Cálculo B - Lista 1

Integração de funções trigonométricas envolvendo potências de seno e cosseno

Resolva cada uma das integrais a seguir

1. $\int \sin^4 x \cos x \; dx$
2. $\int \sin^5 x \; \cos x \; dx$
3. $\int \cos^3 4x \; \sin 4x \; dx$
4. $\int \cos^6 \frac{x}{2} \; \sin \frac{x}{2} \; dx$
5. $\int \sin^3 x dx$
6. $\int \sin^2 3x \; dx$
7. $\int \sin^4 z \; dz$
8. $\int \cos^5 x \; dx$
9. $\int \cos^2 \frac{x}{2} \; dx$
10. $\int \sin^3 x \; \cos^3 x \; dx$
11. $\int \sin^2 x \; \cos^3 x \; dx$
12. $\int \cos^6 x \; dx$
13. $\int \sin^5 x \; \cos^2 x \; dx$
14. $\int \sin^2 2t \; \cos^4 2t \; dt$
15. $\int \sin^2 3t \; \cos^2 3t \; dt$
16. $\int \sqrt{\cos z} \; \sin^3 z \; dz$
17. $\int \frac{\cos^3 3x}{\sqrt[3]{\sin 3x}} \; dx$
18. $\int \sin^3 \frac{y}{2} \; \cos^2 \frac{y}{2} \; dy$
19. $\int \cos 4x \; \cos 3x \; dx$
20. $\int \sin 2x \; \cos 4x \; dx$
21. $\int \sin 3y \; \cos 5y \; dy$
22. $\int \cos t \; \cos 3t \; dt$
23. $\int (\sin 3t - \sin 2t)^2 \; dt$
24. $\int \sin x \; \sin 3x \; \sin 5x \; dx$
25. $\int 2 \sin x \; \cos x \; dx$
26. Mostre que $\int_0^\pi \sin^2 nx \; dx = \frac{\pi}{2}$ para qualquer n inteiro positivo.

27. Mostre que $\int_0^\pi \cos^n x\ dx = 0$ para qualquer n inteiro positivo ímpar.

28. Mostre que

$$\int_{-1}^{1} \cos n\pi x\ \cos m\pi x\ dx = \begin{cases} 0\ , & m \neq n \\ 1\ , & m = n \end{cases}$$

29. Mostre que para n inteiro positivo ímpar tem-se a forma geral

$$\begin{aligned} \int \sin^n x\ dx &= \sum_{r=0}^{k} \frac{k!}{(k-r)!r!} \frac{(-1)^{r+1}}{2r+1} \cos^{2r+1} x + C \\ \int \cos^n x\ dx &= \sum_{r=0}^{k} \frac{k!}{(k-r)!r!} \frac{(-1)^{r}}{2r+1} \sin^{2r+1} x + C \end{aligned}$$

Resposta

1. $\frac{1}{5}\sin^5 x + C$
2. $\frac{1}{6}\sin^6 x + C$
3. $-\frac{1}{16}\cos^4 4x + C$
4. $-\frac{2}{7}\cos^7 \frac{x}{2} + C$
5. $-\cos x + \frac{1}{3}\cos^3 x + C$
6. $\frac{x}{2} - \frac{1}{12}\sin 6x + C$
7. $\frac{3}{8}z - \frac{1}{4}\sin 2z + \frac{1}{32}\sin 4z + C$
8. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$
9. $\frac{1}{2}x + \frac{1}{2}\sin x + C$
10. $-\frac{1}{16}\cos 2x + \frac{1}{48}\cos^3 2x + C$
11. $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$
12. $\frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$
13. $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$
14. $\frac{1}{16}t - \frac{1}{128}\sin 8t + \frac{1}{96}\sin^3 4t + C$
15. $\frac{1}{8}t - \frac{1}{96}\sin 12t + C$
16. $-\frac{2}{3}\cos^{3/2} z + \frac{2}{7}\cos^{7/2} z + C$
17. $\frac{1}{2}(\sin 3x)^{2/3} - \frac{1}{8}(\sin 3x)^{8/3} + C$
18. $-\frac{2}{3}\cos^3 \frac{y}{2} + \frac{2}{5}\cos^5 \frac{y}{2} + C$
19. $\frac{1}{14}\sin 7x + \frac{1}{2}\sin x + C$
20. $\frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$

21. $\frac{1}{4}\cos 2y - \frac{1}{16}\cos 8y + C$

22. $\frac{1}{8}\sin 4t + \frac{1}{4}\sin 2t + C$

23. $t - \sin t - \frac{1}{8}\sin 4t + \frac{1}{5}\sin 5t - \frac{1}{12}\sin 6t + C$

24. $-\frac{1}{12}\cos 3x - \frac{1}{28}\cos 7x + \frac{1}{4}\cos x + \frac{1}{36}\cos 9x + C$

