e^{4x}}{4} + C$$

$$= \frac{1}{4} e^{4x} \left(x - \frac{1}{4} \right) + C$$

Subst. $(*)$ en $(\star)$:

$$\int x^2 e^{4x} dx = \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \left(\frac{1}{4} e^{4x} \left(x - \frac{1}{4} \right) \right) + C$$

$$= \frac{1}{4} x^2 e^{4x} - \frac{1}{8} e^{4x} \left(x - \frac{1}{4} \right) + C$$

$$= \frac{1}{4} x^2 e^{4x} - \frac{1}{8} x e^{4x} + \frac{1}{32} e^{4x} + C \quad //$$

10. $\int x^2 \sin x \, dx =$

$$\begin{cases} u = x^2 \longrightarrow du = 2x \, dx \\ dv = \sin x \, dx \longrightarrow v = -\cos x \end{cases}$$

$$(\star) \begin{cases} \int x^2 \sin x \, dx = -x^2 \cos x - \int -\cos x \, 2x \, dx \\ = -x^2 \cos x + 2 \underbrace{\int \cos x \, x \, dx}_{(*)} \end{cases}$$

$(*) = \int \cos x \, x \, dx$

$u = x \longrightarrow du = dx$

$dv = \cos x \, dx \longrightarrow v = +\sin x$

$\therefore$

$(*) = x\sin x - \int \sin x \, dx$

$= x\sin x - (-\cos x)$

$= x\sin x + \cos x$ //

Subst. $(*) \longrightarrow (\star)$:

$\int x^2 \sin x \, dx = -x^2\cos x + 2(x\sin x + \cos x)$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$ //

11. $\int x^3 \cos x \, dx$

$\begin{cases} u = x^3 \longrightarrow du = 3x^2 dx \\ dv = \cos x \, dx \longrightarrow v = +\sin x \end{cases}$

$(\star)$ $\int x^3 \cos x \, dx = +x^3\sin x + \int -\sin x \, 3x^2 dx$

$= +x^3 \sin x - 3\underbrace{\int \sin x \, x^2 dx}_{(*)}$

Mos

$(*) = \int \sin x \, x^2 \, dx$

$$\begin{cases} u = x^2 \longrightarrow du = 2x\,dx \\ dv = \sin x \, dx \longrightarrow v = -\cos x \end{cases}$$

$$\begin{aligned} (*) &= -x^2 \cos x - \int -\cos x \, 2x \, dx \\ &= -x^2 \cos x + 2 \underbrace{\int \cos x \, x \, dx}_{} \quad \text{(ver item 10)} \\ &= -x^2 \cos x + 2 \left(x \sin x + \cos x \right) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x \end{aligned}$$

Subst. $(*) \rightarrow (\star)$

$$\int x^3 \cos x \, dx = + x^3 \sin x - 3 \left(-x^2 \cos x + 2x \sin x + 2 \cos x \right)$$

$$= + x^3 \sin x + 3x^2 \cos x - 6x \sin x + - 6 \cos x + C$$

12. $\int e^{3x} \cos 3x \, dx$

$$\begin{cases} u = \cos 3x \longrightarrow du = -3 \sin 3x \, dx \\ dv = e^{3x} dx \longrightarrow v = \frac{1}{3} e^{3x} \end{cases}$$

$$(☆) \begin{cases} \int e^{3x} \cos 3x \, dx = \frac{1}{3} e^{3x} \cos 3x - \int \cancel{-3} \sin 3x \, \cancel{\frac{1}{3}} e^{3x} \, dx \\ \quad = \frac{1}{3} e^{3x} \cos 3x + \underbrace{\int \sin 3x \, e^{3x} \, dx}_{⊛} \end{cases}$$

$$⊛ = \int \sin 3x \, e^{3x} \, dx$$

$$\begin{cases} u = \sin 3x \longrightarrow du = 3 \cos 3x \, dx \\ dv = e^{3x} dx \longrightarrow v = \frac{1}{3} e^{3x} \end{cases}$$

$\therefore$

$$⊛ = \frac{1}{3} \sin 3x \, e^{3x} - \int e^{3x} \cos 3x \, dx$$

$\therefore$

$⊛ \rightarrow (☆)$:

$$\underline{\int e^{3x} \cos 3x \, dx} = \frac{1}{3} e^{3x} \cos 3x + \frac{1}{3} \sin 3x \, e^{3x} - \underline{\int e^{3x} \cos 3x \, dx}$$

$$2 \int e^{3x} \cos 3x \, dx = \frac{1}{3} e^{3x} \cos 3x + \frac{1}{3} \sin 3x \, e^{3x}$$

$$\therefore \boxed{\int e^{3x} \cos 3x \, dx = \frac{1}{6} e^{3x} \cos 3x + \frac{1}{6} e^{3x} \sin 3x + C}$$