25. $\sin^2 x + C$

Cálculo B - Lista 1

1. $\int \sin^4 x \cos x \, dx$

Substituição : $\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$

$\therefore$

$$\int \underbrace{\sin^4 x}_{u^4} \underbrace{\cos x \, dx}_{du} = \int u^4 \, du = \frac{1}{5} u^5 + C$$

$$= \frac{1}{5} \sin^5 x + C$$

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2. $\int \sin^5 x \cos x \, dx = \frac{1}{6} \sin^6 x + C$

(Análoga ao caso dado em 1)

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3. $\int \cos^3 4x \sin 4x \, dx$

Seja $u = 4x \therefore du = 4 dx \therefore dx = \frac{1}{4} du$

Daí

$$\int \cos^3 4x \sin 4x \, dx = \int \cos^3 u \sin u \, \frac{1}{4} du$$

$$= \frac{1}{4} \underbrace{\int \cos^3 u \sin u \, du}_{} =$$

$$= \frac{1}{4} (-1) \frac{\cos^4 u}{4} = -\frac{1}{16} \cos^4 4x + C$$

4. $\int \cos^6 \frac{1}{2}x \, \sin \frac{1}{2}x \, dx$

$u = \frac{1}{2}x \longrightarrow du = \frac{1}{2}\,dx$

$\therefore \; dx = 2\,du$

Daí

$\int \cos^6 \frac{1}{2}x \, \sin \frac{1}{2}x \, dx = \int \cos^6 u \, \sin u \; 2\,du$

$= 2 \int \cos^6 u \, \sin u \, du$

$= 2(-1)\dfrac{\cos^7 u}{7}$

$= -\frac{2}{7}\cos^7 u \quad = -\frac{2}{7}\cos^7 \frac{1}{2}x + C$

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5. $\int \sin^3 x \, dx = \int \underbrace{\sin^2 x} \, \sin x \, dx$

$= \int (\underbrace{1-\cos^2 x}) \sin x \, dx$

$= \int \sin x \, dx - \int \cos^2 x \, \sin x \, dx$

$= -\cos x - (-1)\dfrac{\cos^3 x}{3} + C$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

6. $\int \operatorname{sen}^2 3x \, dx$

$u = 3x \rightarrow du = 3\,dx \;\therefore\; dx = \frac{1}{3} du$

Daí

$$\int \operatorname{sen}^2 3x \, dx = \int \operatorname{sen}^2 u \; \frac{1}{3} du = \frac{1}{3} \underbrace{\int \operatorname{sen}^2 u \, du}_{(*)}$$

$$= \frac{1}{3} \int \left(\frac{1 - \cos 2u}{2}\right) du$$

$$= \frac{1}{6} \int (1 - \cos 2u) \, du$$

$$= \frac{1}{6} \int 1 \, du - \frac{1}{6} \int \cos 2u \, du$$

$$= \frac{1}{6} u - \frac{1}{6} \, \frac{1}{2} \operatorname{sen} 2u + C$$

$$= \frac{1}{6} u - \frac{1}{12} \operatorname{sen} 2u + C$$

$$= \frac{1}{6} 3x - \frac{1}{12} \operatorname{sen} 6x + C$$

$$= \frac{1}{2} x - \frac{1}{12} \operatorname{sen} 6x + C$$

(*): Usou-se que $\operatorname{sen}^2 u = \dfrac{1 - \cos 2u}{2}$

7. $\int \sin^4 z \, dz$

Duas soluções:

i. $\int \sin^4 z \, dz = \int \sin^2 z \, \sin^2 z \, dz$

$= \int \left(\frac{1-\cos 2z}{2}\right) \cdot \left(\frac{1-\cos 2z}{2}\right) dz$

$= \int \frac{1}{4}\left(1 - 2\cos 2z + \cos^2 2z\right) dz$

$= \int \left(\frac{1}{4} - \frac{1}{2}\cos 2z + \frac{1}{4}\cos^2 2z\right) dz$

$= \frac{1}{4}\int 1 \, dz - \frac{1}{2}\int \cos 2z \, dz + \frac{1}{4}\int \underbrace{\cos^2 2z}\, dz$

$= \frac{1}{4} z - \frac{1}{2}\,\frac{\sin 2z}{2} + \frac{1}{4}\int \frac{1+\cos 4z}{2}\, dz$

$= \frac{1}{4} z - \frac{1}{4}\sin 2z + \frac{1}{8}\int 1\, dz + \frac{1}{8}\int \cos 4z \, dz$

$= \underbrace{\frac{1}{4} z} - \frac{1}{4}\sin 2z + \underbrace{\frac{1}{8} z} + \frac{1}{8}\,\frac{1}{4}\sin 4z + C$

$= \frac{3}{8} z - \frac{1}{4}\sin 2z + \frac{1}{32}\sin 4z + C$

11°

Usa-se a fórmula:

$$\int \sin^n z \, dz = -\frac{1}{n} \sin^{n-1} z \cos z + \frac{(n-1)}{n} \int \sin^{n-2} z \, dz$$

Daí

$$\int \sin^4 z \, dz = -\frac{1}{4} \sin^3 z \cos z + \frac{3}{4} \int \sin^2 z \, dz$$

$$= -\frac{1}{4} \sin^3 z \cos z + \frac{3}{4} \int \frac{(1 - \cos 2z)}{2} \, dz$$

$$= -\frac{1}{4} \sin^3 z \cos z + \frac{3}{8} \int 1 \, dz - \frac{3}{8} \int \cos 2z \, dz$$

$$= -\frac{1}{4} \sin^3 z \cos z + \frac{3}{8} z - \frac{3}{8} \frac{1}{2} \sin 2z + C$$

$$= -\frac{1}{4} \sin^3 z \cos z + \frac{3}{8} z - \frac{3}{16} \underbrace{\sin 2z}_{2 \sin z \cos z} + C$$

$$= -\frac{1}{4} \sin^3 z \cos z + \frac{3}{8} z - \frac{3}{8} \sin z \cos z + C$$

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8. $\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (\cos^2 x)^2 \cos x \, dx$

$= \int (1 - \text{sen}^2 x)^2 \cos x \, dx$

$= \int (1 - 2\,\text{sen}^2 x + \text{sen}^4 x) \cos x \, dx$

$= \int \cos x \, dx - 2 \int \text{sen}^2 x \cos x \, dx + \int \text{sen}^4 x \cos x \, dx$

$= \text{sen}\, x - 2 \frac{\text{sen}^3 x}{3} + \frac{\text{sen}^5 x}{5} + C$

$= \text{sen}\, x - \frac{2}{3} \text{sen}^3 x + \frac{1}{5} \text{sen}^5 x + C$

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9. $\int \cos^2 \frac{x}{2} \, dx$

$u = \frac{x}{2} \longrightarrow du = \frac{1}{2} dx \longrightarrow dx = 2\,du$

Daí

$\int \cos^2 \frac{x}{2} \, dx = \int \cos^2 u \; 2\,du = 2 \int \cos^2 u \, du =$

$= \cancel{2} \int \frac{1 + \cos 2u}{\cancel{2}} \, du = \int 1\,du + \int \cos 2u \, du$

$= u + \frac{1}{2} \text{sen}\, 2u + C$

$= \frac{1}{2} x + \frac{1}{2} \text{sen}\, x + C$

10. $\displaystyle\int \sin^3 x \cos^3 x \, dx =$

$\displaystyle = \int \sin^3 x \cos^2 x \cos x \, dx$

$\displaystyle = \int \sin^3 x \, (1 - \sin^2 x) \cos x \, dx$

$\displaystyle = \int (\sin^3 x - \sin^5 x) \cos x \, dx$

$\displaystyle = \int (\sin^3 x \cos x - \sin^5 x \cos x) \, dx$

$\displaystyle = \int \sin^3 x \cos x \, dx - \int \sin^5 x \cos x \, dx$

$\displaystyle = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$

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11. $\displaystyle\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$

$\displaystyle = \int \sin^2 x \, (1 - \sin^2 x) \cos x \, dx$

$\displaystyle = \int \sin^2 x \cos x \, dx - \int \sin^3 x \cos x \, dx$

$\displaystyle = \frac{1}{3} \sin^3 x - \frac{1}{4} \sin^4 x + C$

12. $\int \cos^6 x \, dx$

Temos

$$(*): \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{(n-1)}{n} \int \cos^{n-2} x \, dx$$

Mas

$$\int \cos^2 x \, dx = \int \frac{(1+\cos 2x)}{2} dx = \frac{1}{2}\int 1 \, dx + \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{1}{2} x + \frac{1}{2} \frac{1}{2} \sin 2x$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x$$

Tomando $n=4$ em $(*)$ obtem-se:

$$\int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \underbrace{\int \cos^2 x \, dx}$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left(\frac{1}{2} x + \frac{1}{4} \sin 2x\right)$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} x + \frac{3}{16} \sin 2x$$

Daí

$$\int \cos^6 x \, dx = \frac{1}{6} \cos^5 x \sin x + \frac{5}{6} \int \cos^4 x \, dx$$

$$= \frac{1}{6} \cos^5 x \sin x + \frac{5}{6} \left(\frac{1}{4} \cos^3 x \sin x + \frac{3}{8} x + \frac{3}{16} \sin 2x\right)$$

$$= \frac{1}{6} \cos^5 x \sin x + \frac{5}{24} \cos^3 x \sin x + \frac{5}{16} x + \frac{5}{32} \sin 2x$$

$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}x +$$

$$+ \frac{5}{\cancel{32}\,16}\cancel{2}\sin x \cos x$$

$\therefore$

$$\int \cos^6 x \, dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x +$$

$$+ \frac{5}{16}\sin x \cos x + \frac{5}{16}x + C$$

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13. $\int \sin^5 x \cos^2 x \, dx =$

$$= \int \sin x \sin^4 x \cos^2 x \, dx$$

$$= \int \sin x (\sin^2 x)^2 \cos^2 x \, dx$$

$$= \int \sin x \left(1-\cos^2 x\right)^2 \cos^2 x \, dx$$

$$= \int \sin x \left(1 - 2\cos^2 x + \cos^4 x\right) \cos^2 x \, dx$$

$$= \int \left(\sin x - 2\sin x \cos^2 x + \sin x \cos^4 x\right) \cos^2 x \, dx$$

$$= \int \left(\sin x \cos^2 x - 2\sin x \cos^4 x + \sin x \cos^6 x\right) dx$$

$$= \int \cos^2 x \sin x \, dx - 2\int \cos^4 x \sin x \, dx + \int \cos^6 x \sin x \, dx$$

$$= -\frac{1}{3}\cos^3 x + 2\,\frac{1}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