13. $\int \frac{\sin x}{e^x}\,dx$

$$\begin{cases} u = e^{-x} \longrightarrow du = -e^{-x}\,dx \\ dv = \sin x\,dx \longrightarrow v = -\cos x \end{cases}$$

$$(\star)\ \int \frac{\sin x}{e^x}\,dx = -e^{-x}\cos x - \underbrace{\int \cos x\, e^{-x}\,dx}_{(*)}$$

$$(*) = \int \cos x\, e^{-x}\,dx$$

$$\begin{cases} u = e^{-x} \longrightarrow du = -e^{-x}\,dx \\ dv = \cos x\,dx \longrightarrow v = +\sin x \end{cases}$$

$$\therefore\ (*) = +e^{-x}\sin x + \int \frac{\sin x}{e^x}\,dx$$

$(*) \to (\star)$:

$$\int \frac{\sin x}{e^x}\,dx = -e^{-x}\cos x - \left(e^{-x}\sin x + \int \frac{\sin x}{e^x}\,dx\right)$$

$$= -e^{-x}\cos x - e^{-x}\sin x - \int \frac{\sin x}{e^x}\,dx$$

$$\therefore\ 2\int \frac{\sin x}{e^x}\,dx = -e^{-x}(\cos x + \sin x)$$

$$\therefore\ \int \frac{\sin x}{e^x}\,dx = -\frac{1}{2}e^{-x}(\cos x + \sin x) + C$$

14. $\int t\, 2^t\, dt$

$\left(\frac{d}{dt} a^t = a^t \ln a\right)$

$$\begin{cases} u = t \rightarrow du = dt \\ dv = 2^t dt \rightarrow v = \frac{2^t}{\ln 2} \end{cases}$$

$\therefore$

$$\int t\, 2^t dt = \frac{t\, 2^t}{\ln 2} - \int \frac{2^t}{\ln 2} dt$$

$$= \frac{t 2^t}{\ln 2} - \frac{1}{\ln 2} \underline{\int 2^t dt}$$

$$= \frac{t 2^t}{\ln 2} - \frac{1}{\ln 2} \frac{2^t}{\ln 2}$$

$$= \frac{2^t}{\ln 2}\left(t - \frac{1}{\ln 2}\right) + C \; //$$

15. $\int t\, 3^{-t} dt$

$$\begin{cases} u = t \rightarrow du = dt \\ dv = 3^{-t} dt \rightarrow v = -\frac{3^{-t}}{\ln 3} \end{cases}$$

$\therefore$

$$\int t\, 3^{-t} dt = -\frac{t\, 3^{-t}}{\ln 3} + \int \frac{3^{-t}}{\ln 3} dt$$

$$= \frac{-t 3^{-t}}{\ln 3} + \frac{1}{\ln 3} \int 3^{-t} dt$$

$$= \frac{-t 3^{-t}}{\ln 3} - \frac{1}{(\ln 3)^2} 3^{-t} + C \; //$$

16. $\int t^2 4^t \, dt$

$$\begin{cases} u = t^2 \rightarrow du = 2t\,dt \\ dv = 4^t\,dt \rightarrow v = \dfrac{4^t}{\ln 4} \end{cases}$$

$\therefore$

$$\int t^2 4^t \, dt = \frac{t^2 4^t}{\ln 4} - \int 2t \frac{4^t}{\ln 4} \, dt$$

$$= \frac{t^2 4^t}{\ln 4} - \frac{2}{\ln 4} \underbrace{\int t 4^t \, dt}_{(*)}$$

$$(*) = \int t 4^t \, dt$$

$$\begin{cases} u = t \rightarrow du = dt \\ dv = 4^t\,dt \rightarrow v = \dfrac{4^t}{\ln 4} \end{cases}$$

$$(*) = \frac{t 4^t}{\ln 4} - \int \frac{4^t}{\ln 4} \, dt = \frac{t 4^t}{\ln 4} - \frac{1}{\ln 4} \int 4^t \, dt$$

$$= \frac{t 4^t}{\ln 4} - \frac{1}{\ln 4} \frac{4^t}{\ln 4}$$

$\therefore$

$$\int t^2 4^t \, dt = \frac{t^2 4^t}{\ln 4} - \frac{2}{\ln 4} \left(\frac{t 4^t}{\ln 4} - \frac{1}{(\ln 4)^2} 4^t \right)$$

$$= \frac{t^2 4^t}{\ln 4} - \frac{2t 4^t}{(\ln 4)^2} + \frac{2 \cdot 4^t}{(\ln 4)^3} + C \;//$$