14. $\int \sin^2 2t \cos^4 2t \, dt$

$$u = 2t \longrightarrow du = 2dt \longrightarrow dt = \frac{1}{2}du$$

Dai

$$\int \sin^2 2t \cos^4 2t \, dt = \int \sin^2 u \cos^4 u \, \frac{1}{2} du$$

$$= \frac{1}{2} \int \sin^2 u \cos^4 u \, du$$

$$= \frac{1}{2} \int (1-\cos^2 u) \cos^4 u \, du$$

$$= \frac{1}{2} \int \cos^4 u \, du - \frac{1}{2} \int \cos^6 u \, du$$

$$= \frac{1}{2} \int \cos^4 u \, du - \frac{1}{2} \left[\frac{1}{6} \cos^5 u \sin u + \frac{5}{6} \int \cos^4 u \, du \right]$$

$$= \frac{1}{2} \int \cos^4 u \, du - \frac{1}{12} \cos^5 u \sin u - \frac{5}{12} \int \cos^4 u \, du$$

$$= \frac{1}{12} \int \cos^4 u \, du - \frac{1}{12} \cos^5 u \sin u$$

$$= \frac{1}{12} \left[\frac{1}{4} \cos^3 u \sin u + \frac{3}{4} \int \cos^2 u \, du \right] - \frac{1}{12} \cos^5 u \sin u$$

$$= \frac{1}{48} \cos^3 u \sin u + \frac{1}{16} \int \cos^2 u \, du - \frac{1}{12} \cos^5 u \sin u$$

$$= \frac{1}{48}\cos^3 u \sin u + \frac{1}{16}\int \frac{(1+\cos 2u)}{2}\,du +$$

$$- \frac{1}{12}\cos^5 u \sin u$$

$$= \frac{1}{48}\cos^3 u \sin u + \frac{1}{32}\int 1\,du + \frac{1}{32}\int \cos 2u\,du$$

$$- \frac{1}{12}\cos^5 u \sin u$$

$$= \frac{1}{48}\cos^3 u \sin u + \frac{1}{32}u + \frac{1}{32}\frac{1}{2}\sin 2u$$

$$- \frac{1}{12}\cos^5 u \sin u$$

$$= \frac{1}{48}\cos^3 2t \sin 2t + \frac{1}{\cancel{32}}\cancel{2}t + \frac{1}{64}\sin 4t$$

$$- \frac{1}{12}\cos^5 2t \sin 2t$$

$$= -\frac{1}{12}\cos^5 2t \sin 2t + \frac{1}{48}\cos^3 2t \sin 2t +$$

$$+ \frac{1}{64}\sin 4t + \frac{1}{16}t + C$$

15.

$$\int \sin^2 3t \cos^2 3t \, dt =$$

$$= \int (\sin 3t \cos 3t)^2 \, dt$$

usou-se que

$\sin 6t = \sin(2.3t)$

$= 2\sin 3t \cos 3t$

$$= \int \left(\frac{1}{2}\sin 6t\right)^2 dt$$

$$= \int \frac{1}{4} \sin^2 6t \, dt$$

$$= \frac{1}{4}\int \sin^2 6t \, dt$$

usou-se que

$\sin^2 6t = \dfrac{1-\cos(2.6t)}{2}$

$= \dfrac{1-\cos 12t}{2}$

$$= \frac{1}{4}\int \left(\frac{1-\cos 12t}{2}\right) dt$$

$$= \frac{1}{8}\int 1\, dt - \frac{1}{8}\int \cos 12t \, dt$$

$$= \frac{1}{8}t - \frac{1}{8}\,\frac{\sin 12t}{12} + C$$

$$= \frac{1}{8}t - \frac{1}{96}\sin 12t + C$$

16) $\int \sqrt{\cos z}\, \sin^3 z \, dz =$

$= \int \sqrt{\cos z}\, \sin^2 z \, \sin z \, dz$

$= \int \sqrt{\cos z}\, (1-\cos^2 z) \sin z \, dz$

$= \int \sqrt{\cos z}\, \sin z \, dz - \int \sqrt{\cos z}\, \cos^2 z \, \sin z \, dz$

$= \int \cos^{1/2} z \, \sin z \, dz - \int \cos^{5/2} z \, \sin z \, dz$

$= \frac{2}{3(-)} \cos^{3/2} z - \frac{2}{7(-)} \cos^{7/2} z + C$

$= -\frac{2}{3} \cos^{3/2} z + \frac{2}{7} \cos^{7/2} z + C$ //

17) $\int \frac{\cos^3 3x}{\sqrt[3]{\sin 3x}} \, dx =$

$= \int \frac{\cos 3x \, \cos^2 3x}{\sqrt[3]{\sin 3x}} \, dx$

$= \int \frac{\cos 3x \, (1-\sin^2 3x)}{\sqrt[3]{\sin 3x}} \, dx$

$= \int \frac{\cos 3x}{\sqrt[3]{\sin 3x}} \, dx - \int \frac{\cos 3x \, \sin^2 3x}{\sqrt[3]{\sin 3x}} \, dx$

$= \int \cos 3x \, (\sin 3x)^{-\frac{1}{3}} dx - \int \cos 3x \, \sin^2 3x \, (\sin 3x)^{-\frac{1}{3}} dx$

$= \frac{3}{2(3)} (\sin 3x)^{\frac{2}{3}} - \int \cos 3x \, (\sin 3x)^{\frac{5}{3}} dx$

$= \frac{1}{2} (\sin 3x)^{2/3} - \frac{1}{8} (\sin 3x)^{\frac{8}{3}} + C$ //

13

18) $\int \sin^{3} \frac{1}{2}y \cos^2 \frac{1}{2}y \, dy =$

$= \int \sin^2 \frac{y}{2} \sin \frac{y}{2} \cos^2 \frac{y}{2} \, dy$

$= \int (1 - \cos^2 \frac{y}{2}) \sin \frac{y}{2} \cos^2 \frac{y}{2} \, dy$

$= \int \sin \frac{y}{2} \cos^2 \frac{y}{2} \, dy - \int \sin \frac{y}{2} \cos^4 \frac{y}{2} \, dy$

$= \frac{1(2)}{3(-)} \cos^3 \frac{y}{2} - \frac{1(2)}{5(-)} \cos^5 \frac{y}{2} + C$

$= -\frac{2}{3} \cos^3 \frac{y}{2} + \frac{2}{5} \cos^5 \frac{y}{2} + C$

19) $\int \cos \underset{m}{4x} \cos \underset{n}{3x} \, dx =$

$\left(\cos mx \cos nx = \frac{1}{2} \cos (m+n)x + \frac{1}{2} \cos (m-n)x \right)$

$= \frac{1}{2} \int (\cos 7x + \cos x) \, dx$

$= \frac{1}{2} \int \cos 7x \, dx + \frac{1}{2} \int \cos x \, dx$

$= \frac{\sin 7x}{14} + \frac{\sin x}{2} + C$

20) $\int \sin \underset{m}{2x} \cos \underset{n}{4x}\, dx =$

$$\left(\sin mx \cos nx = \frac{1}{2} \sin(m-n)x + \frac{1}{2} \sin(m+n)x \right)$$

$$= \int \left(\frac{1}{2} \sin(-2x) + \frac{1}{2} \sin 6x \right) dx$$

$$= -\frac{1}{2} \int \sin 2x\, dx + \frac{1}{2} \int \sin 6x\, dx$$

$$= -\frac{1}{2} \frac{\cos 2x}{(-2)} + \frac{1}{2} \frac{\cos 6x}{(-6)} + C$$

$$= \frac{1}{4} \cos 2x - \frac{1}{12} \cos 6x + C$$

21) $\int \sin \underset{m}{3y} \cos \underset{n}{5y}\, dy =$

$$= \int \left(\frac{1}{2} \sin(-2y) + \frac{1}{2} \sin 8y \right) dy$$

$$= -\frac{1}{2} \int \sin 2y\, dy + \frac{1}{2} \int \sin 8y\, dy$$

$$= -\frac{1}{2} \frac{\cos 2y}{-2} + \frac{1}{2} \frac{\cos 8y}{-8} + C$$

$$= \frac{1}{4} \cos 2y - \frac{1}{16} \cos 8y + C$$

$\cos mx \cos nx = \frac{1}{2}\cos(m+n)x + \frac{1}{2}\cos(m-$

22) $\int \underset{m}{\cos t}\ \underset{n}{\cos 3t}\, dt =$

$= \frac{1}{2}\int \cos 4t\, dt + \frac{1}{2}\int \cos(-2t)\, dt$

$= \frac{1}{2}\ \frac{\sin 4t}{4} + \frac{1}{2}\int \cos 2t\, dt$

$= \frac{\sin 4t}{8} + \frac{1}{2}\ \frac{\sin 2t}{2} + C$

$= \frac{\sin 4t}{8} + \frac{\sin 2t}{4} + C$ //

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23) $\int (\sin 3t - \sin 2t)^2\, dt =$

$= \int (\sin^2 3t - 2\sin 3t \sin 2t + \sin^2 2t)\, dt$

$= \underset{①}{\int \sin^2 3t\, dt} - \underset{②}{2\int \sin 3t \sin 2t\, dt} + \underset{③}{\int \sin^2 2t\, dt}$

$① = \int \sin^2 3t\, dt = \int \frac{1-\cos 6t}{2}\, dt$

$= \frac{1}{2}t - \frac{1}{2}\ \frac{\sin 6t}{6}$

$= \frac{1}{2}t - \frac{1}{12}\sin 6t$ ✓

$\sin^2 3t = \frac{1-\cos 6t}{2}$

$② = -2\int \sin 3t \sin 2t\, dt$

$\underset{3}{\sin mx}\ \underset{2}{\sin nx} = \frac{1}{2}\cos(m-n)x - \frac{1}{2}\cos(m+n)x$

$= -2\int \left(\frac{1}{2}\cos t - \frac{1}{2}\cos 5t\right) dt$

$= -\int \cos t\, dt + \int \cos 5t\, dt = -\sin t + \frac{\sin 5t}{5}$ ✓

③ $= \int \underbrace{\sin^2 2t}\, dt$

$= \int \frac{1-\cos 4t}{2}\, dt$

$= \frac{1}{2}t - \frac{1}{2}\int \cos 4t\, dt$

$= \frac{t}{2} - \frac{1}{2}\, \frac{\sin 4t}{4}$ ✓

Daí,

$$\int (\sin 3t - \sin 2t)^2\, dt = \frac{1}{2}t - \frac{1}{12}\sin 6t$$

$$- \sin t + \frac{\sin 5t}{5}$$

$$+ \frac{t}{2} - \frac{\sin 4t}{8}$$

$$\boxed{\int (\sin 3t - \sin 2t)^2\, dt = t - \sin t - \frac{\sin 4t}{8} + \frac{\sin 5t}{5} - \frac{\sin 6t}{12}}$$

24) $\int \sin x \sin 3x \sin 5x\, dx$

Temos:

$$\sin mx \sin nx = \frac{1}{2}\cos(m-n)x - \frac{1}{2}\cos(m+n)x$$

$$\Leftarrow$$

$$/\!/ \sin \underset{m}{x} \sin \underset{n}{3x} = \frac{1}{2}\cos(-2x) - \frac{1}{2}\cos(4x)$$

$$= \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x \;/\!/$$

→

Daí:

$\sin x \sin 3x \sin 5x =$

$= \left(\frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x\right)\sin 5x$

(★) $= \frac{1}{2}\cos 2x \sin 5x - \frac{1}{2}\cos 4x \sin 5x$

Mas

$\sin mx \cos nx = \frac{1}{2}\sin(m-n)x + \frac{1}{2}\sin(m+n)x$

$\Leftarrow$

(★★) $\begin{cases} \cos 2x \sin 5x = \frac{1}{2}\sin 3x + \frac{1}{2}\sin 7x \\ \cos 4x \sin 5x = \frac{1}{2}\sin x + \frac{1}{2}\sin 9x \end{cases}$

$\Rightarrow$ subst. (★★) em (★):

$\| \sin x \sin 3x \sin 5x = \frac{1}{4}\sin 3x + \frac{1}{4}\sin 7x - \frac{1}{4}\sin x - \frac{1}{4}\sin 9x \|$

$\therefore \int \sin x \sin 3x \sin 5x \, dx =$

$= \int \left(\frac{1}{4}\sin 3x + \frac{1}{4}\sin 7x - \frac{1}{4}\sin x - \frac{1}{4}\sin 9x\right) dx$

$$\boxed{\int \sin x \sin 3x \sin 5x \, dx = -\frac{1}{12}\cos 3x - \frac{1}{28}\cos 7x + \frac{1}{4}\cos x + \frac{1}{36}\cos 9x + C}$$

25) $\int_0^{\pi/2} \cos^3 x \, dx =$

$$= \int_0^{\pi/2} \cos^2 x \cos x \, dx$$

$$= \int_0^{\pi/2} (1 - \sin^2 x) \cos x \, dx$$

$$= \int_0^{\pi/2} (\cos x - \sin^2 x \cos x) \, dx$$

$$= \underbrace{\sin x \Big|_0^{\pi/2}} - \frac{\sin^3 x}{3} \Big|_0^{\pi/2}$$

$$= 1 - \frac{1}{3} = \frac{2}{3} \,//$$

26) $\int_0^1 \sin^{3} \frac{\pi}{2} t \, dt =$

$$= \int_0^1 \sin^2 \frac{\pi}{2} t \sin \frac{\pi}{2} t \, dt$$

$$= \int_0^1 \left(1 - \cos^2 \frac{\pi}{2} t\right) \sin \frac{\pi}{2} t \, dt$$

$$= \int_0^1 \sin \frac{\pi}{2} t \, dt - \int_0^1 \cos^2 \frac{\pi}{2} t \sin \frac{\pi}{2} t \, dt$$

$$= -\frac{2}{\pi} \cos \frac{\pi}{2} t \Big|_0^1 - \frac{2}{\pi} \frac{1}{3(-)} \cos^3 \frac{\pi}{2} t \Big|_0^1$$

$$= \frac{-2}{\pi} \left[-1 \right] + \frac{2}{3\pi} \left[-1 \right]$$

$$= \frac{2}{\pi} - \frac{2}{3\pi} = \frac{4}{3\pi} \,//$$

27) $\int_0^1 \sin^{4}\left(\frac{1}{2}\pi x\right) \, dx$

$$= \int_0^1 \left[\sin^2\left(\frac{\pi x}{2}\right)\right]^2 dx$$

$$= \int_0^1 \left(\frac{1-\cos \pi x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_0^1 (1 - 2\cos\pi x + \cos^2 \pi x) \, dx$$

$$= \underbrace{\frac{1}{4}\int_0^1 dx} - \frac{1}{2}\int_0^1 \cos\pi x \, dx + \frac{1}{4}\int_0^1 \underbrace{\cos^2 \pi x} \, dx$$

$$= \frac{1}{4} - \frac{1}{2} \left. \cancel{\frac{\sin \pi x}{\pi}}^{\,0} \right|_0^1 + \frac{1}{4}\int_0^1 \frac{1+\cos 2\pi x}{2} \, dx$$

$$= \frac{1}{4} + \frac{1}{8}\left[x + \frac{\sin 2\pi x}{2\pi}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{8}\left[1 + 0\right]$$

$$= \frac{1}{4/2} + \frac{1}{8} = \frac{3}{8}$$

28) $\int_0^{\pi/3} \sin^{3} t \cos^{2} t \, dt$

$= \int_0^{\pi/3} \underline{\sin^2 t} \cos^2 t \sin t \, dt$

$= \int_0^{\pi/3} (1-\cos^2 t) \cos^2 t \sin t \, dt$

$= \int_0^{\pi/3} \cos^2 t \sin t \, dt - \int_0^{\pi/3} \cos^4 t \sin t \, dt$

$= -\frac{\cos^3 t}{3}\Big|_0^{\pi/3} + \frac{\cos^5 t}{5}\Big|_0^{\pi/3}$

$\cos\frac{\pi}{3} = \cos 60^\circ = \frac{1}{2}$

$= -\left[\frac{\cos^3\frac{\pi}{3} - \cos^3 0}{3}\right] + \frac{1}{5}\left(\cos^5\frac{\pi}{3} - \cos^5 0\right)$

$= -\frac{1}{3}\left(\left(\frac{1}{2}\right)^3 - 1\right) + \frac{1}{5}\left(\left(\frac{1}{2}\right)^5 - 1\right)$

$= -\frac{1}{3}\left(\frac{1}{8} - 1\right) + \frac{1}{5}\left(\frac{1}{32} - 1\right)$

$= -\frac{1}{3}\left(-\frac{7}{8}\right) + \frac{1}{5}\left(-\frac{31}{32}\right)$

$= \frac{7}{24} - \frac{31}{160} = \frac{7 \times 160 - 31 \times 24}{24 \times 160}$

$= \frac{1120 - 744}{3840} = \frac{376}{3840} = \cancel{\frac{94}{960}} = \frac{47}{480}$

29) $\int_0^1 \sin^2 \pi t \cos^2 \pi t \, dt =$

$(\sin 2x = 2\sin x \cos x)$

$= \int_0^1 (\sin \pi t \cos \pi t)^2 \, dt$

$= \int_0^1 \left(\frac{1}{2}\sin 2\pi t\right)^2 dt$

$= \int_0^1 \frac{1}{4} \sin^2 2\pi t \, dt$

$= \frac{1}{4} \int_0^1 \sin^2 2\pi t \, dt$

$= \frac{1}{4} \int_0^1 \frac{1 - \cos 4\pi t}{2} \, dt$

$= \frac{1}{8} \left[t - \frac{\sin 4\pi t}{4\pi} \right]_0^1$

$= \frac{1}{8} [1 - 0] = \frac{1}{8}$ //

30) $\int_0^{\pi/6} \sin 2x \cos 4x \, dx =$

de (20)

$= \left[\frac{1}{4} \cos 2x - \frac{1}{12} \cos 6x \right]_0^{\pi/6}$

$= \frac{1}{4}\left(\cos \frac{\pi}{3} - 1\right) - \frac{1}{12}(\cos \pi - \cos 0)$

$= \frac{1}{4}\left(\frac{1}{2} - 1\right) - \frac{1}{12}(-1 - 1)$

$= \frac{1}{4}\left(-\frac{1}{2}\right) - \frac{1}{12}(-2)$

$= -\frac{1}{8} + \frac{1}{6} = \frac{-6+8}{48} = \frac{2}{48} = \frac{1}{24}$ //

33) $\int 2\sin x \cos x \, dx$

a) $u = \sin x \longrightarrow du = \cos x \, dx$

$$\int 2\sin x \cos x \, dx = 2\int u \, du$$

$$= \cancel{2}\frac{u^2}{\cancel{2}} + C$$

$\therefore$

$$\Big/\Big/ \int 2\sin x \cos x \, dx = \sin^2 x + C \Big/\Big/$$

b) $u = \cos x \longrightarrow du = -\sin x \, dx$

$$\int 2\sin x \underline{\cos x} \, dx = -2\int u \, du$$

$$= -2\frac{u^2}{2} + C' = -u^2 + C'$$

$\therefore$

$$\Big/\Big/ \int 2\sin x \cos x \, dx = -\cos^2 x + C' \Big/\Big/$$

c) $2\sin x \cos x = \sin 2x$

$$\int 2\sin x \cos x \, dx = \int \sin 2x \, dx$$

$$= \frac{\cos 2x}{-2} + C''$$

$$\Big/\Big/ \int 2\sin x \cos x \, dx = -\frac{1}{2}\cos 2x + C'' \Big/\Big/$$

$\rightarrow$ isto é temos aqui :

29) $\int 2\sin x \cos x \, dx = \begin{cases} \sin^2 x + C & (\star) \\ -\cos^2 x + C' & (\star\star) \\ -\frac{1}{2}\cos 2x + C'' & (\star\star\star) \end{cases}$

que são equivalentes.

De facto:

$\star\star = \underbrace{-\cos^2 x}_{} + C' = -(1-\sin^2 x) + C'$

$= \sin^2 x \underbrace{-1 + C'}_{}$

$= \sin^2 x + C = (\star)$

$(\star\star\star) = -\frac{1}{2}\underbrace{\cos 2x}_{} + C'' = -\frac{1}{2}(2\cos^2 x - 1) + C''$

$= \underbrace{-\cos^2 x}_{} + \frac{1}{2} + C''$

$= \sin^2 x - 1 + \frac{1}{2} + C''$

$= \sin^2 x \underbrace{-\frac{1}{2} + C''}_{}$

$= \sin^2 x + C = \star$

34) $\int_0^{\pi} \sin^2 nx \, dx = \frac{\pi}{2}$ (n inteiro positivo)

$$\int_0^{\pi} \sin^2 nx \, dx = \int_0^{\pi} \frac{1-\cos 2nx}{2} \, dx$$

$$= \left[\frac{1}{2}x - \frac{1}{2}\frac{\sin 2nx}{2n} \right]_0^{\pi}$$

$$= \frac{\pi}{2} - \frac{1}{4n} \sin \cancel{2n\pi}^{\,0}$$

$$= \frac{\pi}{2}$$

35) $\int_0^{\pi} \cos^n x \, dx = 0$ (n inteiro positivo ímpar)

$n = 2k+1$

$$\int_0^{\pi} \cos^{2k+1} x \, dx = \int_0^{\pi} \cos^{2k} x \cos x \, dx$$

$(a+b)^n = \sum_{p=0}^{n} \binom{n}{p} a^{n-p} b^p$

$a = 1, \; b = -\sin^2 x$

$$= \int_0^{\pi} \underbrace{(\cos^2 x)}^{k} \cos x \, dx$$

$$= \int_0^{\pi} (1-\sin^2 x)^k \cos x \, dx$$

$$= \int_0^{\pi} \sum_{r=0}^{k} \binom{k}{r} (-1)^r \sin^{2r} x \cos x \, dx$$

$$= \sum_{r=0}^{k} \binom{k}{r} (-1)^r \underbrace{\int_0^{\pi} \sin^{2r} x \cos x \, dx}$$

$$= \sum_{r=0}^{k} \binom{k}{r} (-1)^r \left. \frac{\sin^{2r+1} x}{2r+1} \right|_0^{\pi} = 0$$

36) $\int_{-1}^{1} \cos n\pi x \cos m\pi x \, dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$

$\cos n\pi x \cos m\pi x = \frac{1}{2}\cos(n+m)\pi x + \frac{1}{2}\cos(n-m)\pi x$

$\Leftrightarrow$

(A) $\int_{-1}^{1} \cos n\pi x \cos m\pi x \, dx = \frac{1}{2}\int_{-1}^{1} \cos(n+m)\pi x \, dx + \frac{1}{2}\int_{-1}^{1} \cos(n-m)\pi x \, dx$

Se $m \neq n$:

$\frac{1}{2}\int_{-1}^{1} \cos(n+m)\pi x \, dx = \frac{1}{2} \left. \frac{\sin(n+m)\pi x}{(n+m)\pi} \right|_{-1}^{1} = 0$

$\frac{1}{2}\int_{-1}^{1} \cos(n-m)\pi x \, dx = \frac{1}{2} \left. \frac{\sin(n-m)\pi x}{(n-m)\pi} \right|_{-1}^{1} = 0$

$\therefore$ $\int_{-1}^{1} \cos n\pi x \cos m\pi x \, dx = 0$ se $m \neq n$

Se $m = n$:

$\frac{1}{2}\int_{-1}^{1} \cos(n+m)\pi x \, dx = \frac{1}{2}\int_{-1}^{1} \cos 2m\pi x \, dx$

$= \frac{1}{2} \left. \frac{\sin 2m\pi x}{2m\pi} \right|_{-1}^{1} = 0$

$\frac{1}{2}\int_{-1}^{1} \underbrace{\cos(n-m)\pi x}_{\cos 0\pi x} \, dx = \frac{1}{2}\int_{-1}^{1} 1 \, dx = \frac{1}{2} \, 2 = 1$

$\boxed{\int_{-1}^{1} \cos n\pi x \cos m\pi x \, dx = 1 \text{ se } m = n}$